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Uniqueness in Inverse Problems for the Isotropic Lamé System

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Abstract. For isotropic Lamé systems with variable coefficients, we discuss inverse problems of determining force terms or densities from a finite number of measurements of lateral boundary data. We establish uniqueness results by the Carleman estimate. A Lamé system with variable coefficients has different principal parts and usual application of the Carleman estimate is difficult, and for the proof of the uniqueness, we reduce the Lamé system to a system with the same principal part by introducing a divergence component.

§1. Introduction

We consider an isotropic Lamé system with variable coefficients in a bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary is of C^2-class:

$$\begin{aligned}\rho(x)u''(x,t) = {}& \mu(x)\Delta u(x,t)+(\lambda(x)+\mu(x))\nabla(\nabla^T u(x,t))\\ &+\nabla^T u(x,t)\nabla\lambda(x)+(\nabla u(x,t)+(\nabla u(x,t))^T)\nabla\mu(x)+R(x,t)f(x),\\ &x\in\Omega,\ -T<t<T,\end{aligned}\tag{1.1}$$

where we set $u = (u_1, ..., u_n)^T$, $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, $\cdot^T$ denotes the transpose of vectors under consideration and we define an $n \times n$ matrix ∇v and real-valued $\nabla^T v$ by

$$\nabla v = \left(\frac{\partial v_i}{\partial x_j}\right)_{1\le i,j\le n} \quad \text{and} \quad \nabla^T v = \sum_{i=1}^n \frac{\partial v_i}{\partial x_i}$$

for $v = (v_1, ..., v_n)^T$, the Lamé parameters λ, μ and the density ρ satisfy

$$\lambda, \mu, \rho \in C^3(\overline{\Omega}),\tag{1.2}$$

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$$2\mu(x) > \delta_0 > 0, \qquad x \in \overline{\Omega}, \tag{1.3}$$

$$n\lambda(x) + 2\mu(x) > \delta_0 > 0, \quad \lambda(x) + \mu(x) > \delta_0, \qquad x \in \overline{\Omega}, \tag{1.4}$$

and

$$\rho(x) > \delta_0 > 0, \qquad x \in \overline{\Omega}, \tag{1.5}$$

with some constant $\delta_0 > 0$. Furthermore $f = f(x)$ and $R = R(x,t)$ are suitable vector-valued or real-valued fucntions.

REMARK. The condition (1.3) and (1.4) mean that the Lamé operator at the right hand side of (1.1) is positive definite (e.g. Gurtin [6]). If $n \geq 2$, then the second inequality in (1.4) follows from the first one and (1.3).

We define a stress tensor $\sigma(u)$ whose (i,j)-component $\sigma(u)_{ij}$ is defined by

$$\sigma(u)_{ij} = \lambda(x)(\nabla^T u)\delta_{ij} + \mu(x)((\nabla u)_{ij} + (\nabla u)_{ji}), \quad 1 \leq i,j \leq n. \tag{1.6}$$

Henceforth A_{ij} denotes the (i,j)-component of an $n \times n$ matrix A and $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. Let us denote the outward normal vector to $\partial\Omega$ at x by $\nu = \nu(x)$.

Then we formulate

Inverse problem for the isotropic Lamé system. Let $u = u(x,t)$ satisfy (1.1) with suitable initial and boundary conditions on $\partial\Omega$. Let $T > 0$ and $R = R(x,t)$ be given. Then determine some of $\{\lambda, \mu, \rho, f\}$ from

$$\sigma(u)(x,t)\nu(x), \qquad x \in \partial\Omega,\ 0 < t < T \tag{1.7}$$

provided that the other functions are given.

In this paper, we discuss the uniqueness in such inverse problems. Our inverse problems are important parameter identification problems where we are required to determine Lamé parameters, a density and/or a forcing term from the surface traction $\sigma(u)\nu$. Under some strict positivity of initial values, the weighted estimate called a Carleman estimate gives a good uniqueness result to similar inverse problems for a single hyperbolic equation and a single parabolic equation (e.g. Bukhgeim [2], Bukhgeim and Klibanov [3], Isakov [9], [10], [11], Khaĭdarov [13], [14], Klibanov [15], [16], [17], Kubo [18]). Originally Carleman [4] has applied such a weighted estimate to the unique continuation in a nonhyperbolic Cauchy problem and for the Carleman estimate, we can further refer to Hörmander [7], Nirenberg [20] for example. For proving the uniqueness in the inverse problem, the key depends on whether we can establish the Carleman estimate.

An attempt has been made by Isakov [8] to obtain a Carleman estimate for the isotropic Lamé system with variable coefficients. By applying the differential operator whose principal symbol is the cofactor matrix of the principal symbol of our system, he transformed the Lamé system to

$$\square_{\alpha_1}\square_{\alpha_2}u + (\text{terms of "order} \le 3\text{"}) \tag{1.8}$$

with $\alpha_1 = \frac{\rho}{\lambda+2\mu}$, $\alpha_2 = \frac{\rho}{\mu}$ and $\square_\alpha u \equiv \alpha u'' - \Delta u$, so that the Carleman estimate is proved and produces the uniqueness results. Furthermore in Isakov [12], the unique continuation is discussed for the thin elastic plate by the Carleman estimate. For the compensation of using the transform is that it makes difficult to determine the density. His Carleman estimate holds true for more general systems of the fourth order in the form (1.8), not restricted to the Lamé system. However, unlike Isakov [8], we derive a Carleman estimate for the equations for u and $\nabla^T u$ whose principal part is diagonal and its diagonal components are $\square_{\alpha_1}\cdot$ and $\square_{\alpha_2}\cdot$.

The purpose of this paper is to show the uniqueness in two inverse problems: determination of a density ρ and determination of a forcing term f by means of the equations for u and $\nabla^T u$. In a forthcoming paper we discuss the determination of Lamé parameters λ and μ.

For the stationary isotropic Lamé system with variable coefficients, the unique continuation by direct application of Carleman estimates is complicated (e.g. Dehman and Robbiano [5]) because of multiple characteristics. However, the usage of similar equations for u and $\nabla^T u$ makes the argument simple (Ang, Ikehata, Trong and Yamamoto [1]).

This paper is organized as follows:
Section 2: formulation and the uniqueness in determining a forcing term; Theorems 1 - 3
Section 3: the uniqueness in determining a density ; Theorems 4 and 5

Section 4: transform of the Lamé system into the system which is diagonal in its principal part
Section 5: Carleman estimate for the transformed system
Section 6: transform of the boundary conditions
Section 7: proof of Theorem 1
Section 8: proof of Theorem 2
Section 9: proof of Theorem 3
Section 10: proofs of Theorems 4 and 5.

§2. Formulation and the Uniqueness in Determining Forcing Terms

Throughout this paper, in addition to (1.2) - (1.5), we further assume

$$1+\frac{\left(x, \nabla\left(\frac{\rho}{\lambda+2\mu}\right)(x)\right)}{2\left(\frac{\rho}{\lambda+2\mu}\right)(x)}>0, \quad 1+\frac{\left(x, \nabla\left(\frac{\rho}{\mu}\right)(x)\right)}{2\left(\frac{\rho}{\mu}\right)(x)}>0, \quad x \in \overline{\Omega}. \tag{2.1}$$

Here $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$.

Remark 2.1. The condition (2.1) is satisfied if $\nabla\left(\frac{\rho}{\lambda+2\mu}\right)$ and $\nabla\left(\frac{\rho}{\mu}\right)$ are small in comparison with Ω or they are monotonically increasing along the x-direction. The condition (2.1) is necessary for actually establishing a Carleman estimate in §5 and is rather restrictive for general λ, μ and ρ. However, in our case where we assume Dirichlet data on the whole boundary $\partial\Omega$, we do not know whether we can establish a Carleman estimate without (2.1). In Isakov [8], [10], Khaĭdarov [13], [14], Klibanov [17], etc., similar conditions are assumed.

Henceforth we set

$$\begin{aligned}(\mathcal{L}u)(x) &= \mu(x)\Delta u(x) + (\lambda(x)+\mu(x))\nabla(\nabla^T u(x)) + \nabla^T u(x)\nabla\lambda(x) \\ &+(\nabla u(x) + (\nabla u(x))^T)\nabla\mu(x), \qquad x \in \Omega.\end{aligned} \tag{2.2}$$

In this paper, we mainly consider two kinds of systems with the principal part (2.2) in x:

$$
\left\{
\begin{array}{l}
\rho(x)y''(x,t) = (\mathcal{L}y)(x,t) + r(x,t)\begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \\
\hfill x \in \Omega,\ -T < t < T, \\
y(x,0) = 0, \qquad x \in \Omega, \\
y(x,t) = 0, \qquad x \in \partial\Omega,\ -T < t < T,
\end{array}
\right\} \tag{2.3}
$$

where r is a scalar-valued function defined in $\Omega \times (-T,T)$.

$$
\left\{
\begin{array}{ll}
\rho(x)y_k''(x,t) = \mathcal{L}(y_k)(x,t) + f(x)R_k(x,t), & x \in \Omega,\ -T < t < T, \\
y_k(x,0) = 0, \qquad x \in \Omega, & \\
y_k(x,t) = 0, \qquad x \in \partial\Omega,\ -T < t < T, &
\end{array}
\right\} \tag{2.4}
$$

where we set

$$
R_k(x,t) = \begin{pmatrix} r_1^{(k)}(x,t) \\ \vdots \\ r_n^{(k)}(x,t) \end{pmatrix}, \quad x \in \Omega,\ -T < t < T,\ 1 \le k \le n,
$$

and f is a scalar-valued function in Ω. We note that in (2.4) we consider a set of n solutions corresponding to n-kinds of non-homogeneous terms $f(x)R_k(x,t)$, $1 \le k \le n$. In (2.3) and (2.4), we do not assume any conditions on $y'(\cdot,0)$. In fact, like in Isakov [10], Klibanov [17] for example, we can determine also $y'(\cdot,0)$ and $y_k'(\cdot,0)$, $1 \le k \le n$ in (2.3) and (2.4) respectively.

We assume that λ, μ, ρ, r, $r_i^{(k)}$, $1 \le i,k \le n$ are given and consider

Inverse Problem I. Let $T > 0$ be given. In the system (2.3), does $\sigma(y)(x,t)\nu(x) = 0$, $x \in \partial\Omega$, $-T < t < T$ imply $f_1(x) = \ldots = f_n(x) = 0$, $x \in \Omega$?

Inverse Problem II. Let $T > 0$ be given. In the system (2.4), does $\sigma(y_k)(x,t)\nu(x) = 0$, $x \in \partial\Omega$, $-T < t < T$, $1 \le k \le n$ imply $f(x) = 0$, $x \in \Omega$?

For the statements of our uniqueness results for these inverse problems, we introduce notations. We set

$$\alpha_1(x) = \left(\frac{\rho}{\lambda + 2\mu}\right)(x), \quad \alpha_2(x) = \left(\frac{\rho}{\mu}\right)(x), \quad x \in \Omega \tag{2.5}$$

and

$$d_j(x) = |\nabla \alpha_j(x)| \alpha_j(x)^{-\frac{1}{2}}, \quad e_j(x) = 1 + \frac{(\nabla \alpha_j(x), x)}{2\alpha_j(x)}, \quad j = 1, 2. \tag{2.6}$$

For $\Omega \subset \mathbb{R}^n$, we define $\omega > 0$ by

$$\omega = \inf_{\eta \in \Omega} \sup_{x \in \Omega} |x - \eta|.$$

Henceforth after translating Ω by a fixed $x_0 \in \mathbb{R}^n$ if necessary, we can assume that

$$0 \in \Omega \text{ and the infimum in } \eta \text{ is attained at } \eta = 0.$$

That is, throughout this paper, we assume

$$\omega = \inf_{\eta \in \Omega} \sup_{x \in \Omega} |x - \eta| = \sup_{x \in \Omega} |x|. \tag{2.7}$$

Then we choose $\theta > 0$ such that

$$\begin{cases} 0 < \theta < \displaystyle\min_{x \in \overline{\Omega},\, j=1,2} \left(\frac{\sqrt{\omega^2 d_j(x)^2 + 4\alpha_j(x) e_j(x)} - \omega d_j(x)}{2\alpha_j(x)} \right)^2 \\ 0 < \theta \le \displaystyle\min_{x \in \overline{\Omega},\, j=1,2} \frac{1}{\alpha_j(x)}. \end{cases} \tag{2.8}$$

For these inverse problems of determining forcing terms, we are ready to state the uniqueness results, the first group of our main results.

THEOREM 1 (Uniqueness for Inverse Problem I). *Let* $r, \frac{\partial r}{\partial t} \in C^2(\overline{\Omega} \times [-T, T])$, *and* $f_1, ..., f_n \in C(\overline{\Omega})$. *Moreover let us assume that there exists a constant* $r_0 > 0$ *such that*

$$r(x, 0) \ge r_0 > 0, \qquad x \in \overline{\Omega}. \tag{2.9}$$

Let

$$T > \frac{\sup_{x\in\Omega}|x|}{\sqrt{\theta}}. \tag{2.10}$$

If $y \in C^2(\overline{\Omega} \times [-T,T])^n$ *satisfies (2.3),*

$$y \in C^4(\overline{\Omega} \times [-T,T])^n \tag{2.11}$$

and

$$\sigma(y)(x,t)\nu(x) = 0, \qquad x \in \partial\Omega,\ -T < t < T, \tag{2.12}$$

then

$$f_1(x) = \cdots = f_n(x) = 0, \qquad x \in \Omega \tag{2.13}$$

and

$$y(x,t) = 0 \qquad \text{in } \{(x,t) \in \Omega \times (-T,T); |x|^2 - \theta t^2 > 0\}. \tag{2.14}$$

Theorem 2 (Uniqueness for Inverse Problem II). *Let*

$$\left\{ \begin{array}{ll} R_k, R'_k \in C^2(\overline{\Omega} \times [-T,T])^n, & 1 \le k \le n \\ \nabla^T R_k, \nabla^T R'_k \in C^2(\overline{\Omega} \times [-T,T]) & \end{array} \right\} \tag{2.15}$$

and

$$f \in C^1(\overline{\Omega}). \tag{2.16}$$

We assume that there exists a constant $r_0 > 0$ *and some* i, k *with* $1 \le i, k \le n$ *such that*

$$|\det(R_1(x,0), ..., R_n(x,0))| \ge r_0 > 0, \quad x \in \overline{\Omega} \tag{2.17}$$

and

$$|r_i^{(k)}(x,0)| \ge r_0 > 0, \qquad x \in \overline{\Omega}. \tag{2.18}$$

Moreover let

$$T > \frac{\sup_{x\in\Omega}|x|}{\sqrt{\theta}}.$$

If $y_k \in C^2(\overline{\Omega}\times[-T,T])^n$ *satisfies (2.11), (2.4) and*

$$\sigma(y_k)(x,t)\nu(x) = 0, \qquad x\in\partial\Omega,\ -T<t<T,\ 1\le k\le n, \tag{2.19}$$

then

$$f(x) = 0, \qquad x\in\Omega \tag{2.20}$$

and

$$\begin{aligned} &y_k(x,t) = 0 \\ &\qquad\qquad \textit{in } \{(x,t)\in\Omega\times(-T,T); |x|^2-\theta t^2>0\},\ 1\le k\le n. \end{aligned} \tag{2.21}$$

REMARK 2.2. The condition (2.10) for observation time is sufficient for the uniqueness and same as the one in Isakov [8] or [10]. The time length in (2.10) depends on both $\sup_{x\in\Omega}|x|$ and θ given by (2.8). In a simple case where $\frac{\rho}{\lambda+2\mu}$ and $\frac{\rho}{\mu}$ are constants, we can rewrite (2.8) as

$$0<\theta\le \min_{x\in\overline{\Omega},j=1,2}\frac{1}{\alpha_j(x)} \tag{2.8'}$$

so that the condition (2.10) means that $2T$ has to be greater than the time where the P-wave and the S-wave run over the diameter of Ω, and so in this case, (2.10) gives the optimal observation time length for the uniqueness. On the other hand, in the case of variable $\frac{\rho}{\lambda+2\mu}$ and $\frac{\rho}{\mu}$, the optimal time is an open problem.

REMARK 2.3. In Theorems 1 and 2, we require that y, r and R are sufficiently smooth on $\overline{\Omega}\times[-T,T]$. We can relax the regularity condition if we can apply the Carleman estimate in Sobolev spaces of the negative order. As for such a Carleman estimate for the D'Alembertian, we can refer to Ruiz [21].

In order to obtain the uniqueness of a scalar function f in Inverse Problem II, we have to change n-times non-homogeneous terms R_k so that (2.17) is true. This is a very overdetermining formulation for Inverse Problem II.

In particular, in the case of determining 2-dimensional f, we can prove the uniqueness without changing non-homogeneous terms.

Theorem 3 (Uniqueness in the two dimensional Inverse Problem I). *We consider*

$$\left\{\begin{array}{l}\rho(x)y''(x,t) = (\mathcal{L}y)(x,t) + f(x)R(x,t), \\ x = (x_1, ..., x_n) \in \Omega \subset \mathbb{R}^n,\ -T < t < T, \\ y(x,0) = y'(x,0) = 0, \qquad x \in \Omega, \\ y(x,t) = 0, \qquad x \in \partial\Omega,\ -T < t < T,\end{array}\right\} \tag{2.22}$$

where

$$R(x,t) = \begin{pmatrix} r_1(x,t) \\ \vdots \\ r_n(x,t) \end{pmatrix}, \quad x \in \Omega,\ -T < t < T.$$

We assume that

$$f \in C^1(\overline{\Omega}) \text{ and } f(x) \text{ depends only on } x_1 \text{ and } x_2. \tag{2.23}$$

Moreover let R satisfy the regularity condition

$$\left\{\begin{array}{l} R, R', R'' \in C^2(\overline{\Omega} \times [-T,T])^n, \\ \nabla^T R, \nabla^T R', \nabla^T R'' \in C^2(\overline{\Omega} \times [-T,T]) \end{array}\right\} \tag{2.24}$$

and

$$\left|\det \begin{pmatrix} r_1(x,0) & r_1'(x,0) \\ r_2(x,0) & r_2'(x,0) \end{pmatrix}\right| \ge r_0 > 0, \quad x \in \overline{\Omega} \tag{2.25}$$

with some constant $r_0 > 0$ independent of $x \in \overline{\Omega}$. Moreover let

$$T > \frac{\sup_{x\in\Omega} |x|}{\sqrt{\theta}}.$$

If $y = y(x,t)$ satisfies (2.22) with the regularity condition

$$y \in C^5(\overline{\Omega} \times [-T,T])^n \tag{2.26}$$

and

$$\sigma(y)(x,t)\nu(x) = 0, \qquad x \in \partial\Omega,\ -T < t < T, \tag{2.27}$$

then

$$f(x) = 0, \qquad x \in \Omega \tag{2.28}$$

and

$$y(x,t) = 0, \qquad x \in \Omega,\ -T < t < T. \tag{2.29}$$

In Theorem 3, we note that Ω itself is a domain in $\mathbb{R}^n$. We can determine f uniquely in a single system (2.22), while we have to further assume $y'(x,0) = 0$, $x \in \Omega$.

We can restate Theorem 3 in the interval $(0,T)$.

THEOREM 3'. *We consider*

$$\left\{\begin{array}{l} \rho(x)y''(x,t) = (\mathcal{L}y)(x,t) + f(x)R(x,t), \\ x = (x_1,...,x_n) \in \Omega \subset \mathbb{R}^n,\ 0 < t < T, \\ y(x,0) = y'(x,0) = 0, \qquad x \in \Omega, \\ y(x,t) = 0, \qquad x \in \partial\Omega,\ 0 < t < T. \end{array}\right\} \tag{2.22'}$$

Under the same assumptions as in Theorem 3, let $y \in C^2(\overline{\Omega}\times[0,T])^n$ satisfy (2.22)' and

$$y \in C^5(\overline{\Omega} \times [0,T])^n \tag{2.26'}$$

and

$$\sigma(y)(x,t)\nu(x) = 0, \qquad x \in \partial\Omega,\ 0 < t < T. \tag{2.27'}$$

Moreover we assume that $R(x,t)$ *can be extended to* $[-T,0]$ *such that the extension is so smooth that the solution* $\widehat{y} = \widehat{y}(x,t)$ *to*

$$\left\{\begin{array}{l} \rho(x)\widehat{y}''(x,t) = (\mathcal{L}\widehat{y})(x,t) + f(x)R(x,t), \\ x \in \Omega \subset \mathbb{R}^n,\ -T < t < 0, \\ \widehat{y}(x,0) = \widehat{y}'(x,0) = 0, \qquad x \in \Omega, \\ \widehat{y}(x,t) = 0, \qquad x \in \partial\Omega,\ -T < t < 0 \end{array}\right\} \tag{2.30}$$

satisfies $\widehat{y} \in C^5(\overline{\Omega} \times [-T,0])$*. Then* $f(x) = 0$*,* $x \in \Omega$ *and* $y(x,t) = 0$*,* $x \in \Omega$*,* $0 < t < T$ *follows.*

For a sufficient regularity condition in order that $\widehat{y} \in C^5(\overline{\Omega} \times [-T,0])$ in (2.30), we can apply results in Lions and Magenes [19] for example.

We can directly derive Theorem 3' from Theorem 3, noting that

$$Y(x,t) = \left\{\begin{array}{ll} y(x,t), & t \ge 0 \\ \widehat{y}(x,t), & t < 0 \end{array}\right.$$

is in $C^5(\overline{\Omega} \times [-T,T])$.

§3. Uniqueness of Density

In this section, we apply Theorems 2 and 3 for Inverse Problem II of determining a density ρ provided that the Lamé parameters λ and μ are known. For fixed $\theta_0 > 0$, we define an admissible set of densities ρ's in the following way:

$$\begin{aligned} \mathcal{U} = \Bigg\{ & \rho \in C^2(\overline{\Omega}); 1 + \frac{\left(x, \nabla\left(\frac{\rho}{\lambda+2\mu}\right)(x)\right)}{2\left(\frac{\rho}{\lambda+2\mu}\right)(x)} > 0, \\ & 1 + \frac{\left(x, \nabla\left(\frac{\rho}{\mu}\right)(x)\right)}{2\left(\frac{\rho}{\mu}\right)(x)} > 0, \qquad x \in \overline{\Omega}, \\ & \min_{x\in\overline{\Omega},\, j=1,2} \left(\frac{\sqrt{\omega^2 d_j(x)^2 + 4\alpha_j(x)e_j(x)} - \omega d_j(x)}{2\alpha_j(x)}\right)^2 > \theta_0, \\ & \min_{x\in\overline{\Omega},\, j=1,2} \frac{1}{\alpha_j(x)} \ge \theta_0 \Bigg\}, \end{aligned} \tag{3.1}$$

where $\alpha_j = \alpha_j(\rho, \lambda, \mu)$, $d_j = d_j(\rho, \lambda, \mu)$, $e_j = e_j(\rho, \lambda, \mu)$, $j = 1, 2$ are defined by (2.5) and (2.6) for ρ, λ and μ, and ω is given by (2.7). We discuss the Lamé systems without non-homogeneous terms.

$$\left\{\begin{array}{ll} \rho(x)u''(x,t) = (\mathcal{L}u)(x,t), & x \in \Omega,\ -T < t < T, \\ u(x,0) = a_k(x) = (a_1^{(k)}(x), ..., a_n^{(k)}(x))^T, & x \in \Omega, \\ u(x,t) = \eta_k(x,t), & x \in \partial\Omega,\ -T < t < T, \end{array}\right\} \tag{3.2}$$

and

$$\left\{\begin{array}{ll} \widetilde{\rho}(x)\widetilde{u}''(x,t) = (\mathcal{L}\widetilde{u})(x,t), & x \in \Omega,\ -T < t < T, \\ \widetilde{u}(x,0) = a_k(x) = (a_1^{(k)}(x), ..., a_n^{(k)}(x))^T, & x \in \Omega, \\ \widetilde{u}(x,t) = \eta_k(x,t), & x \in \partial\Omega,\ -T < t < T \end{array}\right\} \tag{3.3}$$

for $1 \le k \le n$. Let $u = u_k$ and $\widetilde{u} = \widetilde{u_k}$ satisfy (3.2) and (3.3) respectively for a_k and η_k, $1 \le k \le n$. In this section, a_k, η_k, $1 \le k \le n$ are given suitably.

Henceforth we denote the i-th component of $a \in \mathbb{R}^n$ by $[a]_i$; $[a]_i = a_i$, $1 \le i \le n$ for $a = (a_1, ..., a_n)^T$.

We are concerned with the unique determination of $\widetilde{\rho}$ in (3.3) with $1 \le k \le n$. In other words, we determine the density function by changing n-times initial displacements a_k, $1 \le k \le n$ and boundary values η_k, $1 \le k \le n$. For this formulation, we can state the uniqueness result as follows.

THEOREM 4 (Uniqueness of density by n observations of boundary traction). *Let $\rho, \widetilde{\rho} \in \mathcal{U}$. We assume*

$$T > \frac{\sup_{x\in\Omega} |x|}{\sqrt{\theta_0}} \tag{3.4}$$

and there exists a constant $r_0 > 0$ and some i, k with $1 \le i, k \le n$ such that

$$|\det((\mathcal{L}a_1)(x), ..., (\mathcal{L}a_n)(x))| \ge r_0 > 0, \quad x \in \overline{\Omega} \tag{3.5}$$

and

$$|[(\mathcal{L}a_k)(x)]_i| \ge r_0 > 0, \qquad x \in \overline{\Omega}. \tag{3.6}$$

If for $1 \le k \le n$, u_k *and* $\widetilde{u_k}$ *satisfy (3.2) and (3.3) respectively, and*

$$u_k, \widetilde{u_k} \in C^4(\overline{\Omega} \times [-T, T])^n, \quad 1 \le k \le n, \tag{3.7}$$

$$\begin{aligned} &\sigma(u_k)(x,t)\nu(x) = \sigma(\widetilde{u_k})(x,t)\nu(x), \\ &x \in \partial\Omega,\ -T < t < T,\ 1 \le k \le n, \end{aligned} \tag{3.8}$$

then we have $\rho(x) = \widetilde{\rho}(x)$, $x \in \Omega$ *and*

$$\begin{aligned} &u_k(x,t) = \widetilde{u_k}(x,t) \\ &\qquad \textit{in } \{(x,t) \in \Omega \times (-T,T); |x|^2 - \theta_0 t^2 > 0\},\ 1 \le k \le n. \end{aligned}$$

In the n-dimensional case, we do not know whether we can prove the uniqueness by suitably choosing an initial displacement and boundary value one time, not n-time. This theorem asserts that if we can choose initial displacement and boundary data n-times in an appropriate way such that (3.5) and (3.6) are satisfied, then the uniqueness follows. We should compare Theorem 4 with a result by Isakov [8] which proved the uniqueness in determining a density function in the *three dimensional* case by choosing suitable *four kinds* of data. Moreover our regularity assumption (3.7) is more relaxed.

According to Theorem 3, if we can assume that unknown densities depend only on two components of x, say, x_1 and x_2, then a single adequate choice of intial displacement and boundary data can guarantee the uniqueness. More precisely, we have

Theorem 5 (Uniqueness of two dimensional density by a single observation). *We consider*

$$\left\{ \begin{array}{l} \rho(x)u''(x,t) = (\mathcal{L}u)(x,t), \quad x \in \Omega \subset \mathbb{R}^n,\ -T < t < T, \\ u(x,0) = a(x), \quad u'(x,0) = b(x), \qquad x \in \Omega, \\ u(x,t) = \eta(x,t), \qquad x \in \partial\Omega,\ -T < t < T \end{array} \right\} \tag{3.9}$$

and

$$
\left\{
\begin{array}{ll}
\widetilde{\rho}(x)\widetilde{u''}(x,t) = (\mathcal{L}\widetilde{u})(x,t), & x \in \Omega \subset \mathbb{R}^n,\ -T < t < T, \\
\widetilde{u}(x,0) = a(x), \quad \widetilde{u}'(x,0) = b(x), & x \in \Omega, \\
\widetilde{u}(x,t) = \eta(x,t), & x \in \partial\Omega,\ -T < t < T.
\end{array}
\right\}
\tag{3.10}
$$

Let us set

$$
\mathcal{V} = \{\rho \in \mathcal{U}; \rho \text{ depends only on } x_1 \text{ and } x_2\}
\tag{3.11}
$$

where $\mathcal{U}$ *is defined by (3.1). Let initial data* a, b *and boundary data* η *be given such that*

$$
\left| \det \begin{pmatrix} [(\mathcal{L}a)(x)]_1 & [(\mathcal{L}b)(x)]_1 \\ [(\mathcal{L}a)(x)]_2 & [(\mathcal{L}b)(x)]_2 \end{pmatrix} \right| \geq r_0 > 0, \quad x \in \overline{\Omega}
\tag{3.12}
$$

with some constant $r_0 > 0$ *independent of* $x \in \overline{\Omega}$, *and the solutions* u *and* $\widetilde{u}$ *to (3.9) and (3.10) satisfy the regularity condition*

$$
u, \widetilde{u} \in C^5(\overline{\Omega} \times [-T,T])^n.
\tag{3.13}
$$

Moreover we assume

$$
\rho, \widetilde{\rho} \in \mathcal{V}.
\tag{3.14}
$$

If

$$
\sigma(u)(x,t)\nu(x) = \sigma(\widetilde{u})(x,t)\nu(x), \qquad x \in \partial\Omega,\ -T < t < T,
\tag{3.15}
$$

then

$$
\rho(x) = \widetilde{\rho}(x), \quad u(x,t) = \widetilde{u}(x,t), \qquad x \in \Omega,\ -T < t < T
$$

follows.

§4. Diagonalization of the Lamé System

Our system $\rho y'' - \mathcal{L}y$ has constant multiple characteristics and non-diagonalizable principal parts, so that direct application of the usual Carleman estimate is difficult. Here we reduce (2.2) to a diagonal system for y and $\nabla^T y = \sum_{i=1}^n \frac{\partial y_i}{\partial x_i}(x)$ by introducing $\nabla^T y$ and changing variables. Henceforth I_n denotes the $n \times n$ identity matrix. We set

$$\left\{\begin{array}{l} \gamma(x) = \mu(x)^{-\frac{1}{2}} \\ \beta(x) = \gamma(x)(\lambda(x) + \mu(x)) \\ \zeta(x) = (\lambda + \mu)(x)\nabla\gamma(x) + \gamma(x)\nabla\mu(x), \quad x \in \Omega \end{array}\right\} \tag{4.1}$$

and

$$\left\{\begin{array}{ll} \Gamma(x) = \nabla\lambda(x) \otimes \nabla\gamma(x) + \nabla\gamma(x) \otimes \nabla\mu(x) & \\ +(\lambda+\mu)(x)\nabla^2\gamma(x) + (\nabla\gamma(x)\cdot\nabla\mu(x) + \mu\Delta\gamma(x))I_n & \\ \Lambda(x) = \gamma(x)\{\Gamma(x) - (\nabla\zeta(x))^T\} - \nabla\gamma(x)\otimes\zeta(x), & x \in \Omega, \end{array}\right\} \tag{4.2}$$

$$\left\{\begin{array}{l} \Gamma_1(x) = \begin{pmatrix} -\frac{\gamma\zeta}{1+\gamma\beta} & \frac{\gamma^2\zeta\otimes\zeta}{1+\gamma\beta} + \Lambda \\ \frac{\gamma\beta}{1+\gamma\beta} & \frac{\gamma}{1+\gamma\beta}\zeta^T \end{pmatrix} \\ \Gamma_2(x) = \begin{pmatrix} 0 & 0 \\ -\frac{\gamma\zeta}{1+\gamma\beta} & \frac{\gamma^2\zeta\otimes\zeta}{1+\gamma\beta} + \Lambda \end{pmatrix}. \end{array}\right\} \tag{4.3}$$

Moreover we define a differential operator P_1 of the first order by

$$\begin{aligned} &\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1} \times \left\{\Delta + \begin{pmatrix} \nabla^T & 0 \\ 0 & \nabla \end{pmatrix}\Gamma_1 + \Gamma_2\right\} \\ &\times \begin{pmatrix} 1+\gamma\beta & (\gamma\zeta)^T \\ 0 & I_n \end{pmatrix}\begin{pmatrix} \gamma^{-1} & (\nabla\gamma^{-1})^T \\ 0 & \gamma^{-1}I_n \end{pmatrix} z(x) \\ =&\rho^{-1}\begin{pmatrix} \lambda+2\mu & 2(\nabla\mu)^T - \frac{\mu}{\rho}(\nabla\rho)^T \\ 0 & \mu I_n \end{pmatrix}\Delta z(x) \\ &+(P_1 z)(x), \qquad x \in \Omega \end{aligned} \tag{4.4}$$

for $z(x) = (z_1(x), ..., z_n(x), z_{n+1}(x))^T$. We note that all the coefficients of P_1 are independent of t. Here we note

$$\begin{aligned}
&\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1} \begin{pmatrix} 1+\gamma\beta & (\gamma\zeta)^T \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \gamma^{-1} & (\nabla\gamma^{-1})^T \\ 0 & \gamma^{-1}I_n \end{pmatrix} \\
&= \rho^{-1} \begin{pmatrix} \lambda+2\mu & 2(\nabla\mu)^T - \frac{\mu}{\rho}(\nabla\rho)^T \\ 0 & \mu I_n \end{pmatrix}.
\end{aligned} \tag{4.5}$$

Let us set

$$A_0(x) = \rho^{-1} \begin{pmatrix} \lambda+2\mu & 2(\nabla\mu)^T - \frac{\mu}{\rho}(\nabla\rho)^T \\ 0 & \mu I_n \end{pmatrix}(x). \tag{4.6}$$

By (1.4), we see that $\lambda(x) + \mu(x) > 0$, $x \in \overline{\Omega}$. Moreover in view of (1.2), we can show

LEMMA 4.1. *We set*

$$Q(x) = \begin{pmatrix} 1 & 2(\nabla\mu)^T - \frac{\mu}{\rho}(\nabla\rho)^T \\ 0 & -(\lambda+\mu)I_n \end{pmatrix}, \quad x \in \Omega. \tag{4.7}$$

Then $Q, Q^{-1} \in C^2(\overline{\Omega})$ *and*

$$Q(x)^{-1}A_0(x)Q(x) = \begin{pmatrix} \alpha_1(x)^{-1} & 0 \\ 0 & \alpha_2(x)^{-1}I_n \end{pmatrix}, \quad x \in \Omega.$$

Here we recall (2.5). Moreover we set

$$\Pi(x) = \begin{pmatrix} \alpha_1(x) & 0 \\ 0 & \alpha_2(x)I_n \end{pmatrix}, \quad x \in \Omega, \tag{4.8}$$

and we define differential operators P_2, D_1 and P by

$$P_2 w = \Delta(Qw) - Q\Delta w, \tag{4.9}$$

$$D_1 w = -\Pi(Q^{-1}P_1 Q + Q^{-1}A_0 P_2)w, \tag{4.10}$$

and

$$Pw = (\Pi\partial_t^2 - \Delta)w + D_1 w \tag{4.11}$$

for $w \in C^2(\overline{\Omega} \times [-T,T])^{n+1}$. Here Q, P_1 and A_0 are defined by (4.7), (4.4) and (4.6). Then we note that D_1 is of order 1 and that all the coefficients of D_1 and P are independent of t.

We are ready to state the diagonalization:

PROPOSITION 4.1. *Let us set*

$$w(x,t) = Q(x)^{-1}\begin{pmatrix} \nabla^T y(x,t) \\ y(x,t) \end{pmatrix}. \tag{4.12}$$

Then

$$\begin{aligned}
&\Pi Q^{-1}\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1}\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix}\gamma[\rho y'' - \mathcal{L}y - \Phi(x,t)] \\
=&(\Pi\partial_t^2 - \Delta)w - \Pi(Q^{-1}P_1Q + Q^{-1}A_0P_2)w \\
-&\Pi Q^{-1}\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1}\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix}\gamma\Phi(x,t) \\
=&Pw - \Pi Q^{-1}\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1}\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix}\gamma\Phi(x,t) \\
&\qquad\qquad\qquad\qquad \text{in } \Omega \times (-T,T).
\end{aligned} \tag{4.13}$$

Here $\Phi = \Phi(x,t)$ is an $n \times 1$ matrix and

$$\Phi, \frac{\partial\Phi}{\partial x_i} \in C(\overline{\Omega} \times [-T,T]), \quad 1 \le i \le n.$$

Henceforth for a $k \times l$ matrix-valued function Ψ, we simply write $\Psi \in C(\overline{\Omega} \times [-T,T])$, not $\Psi \in C(\overline{\Omega} \times [-T,T])^{k\times l}$, when all the components of Ψ are in $C(\overline{\Omega} \times [-T,T])$.

PROOF OF PROPOSITION 4.1. Here we recall that the operator $\mathcal{L}$ is defined by (2.2). First lengthy but direct calculations yield

Lemma 4.2.

$$\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma(\rho y'' - \mathcal{L}y) = \begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix} \left[\partial_t^2 - A_0(x)\Delta - P_1\right] \begin{pmatrix} \nabla^T y \\ y \end{pmatrix}.$$

By Lemma 4.2, we have

$$\begin{aligned}
&\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma(\rho y'' - \mathcal{L}y - \Phi) \\
=&(\partial_t^2 - A_0(x)\Delta - P_1) \begin{pmatrix} \nabla^T y \\ y \end{pmatrix} \\
-&\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma\Phi,
\end{aligned}$$

and

$$\begin{aligned}
&Q^{-1}\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma(\rho y'' - \mathcal{L}y - \Phi) \\
=&Q^{-1}\partial_t^2 \begin{pmatrix} \nabla^T y \\ y \end{pmatrix} - Q^{-1}A_0\Delta \begin{pmatrix} \nabla^T y \\ y \end{pmatrix} \\
&-Q^{-1}P_1 \begin{pmatrix} \nabla^T y \\ y \end{pmatrix} - Q^{-1}\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma\Phi \\
=&Q^{-1}\partial_t^2(Qw) - Q^{-1}A_0\Delta(Qw) - Q^{-1}P_1Qw \\
&-Q^{-1}\begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma\Phi
\end{aligned}$$

by (4.12). By noting (4.9), multiplication of $\Pi = \Pi(x)$ from the left completes the proof of Proposition 4.1. □

§5. Carleman Estimate for a Diagonal System

By Proposition 4.1, we can reduce the Lamé system to a diagonal system of hyperbolic equations. Therefore on the basis of an existing Carleman estimate, we can establish the Carleman estimate for the Lamé system.

We set

$$\phi = \phi(x,t) = |x|^2 - \theta t^2 \tag{5.1}$$

where $\theta > 0$ is given by (2.8), and

$$\phi_\epsilon = \{(x,t) \in \Omega \times (-\infty, \infty); \phi(x,t) > \epsilon^2\} \tag{5.2}$$

with a constant $\epsilon > 0$. Henceforth we further set

$$\|w\|^2_{L^2(\phi_\epsilon)} = \sum_{k=1}^{n+1} \int_{\phi_\epsilon} |w_k(x,t)|^2 dxdt \tag{5.3}$$

for $w = (w_1, ..., w_n, w_{n+1})^T$, and

$$x_{n+1} = t, \quad x = (x_1, ..., x_n), \tag{5.4}$$

$$D^\alpha = \partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_{n+1}}_{x_{n+1}}, \quad \alpha = (\alpha_1, ..., \alpha_{n+1}) \in (\mathbb{N} \cup \{0\})^{n+1}, \tag{5.5}$$

$$|\alpha| = \alpha_1 + \dots + \alpha_{n+1}. \tag{5.6}$$

For the operator P, we show the following Carleman estimate.

PROPOSITION 5.1. *In addition to (2.10), we assume that $\phi_\epsilon \subset \Omega \times (-T,T)$ and $T - \frac{\sup_{x\in\Omega}|x|}{\sqrt{\theta}}$ is sufficiently small. Let $\epsilon > 0$ be given. Then there exist constants $\tau = \tau(\epsilon) > 0$, $M = M(\epsilon) > 0$ and $\Xi = \Xi(\epsilon) > 0$ such that if $\xi > \Xi(\epsilon)$, then*

$$\begin{aligned}&\xi^3 \|w \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} + \xi \sum_{|\alpha|=1} \|(D^\alpha w) \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} \\ \le& M \|(Pw) \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)}\end{aligned} \tag{5.7}$$

for any $w \in H^2_0(\phi_\epsilon)^{n+1}$.

Here and henceforth $\sum_{|\alpha|=1}$ means the sum when the multi-index α varies over $|\alpha| = 1$.

PROOF OF PROPOSITIO 5.1. First we can restate the Carleman estimate by Isakov [10], [11] as follows:

PROPOSITION 5.2. *Let*

$$c \in C^2(\overline{\Omega}), \quad c(x) > 0, \quad 1 + \frac{(\nabla c(x), x)}{2c(x)} > 0, \quad x \in \overline{\Omega} \tag{5.8}$$

and $\phi_\epsilon \subset \Omega \times (-T, T)$. *We set* $\widetilde{d}(x) = |\nabla c(x)| c(x)^{-\frac{1}{2}}$ *and* $\widetilde{e}(x) = 1 + \frac{(\nabla c(x),x)}{2c(x)} > 0$. *Moreover we fix* $\theta > 0$ *such that*

$$0 < \theta < \min_{x \in \overline{\Omega}} \left(\frac{\sqrt{\omega^2 \widetilde{d}(x)^2 + 4c(x)\widetilde{e}(x)} - \omega \widetilde{d}(x)}{2c(x)} \right)^2 \tag{5.9}$$

and

$$0 < \theta \le \min_{x \in \overline{\Omega}} \frac{1}{c(x)}. \tag{5.10}$$

Then there exists $\eta_0 = \eta_0(c) > 0$ *such that if*

$$0 < T - \frac{\sup_{x \in \Omega} |x|}{\sqrt{\theta}} < \eta_0, \tag{5.11}$$

then for $\epsilon > 0$, *there exist constants* $\tau = \tau(\epsilon) > 0$, $M = M(\epsilon) > 0$ *and* $\Xi = \Xi(\epsilon) > 0$ *such that if* $\xi > \Xi(\epsilon)$, *then*

$$\begin{aligned} &\xi^3 \|u \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} + \xi \sum_{|\alpha|=1} \|(D^\alpha u) \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} \\ \le& M \|(c(x) \frac{\partial^2 u}{\partial t^2} - \Delta u) \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} \end{aligned}$$

for any $u \in H^2_0(\phi_\epsilon)$.

In fact, the Carleman estimate by Isakov (e.g. Corollary 1.2.5 in [10]) reads as follows: If

$$\begin{aligned}
&\theta\left(c(x)+|t||\nabla c(x)|c(x)^{-\frac{1}{2}}\right)\\
&<1+\frac{(x,\nabla c(x))}{2c(x)}, \quad x\in\overline{\Omega},\ -T<t<T
\end{aligned} \tag{5.12}$$

$$0<c(x)\le\frac{1}{\theta}, \qquad x\in\overline{\Omega} \tag{5.13}$$

and

$$\omega^2<\theta T^2, \tag{5.14}$$

then the conclusion of Proposition 5.2 holds.

For the proof of Proposition 5.2, it is sufficient to verify that if θ satisfies (5.9) and (5.10), and T satisfies (5.11), then (5.12) - (5.14) hold. First (5.13) and (5.14) follow from (5.10) and (5.11). Now we have to verify (5.12). We define a function $\Psi=\Psi(x,\eta)$ by

$$\Psi(x,\eta)=\frac{-\omega\widetilde{d}(x)+\sqrt{\omega^2\widetilde{d}^2(x)+4c(x)\widetilde{e}(x)+4\eta\widetilde{d}(x)\widetilde{e}(x)}}{2c(x)+2\eta\widetilde{d}(x)}$$

for $x\in\overline{\Omega}$ and $\eta\ge 0$. The condition (5.9) implies that $\min_{x\in\overline{\Omega}}\Psi(x,0)>\sqrt{\theta}$. Since $\Psi(x,\eta)$ is continuous in $x\in\overline{\Omega}$ and $\eta\ge 0$, we see that

$$\Psi(x,\eta)>\sqrt{\theta}, \quad x\in\overline{\Omega} \tag{5.15}$$

for sufficiently small $\eta>0$. By (5.11) we can set $T=\frac{\omega}{\sqrt{\theta}}+\eta$, where $\eta>0$ makes (5.15) valid. That is,

$$\Psi\left(x,T-\frac{\omega}{\sqrt{\theta}}\right)>\sqrt{\theta}, \quad x\in\overline{\Omega}.$$

This inequality means

$$\theta\left(c(x)+\left(T-\frac{\omega}{\sqrt{\theta}}\right)\widetilde{d}(x)\right)+\omega\widetilde{d}(x)\sqrt{\theta}-\widetilde{e}(x)<0,$$

namely,

$$\theta(c(x)+T\widetilde{d}(x)) < \widetilde{e}(x), \quad x \in \overline{\Omega}.$$

Recalling the definition of $\widetilde{d}$ and $\widetilde{e}$, by $|t| \le T$, we can see (5.12). Thus the derivation of Proposition 5.2 from Corollary 1.2.5 in [10] is complete.

Proposition 5.1 can be easily derived from Proposition 5.2. In fact, by the assumption (2.8), we can apply Proposition 5.2 separately to $\alpha_1(x)\partial_t^2 - \Delta$ and $\alpha_2(x)\partial_t^2 - \Delta$, so that

$$\begin{aligned}&\xi^3\|w_1 \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} + \xi \sum_{|\alpha|=1} \|(D^\alpha w_1)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} \\ \le &M\|(\alpha_1(x)\frac{\partial^2 w_1}{\partial t^2} - \Delta w_1)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)}\end{aligned}$$

and

$$\begin{aligned}&\xi^3\|w_k \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} + \xi \sum_{|\alpha|=1} \|(D^\alpha w_k)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} \\ \le &M\|(\alpha_2(x)\frac{\partial^2 w_k}{\partial t^2} - \Delta w_k)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)}, \quad 2 \le k \le n+1\end{aligned}$$

for $w = (w_1, ..., w_{n+1})^T \in H_0^2(\phi_\epsilon)^{n+1}$. Therefore we have

$$\begin{aligned}&\xi^3\|w \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} + \xi \sum_{|\alpha|=1} \|(D^\alpha w)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} \\ \le &M\|(\Pi(x)\partial_t^2 - \Delta)w \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)}, \end{aligned}\tag{5.16}$$

for $w \in H_0^2(\phi_\epsilon)^{n+1}$. Since D_1 is a differential operator of the first order with bounded coefficients, we have

$$\|(D_1 w)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)} \le M \sum_{|\alpha|\le 1} \|(D^\alpha w)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_\epsilon)}. \tag{5.17}$$

Recalling (4.11), we see that

$$\begin{aligned}\|(\Pi(x)\partial_t^2 - \Delta)w \exp(\xi e^{\tau\phi})\|_{L^2(\phi_\epsilon)} \le &\|(Pw)\exp(\xi e^{\tau\phi})\|_{L^2(\phi_\epsilon)} \\ &+ \|(D_1 w)\exp(\xi e^{\tau\phi})\|_{L^2(\phi_\epsilon)},\end{aligned}$$

so that in view of (5.16) and (5.17), we complete the proof of Proposition 5.1. □

§6. Relation between the Boundary Conditions for the Lamé System and the Diagonalized System

For the Carleman estimate for the diagonalized system P, we have to take $w_{|\partial\Omega}$ and $\frac{\partial w}{\partial \nu}_{|\partial\Omega}$ into consideration. For this, in this section we prove

LEMMA 6.1. *Let $y \in C^2(\overline{\Omega} \times [-T,T])^n$ satisfy*

$$\rho(x)y''(x,t) - (\mathcal{L}y)(x,t) = 0, \quad x \in \partial\Omega,\ -T < t < T \tag{6.1}$$

$$y(x,t) = 0, \qquad x \in \partial\Omega,\ -T < t < T \tag{6.2}$$

and

$$\sigma(y)(x,t)\nu(x) = 0, \qquad x \in \partial\Omega,\ -T < t < T. \tag{6.3}$$

Then

$$\frac{\partial y}{\partial x_i} = \frac{\partial^2 y}{\partial x_j \partial x_k} = 0 \quad \textit{on } \partial\Omega \times (-T,T),\ 1 \le i,j,k \le n. \tag{6.4}$$

Therefore, setting

$$w(x,t) = Q(x)^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} y(x,t), \quad x \in \Omega,\ -T < t < T$$

where Q is defined by (4.7), we see that

$$w(x,t) = 0, \qquad x \in \partial\Omega,\ -T < t < T \tag{6.5}$$

and

$$\frac{\partial w}{\partial \nu}(x,t) = 0, \qquad x \in \partial\Omega,\ -T < t < T. \tag{6.6}$$

PROOF. We divide the proof into three steps.

First Step. For $a = (a_1, ..., a_n)^T$ and $b = (b_1, ..., b_n)^T$, we define the tensor product $a \otimes b$ by

$$a \otimes b = (a_i b_j)_{1 \le i,j \le n}. \tag{6.7}$$

We note that $a \otimes b$ is an $n \times n$ matrix. Then, under the assumption (6.2), we have

$$\nabla y = \{(\nabla y)\nu\} \otimes \nu \quad \text{on } \partial\Omega \times (-T, T) \tag{6.8}$$

and

$$\nabla^T y = (\nabla y)\nu \cdot \nu \quad \text{on } \partial\Omega \times (-T, T). \tag{6.9}$$

In fact, setting $y = (y_1, ..., y_n)^T$ and $\nu = (\nu_1, ..., \nu_n)^T$, we see that the condition (6.2) implies

$$\nabla y_i = \begin{pmatrix} \frac{\partial y_i}{\partial x_1} \\ \vdots \\ \frac{\partial y_i}{\partial x_n} \end{pmatrix} = (\nabla y_i \cdot \nu)\nu = \left(\frac{\partial y_i}{\partial \nu}\right)\nu, \quad 1 \le i \le n.$$

Therefore we have

$$\nabla y = \begin{pmatrix} (\nabla y_1)^T \\ \vdots \\ (\nabla y_n)^T \end{pmatrix} = \begin{pmatrix} (\nabla y_1 \cdot \nu)\nu^T \\ \vdots \\ (\nabla y_n \cdot \nu)\nu^T \end{pmatrix} = ((\nabla y_i \cdot \nu)\nu_j)_{1 \le i,j \le n}. \tag{6.10}$$

By the definition (6.7), this means (6.8). Moreover by $\nu\nu^T = 1$, we have

$$(\nabla y)\nu = \begin{pmatrix} (\nabla y_1 \cdot \nu)\nu^T \\ \vdots \\ (\nabla y_n \cdot \nu)\nu^T \end{pmatrix} \nu = \begin{pmatrix} (\nabla y_1 \cdot \nu) \\ \vdots \\ (\nabla y_n \cdot \nu) \end{pmatrix},$$

and so

$$(\nabla y)\nu \cdot \nu = (\nabla y_1 \cdot \nu)\nu_1 + \cdots + (\nabla y_n \cdot \nu)\nu_n = \mathrm{Trace}\, \nabla y = \nabla^T y$$

by (6.10). Thus we see (6.8) and (6.9).

Second Step. In this step, we will prove that (6.2) and (6.3) imply

$$\nabla y = 0 \quad \text{on } \partial\Omega \times (-T, T). \tag{6.11}$$

Proof of (6.11). We define an $n \times n$ matrix $B = B(x)$ by

$$B(x)a = \lambda(x)(a \cdot \nu(x))\nu(x) + 2\mu(x)\{\mathrm{Sym}\,(a \otimes \nu(x))\}\nu(x) \tag{6.12}$$

for $a \in \mathbb{R}^n$. Here and henceforth, for square matrices $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$, we set $\mathrm{Sym}\,A = \frac{1}{2}(A + A^T)$ and $A \cdot B = \sum_{i,j=1}^n a_{ij}b_{ij}$, $|A|^2 = \sum_{i,j=1}^n a_{ij}^2$. Then

$$B = B(x) \text{ is invertible for all } x \in \partial\Omega. \tag{6.13}$$

In fact, since

$$\begin{aligned}&(\mathrm{Sym}\,(a \otimes \nu))\nu \cdot a = (\mathrm{Sym}\,(a \otimes \nu)) \cdot (a \otimes \nu)\\ =&|\mathrm{Sym}\,(a \otimes \nu)|^2\end{aligned}$$

by direct calculations, we see

$$\begin{aligned}&(Ba \cdot a) = \lambda|a \cdot \nu|^2 + 2\mu|\mathrm{Sym}\,(a \otimes \nu)|^2\\ =&\lambda|\mathrm{tr}\,A|^2 + 2\mu|A|^2.\end{aligned} \tag{6.14}$$

Here we set

$$A = \mathrm{Sym}\,(a \otimes \nu)$$

and

$$C = A - \frac{\mathrm{tr}\,A}{n} I_n. \tag{6.15}$$

Then $\mathrm{tr}\,C = 0$, so that

$$C \cdot I_n = 0 \tag{6.16}$$

by the identity $C \cdot I_n = \mathrm{tr}\, C$. Therefore (6.14) - (6.16) imply

$$
\begin{aligned}
Ba \cdot a =& \lambda |\mathrm{tr}\, A|^2 + 2\mu \left| \frac{\mathrm{tr}\, A}{n} I_n + C \right|^2 \\
=& \lambda |\mathrm{tr}\, A|^2 + 2\mu \left(\left| \frac{\mathrm{tr}\, A}{n} I_n \right|^2 + |C|^2 + 2 \frac{\mathrm{tr}\, A}{n} I_n \cdot C \right) \\
=& \frac{n\lambda + 2\mu}{n} |\mathrm{tr}\, A|^2 + 2\mu |C|^2 \geq \frac{\delta_0}{n} |\mathrm{tr}\, A|^2 + \delta_0 |C|^2.
\end{aligned}
$$

At the last inequality, we have used (1.3) and (1.4). By (6.15), we have $A = C + \frac{\mathrm{tr}\, A}{n} I_n$, so that $\delta_0 |A|^2 = \frac{\delta_0}{n} |\mathrm{tr}\, A|^2 + \delta_0 |C|^2$ by (6.16). Therefore

$$
Ba \cdot a \geq \delta_0 |\mathrm{Sym}\,(a \otimes \nu)|^2.
$$

Moreover we have

$$
\begin{aligned}
|\mathrm{Sym}\,(a \otimes \nu)|^2 =& \frac{1}{4} (|a \otimes \nu|^2 + 2(a \otimes \nu) \cdot (\nu \otimes a) + |\nu \otimes a|^2) \\
=& \frac{1}{4} (|a|^2 + 2|a \cdot \nu| + |a|^2) = \frac{1}{2} (|a|^2 + |a \cdot \nu|) \geq \frac{1}{2} |a|^2,
\end{aligned}
$$

so that

$$
Ba \cdot a \geq \frac{\delta_0}{2} |a|^2 \quad \text{on } \partial\Omega. \tag{6.17}
$$

Moreover direct calculations verify that $Ba \cdot b = Bb \cdot a$ for every $a, b \in \mathbb{R}^n$, which means that B is a symmetric matrix. Therefore (6.17) implies (6.13).

On the other hand, by (6.8) and (6.9) we have

$$
\begin{aligned}
B((\nabla y)\nu) =& \lambda \{(\nabla y)\nu \cdot \nu\} \nu + 2\mu \{\mathrm{Sym}\,((\nabla y)\nu \otimes \nu)\} \nu \\
=& \lambda (\nabla^T y) \nu + 2\mu \{\mathrm{Sym}\,(\nabla y)\} \nu = \sigma(y) \nu \quad \text{on } \partial\Omega \times (-T, T).
\end{aligned}
$$

Therefore by (6.3) we obtain $B((\nabla y)\nu) = 0$ on $\partial\Omega \times (-T, T)$, which is $(\nabla y)\nu = 0$ by (6.13). Therefore, in view of (6.8), we see that $\nabla y = 0$. Thus the proof of (6.11) is complete. $\square$

Third Step. Recalling (2.2), since $y \in C^2(\overline{\Omega} \times [-T,T])^n$, we see from (6.11) that $\mathcal{L}y = \mu\Delta y + (\lambda+\mu)\nabla(\nabla^T y)$ on $\partial\Omega \times (-T,T)$. Therefore by (6.1) and (6.2), we obtain

$$\mu\Delta y + (\lambda+\mu)\nabla(\nabla^T y) = 0 \quad \text{on } \partial\Omega \times (-T,T). \tag{6.18}$$

Since $\partial\Omega$ is of C^2-class, for any $x^0 = (x_1^0, x_2^0, ..., x_n^0) \in \partial\Omega$, we can take neighbourhoods $\mathcal{V}$ in $\mathbb{R}^n$ of x^0 and $\mathcal{U}$ in $\mathbb{R}^{n-1}$ of $(x_1^0, ..., x_{n-1}^0)$, a function $\sigma = \sigma(x_1, ..., x_{n-1}) \in C^2(\mathcal{U})$ such that

$$(x_1, ..., x_{n-1}, x_n) \in \mathcal{V} \cap \partial\Omega \text{ if and only if } x_n = \sigma(x_1, ..., x_{n-1}). \tag{6.19}$$

We introduce a new coordinate $\eta = \eta(x) = (\eta_1, ..., \eta_{n-1}, \eta_n)$ by

$$\eta_1 = x_1, \quad ..., \quad \eta_{n-1} = x_{n-1}, \quad \eta_n = x_n - \sigma(x_1, ..., x_{n-1}) \tag{6.20}$$

for $(x_1, ..., x_{n-1}) \in \mathcal{U}$. We define a set $\mathcal{W}$ of $(\eta_1, ..., \eta_n)$ by $\mathcal{W} = \eta(\mathcal{V} \cap \partial\Omega)$. Henceforth we locally regard $y = y(x_1, ..., x_n)$ as a function in $(\eta_1, ..., \eta_n) \in \eta(\mathcal{V})$. Then the boundary conditions (6.2) and (6.11) imply

$$y = \frac{\partial y}{\partial \eta_j} = \frac{\partial^2 y}{\partial \eta_i \partial \eta_j} = 0, \quad 1 \le i \le n-1,\ 1 \le j \le n \tag{6.21}$$

in $\mathcal{W}$. For simplicity, we set

$$\sigma_i = \frac{\partial \sigma}{\partial x_i}(x_1, ..., x_{n-1}), \quad 1 \le i \le n-1$$

and $\sigma_n = -1$. Then noting that

$$\frac{\partial y}{\partial x_i} = \frac{\partial y}{\partial \eta_i} - \sigma_i \frac{\partial y}{\partial \eta_n}, \qquad 1 \le i \le n-1, \quad \frac{\partial y}{\partial x_n} = \frac{\partial y}{\partial \eta_n},$$

we see

$$\begin{aligned} \frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial \eta_i \partial \eta_j} - \sigma_j \frac{\partial^2 y}{\partial \eta_i \partial \eta_n} - \frac{\partial^2 \sigma}{\partial x_i \partial x_j} \frac{\partial y}{\partial \eta_n} \\ -\sigma_i \frac{\partial^2 y}{\partial \eta_j \partial \eta_n} + \sigma_i \sigma_j \frac{\partial^2 y}{\partial \eta_n^2}, \quad 1 \le i, j \le n-1, \end{aligned}$$

$$\frac{\partial^2 y}{\partial x_n^2} = \frac{\partial^2 y}{\partial \eta_n^2}, \quad \frac{\partial^2 y}{\partial x_i \partial x_n} = \frac{\partial^2 y}{\partial \eta_i \partial \eta_n} - \sigma_i \frac{\partial^2 y}{\partial \eta_n^2}, \quad 1 \le i \le n-1.$$

Therefore (6.21) implies

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \sigma_i \sigma_j \frac{\partial^2 y}{\partial \eta_n^2}, \quad 1 \le i, j \le n \tag{6.22}$$

in $\mathcal{W}$. Here we note that we set $\sigma_n = -1$. We substitute (6.22) into (6.18), and we obtain

$$\mu(\sigma_1^2 + \cdots + \sigma_n^2)\frac{\partial^2 y_i}{\partial \eta_n^2} + (\lambda + \mu)\left(\sigma_i \sigma_1 \frac{\partial^2 y_1}{\partial \eta_n^2} + \cdots + \sigma_i \sigma_n \frac{\partial^2 y_n}{\partial \eta_n^2}\right) = 0, \quad 1 \le i \le n$$

in $\mathcal{W}$. We can rewrite the above equalities as

$$D(\eta)\left(\frac{\partial^2 y_1}{\partial \eta_n^2}, \cdots, \frac{\partial^2 y_n}{\partial \eta_n^2}\right)^T = 0 \quad \text{in } \mathcal{W} \tag{6.23}$$

where we define an $n \times n$ matrix $D = D(\eta)$ by

$$D(\eta) = (\lambda + \mu)\kappa \otimes \kappa + \mu |\kappa|^2 I_n$$

with $\kappa = (\sigma_1, \cdots, \sigma_n)^T$. For any $a \in \mathbb{R}^n$, we have

$$\begin{aligned} Da \cdot a &= (\lambda + \mu)((\kappa \otimes \kappa)a) \cdot a + \mu|\kappa|^2 (a \cdot a) \\ &= (\lambda + \mu)(a \cdot \kappa)^2 + \mu |\kappa|^2 |a|^2. \end{aligned} \tag{6.24}$$

On the other hand, (1.4) implies

$$\lambda(x) + 2\mu(x) > \delta_0, \qquad x \in \partial\Omega. \tag{6.25}$$

In fact, if $\lambda \ge 0$, then (6.25) is straightforward from (1.3). If $\lambda(x) < 0$, then $n\lambda(x) + 2\mu(x) < \lambda(x) + 2\mu(x)$ implies (6.25).

Since $\mu \in C(\overline{\Omega})$, by (6.25) we can choose a sufficiently small $\epsilon > 0$ such that

$$\lambda(x) + (2 - \epsilon)\mu(x) > \frac{\delta_0}{2}, \quad x \in \partial\Omega. \tag{6.26}$$

Applying Schwarz's inequality in (6.24), by (6.26) and (1.3), we obtain

$$\begin{aligned} Da\cdot a = &(\lambda+\mu)(a\cdot\kappa)^2 + (\mu-\mu\epsilon)|\kappa|^2|a|^2 + \mu\epsilon|\kappa|^2|a|^2 \\ \geq&(\lambda+(2-\epsilon)\mu)(a\cdot\kappa)^2 + \mu\epsilon|\kappa|^2|a|^2 \geq \mu\epsilon|\kappa|^2|a|^2 \\ \geq&\frac{\delta_0\epsilon}{2}|\kappa|^2|a|^2 \geq \frac{\delta_0\epsilon}{2}|a|^2. \end{aligned} \tag{6.27}$$

At the last inequality, we use

$$|\kappa|^2 = \sigma_1^2 + \cdots + \sigma_n^2 = 1 + \left(\frac{\partial\sigma}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial\sigma}{\partial x_{n-1}}\right)^2 \geq 1.$$

By the definition, $D(\eta)$ is symmetric, the inequality (6.27) implies that $D = D(\eta)$ is invertible in $\mathcal{W}$. Therefore (6.23) yields

$$\frac{\partial^2 y_1}{\partial\eta_n^2} = \cdots = \frac{\partial^2 y_n}{\partial\eta_n^2} = 0 \quad \text{in } \mathcal{W},$$

with which we combine (6.22) to obtain

$$\frac{\partial^2 y}{\partial x_i\partial x_j}(x,t) = 0, \quad x\in\mathcal{V}\cap\partial\Omega,\ -T<t<T,\ 1\leq i,j\leq n.$$

Since $x^0\in\mathcal{V}$ is an arbitrary point of $\partial\Omega$, we see that

$$\frac{\partial^2 y}{\partial x_i\partial x_j}(x,t) = 0, \quad x\in\partial\Omega,\ -T<t<T,\ 1\leq i,j\leq n.$$

Thus the proof of the former part of the lemma is complete. $\square$

Now we will complete the proof of the latter part. First (6.5) is straightforward from (6.2) and (6.11). Next as for (6.6), we obtain

$$\frac{\partial w}{\partial\nu} = \sum_{i=1}^n \frac{\partial Q^{-1}}{\partial x_i}\nu_i \begin{pmatrix}\nabla^T\\ I_n\end{pmatrix} y + \sum_{i=1}^n Q^{-1}(x)\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix}\frac{\partial y}{\partial x_i}\nu_i$$

on $\partial\Omega\times(-T,T)$, and the equalities (6.2), (6.11) and (6.4) yield the conclusion (6.6).

§7. Proof of Theorem 1

Our proof is a variant of Isakov [10] and Klibanov [17], because we establish the Carleman estimate (Proposition 5.1) for the Lamé system and we reduce the original boundary conditions to the ones of H_0^2-class (Lemma 6.1). We here point that the proof of Theorem 1 is done by first eliminating the non-homogeneous term $r(f_1, ..., f_n)^T$ and then diagonalizing. On the other hand, for the proof of Theorem 2 in §8, we first diagonalize the system and then eliminate the non-homogeneous term.

We divide the proof into four steps. We can assume that

$$T - \frac{\sup_{x\in\Omega}|x|}{\sqrt{\theta}} > 0 \text{ is sufficiently small for } \alpha_1 \text{ and } \alpha_2,$$

so that Proposition 5.1 is applicable.

First Step. We set $f = (f_1, ..., f_n)^T$. By the assumption (2.9), we can take small $\delta \in (0, T)$ such that

$$r(x,t) \neq 0, \quad x \in \overline{\Omega}, \ |t| < \delta. \tag{7.1}$$

For simplicity, we set

$$\begin{aligned} Ly \equiv &\rho y'' - \mu\Delta y - (\lambda+\mu)\nabla(\nabla^T y) - \nabla^T y(\nabla\lambda) - (\nabla y + (\nabla y)^T)\nabla\mu \\ = &\rho y'' - \mathcal{L}y. \end{aligned} \tag{7.2}$$

Since

$$(Ly)(x,t) = r(x,t)f(x), \quad x \in \Omega, \ |t| < T,$$

we see by (7.1) that

$$r(x,t)^{-1}(Ly)(x,t) = f(x), \quad x \in \Omega, \ |t| < \delta.$$

We differentiate the both sides with respect to t, so that

$$r(x,t)^{-1}(Ly)'(x,t) - r(x,t)^{-2}r'(x,t)(Ly)(x,t) = 0, \quad x \in \Omega, \ |t| < \delta,$$

namely,

$$(Ly)'(x,t) - h(x,t)(Ly)(x,t) = 0, \quad x \in \Omega, \ |t| < \delta, \tag{7.3}$$

where

$$h(x,t) = r'(x,t)r(x,t)^{-1}, \quad x \in \Omega,\ |t| < \delta. \tag{7.4}$$

We define a differential operator N in t by

$$(Ny)(x,t) = y'(x,t) - h(x,t)y(x,t), \quad x \in \Omega,\ |t| < \delta. \tag{7.5}$$

Consequently we obtain

$$(NLy)(x,t) = 0, \quad x \in \Omega,\ |t| < \delta. \tag{7.6}$$

Second Step. We recall

$$\omega = \sup_{x \in \Omega} |x|. \tag{7.7}$$

For a sufficiently small $\epsilon > 0$, we set

$$c(\epsilon) = (\omega^2 - \theta\delta^2 + \epsilon^2)^{\frac{1}{2}} \tag{7.8}$$

where $\delta \in (0,T)$ is chosen such that (7.1) is true. Let $\chi \in C^\infty(\overline{\Omega} \times [-\delta,\delta])$ such that $0 \le \chi(x,t) \le 1$, $x \in \Omega$, $|t| \le \delta$ and

$$\chi(x,t) = \begin{cases} 1, & (x,t) \in \phi_{c(3\epsilon)} \\ 0, & (x,t) \in (\overline{\Omega} \times [-\delta,\delta]) \setminus \phi_{c(2\epsilon)}. \end{cases} \tag{7.9}$$

We set

$$v(x,t) = \chi(x,t)y(x,t), \qquad (x,t) \in \phi_{c(\epsilon)}. \tag{7.10}$$

In this step, we will prove

$$(Nv)(x,t) = 0, \quad (x,t) \in \phi_{c(3\epsilon)}. \tag{7.11}$$

First we directly verify that

$$\begin{aligned} &\phi_{c(\epsilon)} \subset \phi_{c(0)} \subset \{(x,t); x \in \Omega,\ \sqrt{\omega^2 - \theta\delta^2} < |x| < \omega,\ |t| < \delta\} \\ &\subset \Omega \times (-T,T). \end{aligned} \tag{7.12}$$

Then we have

$$\begin{aligned}\gamma LNv &= \gamma NLv + \gamma[L,N]v\\ &= \gamma[NL,\chi]y + \gamma\chi NLy + \gamma[L,N]v \quad \text{in } \phi_{c(\epsilon)}.\end{aligned}$$

Here $[L,N]$ denotes the commutator of the two operators:

$$[L,N]v = L(Nv) - N(Lv). \tag{7.13}$$

By (7.6) we have

$$\gamma LNv = \gamma[NL,\chi]y + \gamma[L,N]v \quad \text{in } \phi_{c(\epsilon)}. \tag{7.14}$$

We set

$$w = Q^{-1}\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix}Nv = Q^{-1}\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix}N(\chi y) \quad \text{in } \phi_{c(\epsilon)}. \tag{7.15}$$

Then we can prove

$$w \in H_0^2(\phi_{c(\epsilon)})^{n+1}. \tag{7.16}$$

Proof of (7.16). By the definition (7.9) of χ, we readily see that $w = \frac{\partial w}{\partial \nu} = 0$ on $\partial\phi_{c(\epsilon)} \cap (\Omega \times (-T,T))$. Therefore for the proof of (7.16), it is sufficient to prove $w = \frac{\partial w}{\partial \nu} = 0$ on $\partial\phi_{c(\epsilon)} \cap (\partial\Omega \times (-T,T))$, namely,

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (-T,T).$$

In view of Lemma 6.1, we have to verify

$$N(\chi y) \in C^2(\overline{\Omega} \times [-T,T])^n \tag{7.17}$$

$$L(N\chi y) = 0 \quad \text{on } \partial\Omega \times (-T,T) \tag{7.18}$$

$$N(\chi y) = 0 \quad \text{on } \partial\Omega \times (-T,T) \tag{7.19}$$

and

$$\sigma(N(\chi y))\nu = 0 \quad \text{on } \partial\Omega \times (-T,T). \tag{7.20}$$

First, since $N(\chi y) = (\chi y)' - h\chi y$, the equalities (7.17) and (7.19) follow from $y, y' \in C^2(\overline{\Omega} \times [-T,T])^n$ and $y = 0$ on $\partial\Omega \times (-T,T)$.

Next we see

$$\frac{\partial y}{\partial x_i} = \frac{\partial^2 y}{\partial x_j \partial x_k} = 0 \quad \text{on } \partial\Omega \times (-T,T),\ 1 \le i,j,k \le n. \tag{7.21}$$

In fact, since $y \in C^2(\overline{\Omega} \times [-T,T])^n$ satisfies the first equation in (2.3), we have

$$(Ly)(x,t) = r(x,t) \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} \quad \text{on } \partial\Omega \times (-T,T). \tag{7.22}$$

Consequently at $t = 0$, the boundary condition and the initial condition in (2.3) yield $r(x,0)(f_1(x), ..., f_n(x))^T = 0$, $x \in \partial\Omega$. The assumption (2.9) implies

$$f_1(x) = ... = f_n(x) = 0, \qquad x \in \partial\Omega. \tag{7.23}$$

Hence, by (7.22) and (7.23), we obtain

$$(Ly)(x,t) = 0, \qquad x \in \partial\Omega,\ -T < t < T. \tag{7.24}$$

Thus application of Lemma 6.1 with (2.12) and the boundary condition in (2.3), yields (7.21). Let us proceed to completing the proof of (7.18) and (7.20). By the definition (1.6) of $\sigma(\cdot)$, we see that $\sigma(N(\chi y))\nu$ is given by a linear combination of $\frac{\partial y}{\partial x_i}, \frac{\partial y'}{\partial x_j}$, $1 \le i,j \le n$ with variable coefficients on $\partial\Omega \times (-T,T)$. Therefore the conclusion (7.20) is straigthforward from (7.21). Similarly we readily verify that $L(N\chi y)$ is given by a linear combination of $\left(\frac{\partial}{\partial t}\right)^l y$, $0 \le l \le 3$, $\left(\frac{\partial}{\partial t}\right)^m \frac{\partial y}{\partial x_i}$, $\left(\frac{\partial}{\partial t}\right)^m \frac{\partial^2 y}{\partial x_j \partial x_k}$, $m = 0, 1$, $1 \le i,j,k \le n$, so that (7.21) and the regularity condition (2.11) imply (7.18). Thus the proof of (7.16) is complete. $\square$

We set

$$S(x) = \begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma)I_n \end{pmatrix}^{-1}, \quad x \in \Omega. \tag{7.25}$$

Then with P and L given by (4.11) and (7.2) we can rewrite Proposition 4.1 for Nv:

$$\Pi(x)Q(x)^{-1}S(x)\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix}\gamma(x)(LNv)(x,t) = (Pw)(x,t), \qquad (x,t) \in \phi_{c(\epsilon)}. \tag{7.26}$$

In view of (7.12) and (7.16), we can apply Proposition 5.1 to Pw, so that there exist constants $\tau = \tau(c(\epsilon)) > 0$, $M = M(c(\epsilon)) > 0$ and $\Xi = \Xi(c(\epsilon)) > 0$ such that if $\xi > \Xi$, then

$$\begin{aligned}&\xi^3\|w\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} + \xi\sum_{|\alpha|=1}\|(D^\alpha w)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})}\\ \le& M\|(Pw)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})},\end{aligned}$$

with which we combine (7.26) to obtain

$$\begin{aligned}&\xi^3\|w\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} + \xi\sum_{|\alpha|=1}\|(D^\alpha w)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})}\\ \le& M\left\|\Pi Q^{-1}S\left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix}\gamma LNv\right)\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}.\end{aligned} \tag{7.27}$$

Since $S = S(x)$ is a matrix whose components are smooth on $\overline{\Omega}$, we have $|(\Pi Q^{-1}S)(x)w| \le M_0|w|$, $x \in \overline{\Omega}$, $w \in \mathbb{R}^{n+1}$ with a constant M_0. Consequently

$$[\text{the right hand side of (7.27)}] \le M\left\|\left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix}\gamma LNv\right)\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}.$$

Substitution of (7.14) and (7.15) into (7.27) yields

$$\begin{aligned}
&\xi^3 \left\| \left(Q^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})} \\
&+\xi \sum_{|\alpha|=1} \left\| D^\alpha \left(Q^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})} \\
&\le M \left\| \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma[NL, \chi]y\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})} \\
&+M \left\| \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma[L, N]v\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})}.
\end{aligned} \tag{7.28}$$

Since $Q(x)^{-1}$ is a regular matrix on $\overline{\Omega}$, we have $|Q(x)^{-1}w| \ge M_0|w|$ and $|(D^\alpha Q^{-1})(x)w| \le M_1|w|$, $x \in \overline{\Omega}$, $w \in \mathbb{R}^{n+1}$ with constants $M_0 > 0$ and $M_1 > 0$, which are independent of $x \in \overline{\Omega}$ and $w \in \mathbb{R}^{n+1}$. Then, noting $D^\alpha(Q^{-1}w) = Q^{-1}D^\alpha w + (D^\alpha Q^{-1})w$, we have

$$\begin{aligned}
&\xi \left\| D^\alpha \left(Q^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})} \\
&\ge \xi \left\| Q^{-1} \left(D^\alpha \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})} \\
&-\xi \left\| (D^\alpha Q^{-1}) \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})} \\
&\ge M_0 \xi \left\| D^\alpha \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})} \\
&-M_1 \xi \left\| \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})}
\end{aligned}$$

and

$$\begin{aligned}
&\xi^3 \left\| \left(Q^{-1} \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})} \\
&\ge M_0 \xi^3 \left\| \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv\right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})}.
\end{aligned}$$

Therefore from (7.28) we obtain

$$
\begin{aligned}
&(M_0\xi^3 - M_1(n+1)\xi)\left\|\left(\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix} Nv\right)\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}\\
&+M_0\xi\sum_{|\alpha|=1}\left\|D^\alpha\left(\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix} Nv\right)\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}\\
&\le M\left\|\left(\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix}\gamma[NL,\chi]y\right)\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}\\
&+M\left\|\left(\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix}\gamma[L,N]v\right)\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})},
\end{aligned}
$$

namely,

$$
\begin{aligned}
&\xi\sum_{|\alpha|\le 1}\left\|D^\alpha\left(\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix} Nv\right)\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}\\
&\le M\left\|\left(\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix}\gamma[NL,\chi]y\right)\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}\\
&+M\left\|\left(\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix}\gamma[L,N]v\right)\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}
\end{aligned}
\tag{7.29}
$$

for all sufficiently large $\xi > 0$. Now we give an upper estimate of the right hand side.

Upper estimate of the second term of the right hand side of (7.29). We will prove

$$
\begin{aligned}
&\left\|\left(\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix}\gamma[L,N]v\right)\exp(\xi e^{\tau\phi})\right\|_{L^2(\phi_{c(\epsilon)})}\\
&\le M\sum_{|\alpha|\le 1}\left\|D^\alpha\left(\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix} Nv\right)\exp(\xi e^{\tau\phi})\right\|_{L^2(\phi_{c(\epsilon)})}
\end{aligned}
\tag{7.30}
$$

for all sufficiently large $\xi > 0$. Because the coefficients of L are independent of t, we see by direct calculations that

$$
(\gamma[L,N]v)(x,t) = \sum_{|\alpha|\le 1} p_\alpha(x,t)(D^\alpha v)(x,t)
$$

where $p_\alpha \in (L^\infty(\phi_{c(\epsilon)}))^{(n+1)\times(n+1)}$, $|\alpha| \le 1$. For the proof of (7.30), it is sufficient to verify

$$\begin{aligned}
&\|(D^\kappa v)\exp(\xi e^{\tau\phi})\|_{L^2(\phi_{c(\epsilon)})}\\
\le& M\sum_{|\alpha|\le 1}\|D^\alpha(Nv)\exp(\xi e^{\tau\phi})\|_{L^2(\phi_{c(\epsilon)})}, \quad |\kappa| \le 1,
\end{aligned}\tag{7.31}$$

and

$$\begin{aligned}
&\|\nabla^T(D^\kappa v)\exp(\xi e^{\tau\phi})\|_{L^2(\phi_{c(\epsilon)})}\\
\le& M\sum_{|\alpha|\le 1}\|D^\alpha(Nv)\exp(\xi e^{\tau\phi})\|_{L^2(\phi_{c(\epsilon)})}\\
+& M\sum_{|\alpha|\le 1}\|D^\alpha(\nabla^T(Nv))\exp(\xi e^{\tau\phi})\|_{L^2(\phi_{c(\epsilon)})}, \quad |\kappa| \le 1.
\end{aligned}\tag{7.32}$$

By the definition of (7.5) of N, we have

$$v'(x,t) - h(x,t)v(x,t) = (Nv)(x,t), \quad (x,t) \in \phi_{c(\epsilon)}. \tag{7.33}$$

Moreover we see that

$$v(x,0) = 0, \qquad (x,0) \in \phi_{c(\epsilon)} \cap \{t=0\}$$

by $y(x,0) = 0$, $x \in \Omega$. Therefore in terms of the fundamental solution of an ordinary differential equation (7.33) in t, we can take a scalar-valued function $K = K(x,t,s)$ such that $K \in C^1(\{(x,t,s);(x,t) \in \phi_{c(\epsilon)}, 0 < s < t\})$ and $D^\alpha K$ are bounded there for $|\alpha| \le 1$, and

$$v(x,t) = \int_0^t K(x,t,s)(Nv)(x,s)ds, \quad (x,t) \in \phi_{c(\epsilon)}. \tag{7.34}$$

Now we can proceed to

Verification of (7.31). By (7.33) and (7.34), we have

$$\begin{aligned}
(\partial_{x_i}v)(x,t) =& \int_0^t (\partial_{x_i}K)(x,t,s)(Nv)(x,s)ds\\
&+\int_0^t K(x,t,s)(\partial_{x_i}(Nv))(x,s)ds \quad \text{in } \phi_{c(\epsilon)}, \quad 1 \le i \le n
\end{aligned}$$

and

$$\begin{aligned}(\partial_t v)(x,t) &= (Nv)(x,t) + h(x,t)v(x,t)\\ &=(Nv)(x,t) + h(x,t)\int_0^t K(x,t,s)(Nv)(x,s)ds \quad \text{in } \phi_{c(\epsilon)}.\end{aligned}$$

Therefore

$$|(D^\kappa v)(x,t)| \le M|(Nv)(x,t)| + M\sum_{|\alpha|\le 1}\int_0^t |D^\alpha (Nv)(x,s)|ds, \quad (x,t)\in \phi_{c(\epsilon)}$$

for all $\kappa \in (\mathbb{N}\cup\{0\})^{n+1}$ with $|\kappa| \le 1$. Thus

$$\begin{aligned}&\|(D^\kappa v)\exp(\xi e^{\tau\phi})\|_{L^2(\phi_{c(\epsilon)})} \le M\|(Nv)\exp(\xi e^{\tau\phi})\|_{L^2(\phi_{c(\epsilon)})}\\ &+M\sum_{|\alpha|\le 1}\left(\int_{\phi_{c(\epsilon)}}\left|\int_0^t |D^\alpha(Nv)(x,s)|ds\right|^2 \exp(2\xi e^{\tau\phi})dxdt\right)^{\frac{1}{2}}. \qquad (7.35)\end{aligned}$$

Here we show

Lemma 7.1. *Let $\psi \in C^1(\overline{\phi_{c(\epsilon)}})$ and $t\frac{\partial\psi}{\partial t} \le 0$. Then*

$$\int_{\phi_{c(\epsilon)}} e^{2\xi\psi}\left|\int_0^t |p(x,s)|ds\right|^2 dxdt \le \sup_{(x,t)\in\phi_{c(\epsilon)}} |t|^2 \int_{\phi_{c(\epsilon)}} e^{2\xi\psi}|p(x,t)|^2 dxdt$$

for $p \in L^2(\phi_{c(\epsilon)})$.

This lemma is proved for example in Klibanov [17], and we will give the proof in Appendix for convenience.

Since in Lemma 7.1 we can set $\psi(x,t) = e^{\tau\phi(x,t)}$, we can derive (7.31) from (7.35).

Verification of (7.32). By (7.34) we have

$$\begin{aligned}(\nabla^T v)(x,t) &= \int_0^t \sum_{i=1}^n (\partial_{x_i}K)(x,t,s)[Nv]_i(x,s)ds\\ &+\int_0^t K(x,t,s)(\nabla^T(Nv))(x,s)ds \quad \text{in } \phi_{c(\epsilon)}. \qquad (7.36)\end{aligned}$$

Therefore similarly to (7.31), in view of Lemma 7.1, we can see (7.32) with $|\kappa| = 0$. Let us prove (7.32) with $|\kappa| = 1$. In fact, by (7.36) we obtain

$$
\begin{aligned}
&\nabla^T(D^\kappa v) = D^\kappa(\nabla^T v)\\
&= \int_0^t \sum_{i=1}^n D^\kappa(\partial_{x_i}K)(x,t,s)[Nv]_i(x,s)ds\\
&+ \int_0^t \sum_{i=1}^n (\partial_{x_i}K)(x,t,s)D^\kappa[Nv]_i(x,s)ds\\
&+ \int_0^t (D^\kappa K)(x,t,s)(\nabla^T(Nv))(x,s)ds + \int_0^t K(x,t,s)(D^\kappa \nabla^T(Nv))(x,s)ds.
\end{aligned}
$$

Again by Lemma 7.1 we similarly obtain (7.32) with $|\kappa| = 1$.

Next we proceed to

Upper estimate of the first term of the right hand side of (7.29). Since $[NL, 1]y = 0$, by (7.9) we have

$$
\begin{aligned}
&\left\| \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma[NL,\chi]y \right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})}\\
&= \left\| \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma[NL,\chi]y \right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)}\setminus\phi_{c(3\epsilon)})}\\
&\le \exp(2\xi e^{\tau c(3\epsilon)^2}) \left\| \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma[NL,\chi]y \right\|^2_{L^2(\phi_{c(\epsilon)}\setminus\phi_{c(3\epsilon)})}.
\end{aligned}
\tag{7.37}
$$

Consequently applying (7.30) and (7.37) in (7.29), we obtain

$$
\begin{aligned}
&\xi \sum_{|\alpha|\le 1} \left\| D^\alpha \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv \right) \exp(\xi e^{\tau\phi}) \right\|^2_{L^2(\phi_{c(\epsilon)})}\\
&\le M \exp(2\xi e^{\tau c(3\epsilon)^2}) \left\| \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma[NL,\chi]y \right\|^2_{L^2(\phi_{c(\epsilon)}\setminus\phi_{c(3\epsilon)})}
\end{aligned}
\tag{7.38}
$$

for all sufficiently large $\xi > 0$. By $\phi_{c(3\epsilon)} \subset \phi_{c(\epsilon)}$ and $\phi(x,t) \geq c(3\epsilon)^2$ in $\phi_{c(3\epsilon)}$, the estimate (7.38) immediately implies

$$\begin{aligned} &\sum_{|\alpha|\leq 1} \left\| D^\alpha \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} Nv \right) \right\|^2_{L^2(\phi_{c(3\epsilon)})} \\ \leq &\frac{M}{\xi} \left\| \left(\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma[NL, \chi] y \right) \right\|^2_{L^2(\phi_{c(\epsilon)} \setminus \phi_{c(3\epsilon)})} \end{aligned} \tag{7.39}$$

for all sufficiently large $\xi > 0$. Hence we can let $\xi > 0$ tend to ∞ in (7.39), so that we obtain $Nv = 0$ in $\phi_{c(3\epsilon)}$, namely, (7.11) is seen.

Third Step. By (7.11) and (7.34) we have $v(x,t) = 0$, $(x,t) \in \phi_{c(3\epsilon)}$. Consequently by the definition (7.9) and (7.10) of χ and v, we see $y(x,t) = 0$, $(x,t) \in \phi_{c(3\epsilon)}$. Since $\epsilon > 0$ is arbitrarily small, it follows that

$$\begin{aligned} &y(x,t) = 0, \\ &(x,t) \in \phi_{c(0)} = \{(x,t) \in \Omega \times (-T,T);\ |x|^2 - \theta t^2 > \omega^2 - \theta\delta^2\}. \end{aligned} \tag{7.40}$$

Substitution of (7.40) into the first equation in (2.3) in $(x,t) \in \phi_{c(\epsilon)}$ for an arbitrarily small $\epsilon > 0$, yields

$$r(x,t) \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} = 0, \qquad (x,t) \in \phi_{c(\epsilon)},$$

namely

$$r(x,t) \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} = 0, \qquad (x,t) \in \phi_{c(0)}.$$

Since $(x,t) \in \phi_{c(0)}$ implies $|t| < \delta$, in view of (7.1) we obtain

$$f_1(x) = \cdots = f_n(x) = 0, \quad x \in \Omega \quad \text{with } \sqrt{\omega^2 - \theta\delta^2} \leq |x| \leq \omega. \tag{7.41}$$

Fourth Step. By (7.41), near the boundary $\partial\Omega$, we can write (2.3):

$$(Ly)(x,t) = 0, \quad \sqrt{\omega^2 - \theta\delta^2} \leq |x| \leq \omega,\ -T < t < T \tag{7.42}$$

$$y(x,0) = 0, \qquad x \in \Omega \tag{7.43}$$

$$y(x,t) = 0, \qquad x \in \partial\Omega,\ -T < t < T \tag{7.44}$$

and

$$\sigma(y)(x,t)\nu(x) = 0, \qquad x \in \partial\Omega,\ -T < t < T. \tag{7.45}$$

In this step, we will prove

$$f_1(x) = \cdots = f_n(x) = 0, \quad x \in \Omega \quad \text{with } \sqrt{\omega^2 - 2\theta\delta^2} \le |x| \le \omega \tag{7.46}$$

and

$$y(x,t) = 0, \qquad (x,t) \in \phi_{c_1(\epsilon)} \tag{7.47}$$

for sufficiently small $\epsilon > 0$. Here and henceforth we set

$$c_1(\epsilon) = \sqrt{\omega^2 - 2\theta\delta^2 + \theta\epsilon^2}. \tag{7.48}$$

In other words, we will expand a domain where the uniqueness holds. For this, we will extend r in t to $\widetilde{r}$, keeping the positivity and the equality $Ly = \widetilde{r}f$ simultaneously (see (7.53) and (7.55)).

We define a function $\kappa = \kappa(t)$ such that

$$\begin{gathered} \kappa \in C_0^\infty(\mathbb{R}), \quad 0 \le \kappa(t) \le 1, \\ \kappa(t) = \begin{cases} 1 & |t| \le \sqrt{\delta^2 - \epsilon^2} \\ 0 & |t| > \delta \end{cases} \end{gathered} \tag{7.49}$$

and we define an extension $\widetilde{r}$ of r by

$$\widetilde{r}(x,t) = r(x,0) + \kappa(t)(r(x,t) - r(x,0)), \quad (x,t) \in \phi_{c_1(\epsilon)}. \tag{7.50}$$

Then by (7.49), we have

$$\widetilde{r}(x,t) = \begin{cases} r(x,t), & |t| \le \sqrt{\delta^2 - \epsilon^2} \\ r(x,0), & |t| > \delta. \end{cases} \tag{7.51}$$

Furthermore for $|t| \le \delta$, we have $\widetilde{r}(x,t) - r(x,t) = (1-\kappa(t))(r(x,0) - r(x,t))$, so that $|\widetilde{r}(x,t) - r(x,t)| \le \sup_{|t|\le\delta} |r(x,0) - r(x,t)| \le \|r'\|_{L^\infty(\Omega\times(-T,T))}\delta$. Therefore in view of (2.9), we can actually take $\delta > 0$ so that

$$\widetilde{r}(x,t) \ge \frac{r_0}{4}, \qquad x \in \overline{\Omega},\ |t| \le \delta$$

and

$$\delta = \frac{T}{\sqrt{l}} \quad \text{for large } l \in \mathbb{N} \tag{7.52}$$

as well as (7.1). Hence by (7.51) we obtain

$$\widetilde{r}(x,t) \ge \frac{r_0}{4}, \qquad (x,t) \in \overline{\phi_{c_1(\epsilon)}}. \tag{7.53}$$

By (7.41), (7.51) and the geometry of $\phi_{c_1(\epsilon)}$, we see that

$$\begin{aligned} &\widetilde{r}(x,t)f(x) \\ =&\begin{cases} 0, & (x,t) \in \phi_{c_1(\epsilon)} \cap \{(x,t); \sqrt{\omega^2 - \theta\delta^2} \le |x| \le \omega\} \\ r(x,t)f(x), & (x,t) \in \phi_{c_1(\epsilon)} \cap \{(x,t); |x| < \sqrt{\omega^2 - \theta\delta^2}\}. \end{cases} \end{aligned} \tag{7.54}$$

In fact, as direct calculations show, if $(x,t) \in \phi_{c_1(\epsilon)} \cap \{(x,t); |x| < \sqrt{\omega^2 - \theta\delta^2}\}$, then $|t| < \sqrt{\delta^2 - \epsilon^2}$, and so (7.51) implies that $\widetilde{r}(x,t)f(x) = r(x,t)f(x)$ for $(x,t) \in \phi_{c_1(\epsilon)} \cap \{(x,t); |x| < \sqrt{\omega^2 - \theta\delta^2}\}$. If $(x,t) \in \phi_{c_1(\epsilon)} \cap \{(x,t); \sqrt{\omega^2 - \theta\delta^2} \le |x| \le \omega\}$, then (7.41) implies $\widetilde{r}(x,t)f(x) = 0$.

Moreover we note

$$\begin{aligned} &\phi_{c_1(\epsilon)} \equiv \{(x,t) \in \Omega \times (-\infty,\infty); \phi(x,t) > c_1(\epsilon)^2\} \\ \subset &\Omega \times (-\sqrt{2}\delta, \sqrt{2}\delta), \end{aligned}$$

so that

$$\phi_{c_1(\epsilon)} = \{(x,t) \in \Omega \times (-T,T); \phi(x,t) > c_1(\epsilon)^2\},$$

provided that $\delta > 0$ is sufficiently small. Therefore by (7.42) and (2.10) we obtain

$$(Ly)(x,t) = \widetilde{r}(x,t)f(x), \qquad (x,t) \in \phi_{c_1(\epsilon)} \tag{7.55}$$

and

$$(7.56)\quad y(x,t)=\sigma(y)(x,t)\nu(x)=0,\quad (x,t)\in\partial\phi_{c_1(\epsilon)}\cap(\partial\Omega\times(-\infty,\infty)).$$

Hence in view of (7.53), we can repeat the arguments in First, Second and Third Steps to the system (7.55) with (7.43) and (7.56), so that we obtain (7.46) and (7.47).

Repeating $(m+1)$-times the above argument, we see

$$f(x)=0,\quad x\in\Omega\quad\text{with } \sqrt{\omega^2-m\theta\delta^2}<|x|<\omega \tag{7.57}$$

and

$$\begin{aligned} y(x,t)=0 \quad &\text{in } \phi_{\sqrt{\omega^2-m\theta\delta^2}} \\ \equiv&\{(x,t)\in\Omega\times(-\infty,\infty); |x|^2-\theta t^2>\omega^2-m\theta\delta^2\}. \end{aligned} \tag{7.58}$$

We have $\phi_{\sqrt{\omega^2-m\theta\delta^2}}\subset\Omega\times\{|t|<\delta\sqrt{m}\}$. Therefore we can repeat the argument until a natural number m satisfies $\delta\sqrt{m}\le T<\delta\sqrt{m+1}$, namely, $m=l$ by (7.52). Then $\omega^2-\theta m\delta^2=\omega^2-\theta l\delta^2=\omega^2-\theta T^2<0$ by (2.10).

Hence our process can be repeated until $\omega^2-m\theta\delta^2=0$. Therefore we have $f(x)=0$, $x\in\Omega$ with $|x|<\omega$, that is, $f(x)=0$ for all $x\in\Omega$. Then

$$y(x,t)=0,\quad \text{if } |x|^2-\theta t^2>0.$$

Thus the proof of Theorem 1 is complete. □

§8. Proof of Theorem 2

First Step. In contrast with the proof of Theorem 1, for the proof of Theorem 2, we first apply Proposition 4.1. We set

$$w_k(x,t)=Q(x)^{-1}\begin{pmatrix}\nabla^T y_k(x,t)\\ y_k(x,t)\end{pmatrix},\quad x\in\Omega,\ -T<t<T,\ 1\le k\le n.$$

Here the matrix fucntions $Q=Q(x)$, $S=S(x)$ and the differential operator P are given by (4.7), (7.25) and (4.11) respectively. Since $Q^{-1}\in C^2(\overline{\Omega})$ by Lemma 4.1, from (2.11) we see

$$\begin{aligned} w_k\in& C^1([-T,T];C^2(\overline{\Omega})^{n+1})\cap C^2([-T,T];C^1(\overline{\Omega})^{n+1}) \\ \cap& C^3([-T,T];C(\overline{\Omega})^{n+1}),\quad 1\le k\le n. \end{aligned} \tag{8.1}$$

By (2.4) we apply Proposition 4.1, so that

$$(Pw_k)(x,t) - \Pi(x)Q^{-1}(x)S(x)\begin{pmatrix}\nabla^T\\ I_n\end{pmatrix}(\gamma f)(x)R_k(x,t) = 0 \tag{8.2}$$
$$x \in \Omega,\ -T < t < T,\ 1 \le k \le n.$$

For $R_k(x,t) = (r_1^{(k)}(x,t), ..., r_n^{(k)}(x,t))^T$, we set

$$r_0^{(k)} = r_0^{(k)}(x,t) = (\nabla^T R_k)(x,t), \quad x \in \Omega,\ -T < t < T,\ 1 \le k \le n, \tag{8.3}$$

and define an $(n+1)\times(n+1)$ matrix C_k and an $(n+1)$-dimensional vector g_k respectively by

$$C_k(x,t) = \begin{pmatrix} r_0^{(k)} & r_1^{(k)} & \cdots & r_n^{(k)} \\ r_1^{(k)} & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots \\ r_n^{(k)} & 0 & \cdots & 0 \end{pmatrix} \tag{8.4}$$

and

$$g(x) = \begin{pmatrix} (\gamma f)(x) \\ \partial_{x_1}(\gamma f)(x) \\ \vdots \\ \partial_{x_n}(\gamma f)(x) \end{pmatrix} \tag{8.5}$$

for $x \in \Omega$, $-T < t < T$ and $1 \le k \le n$. Then by direct calculations from (8.2) we can derive

$$(Pw_k)(x,t) - \Pi(x)Q^{-1}(x)S(x)C_k(x,t)g(x) = 0, \tag{8.6}$$
$$x \in \Omega,\ -T < t < T,\ 1 \le k \le n.$$

Moreover we can prove

$$w_k(x,t) = \frac{\partial w_k}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega,\ -T < t < T,\ 1 \le k \le n. \tag{8.7}$$

PROOF OF (8.7). Since $y_k \in C^2(\overline{\Omega} \times [-T,T])^n$ satisfies (2.4), we see

$$(Ly_k)(x,t) = f(x)R_k(x,t), \quad x \in \partial\Omega,\ -T < t < T,\ 1 \le k \le n, \tag{8.8}$$

so that $0 = (Ly_k)(x, 0) = f(x)R_k(x, 0)$, $x \in \partial\Omega$, $1 \le k \le n$ by $y_k(x, 0) = 0$, $x \in \Omega$, $1 \le k \le n$. Therefore (2.17) yields

$$f(x) = 0, \qquad x \in \partial\Omega. \tag{8.9}$$

Hence

$$(Ly_k)(x, t) = 0, \quad x \in \partial\Omega,\ -T < t < T. \tag{8.10}$$

Consequently by $y_k(x, t) = 0$, $x \in \partial\Omega$, $-T < t < T$, $1 \le k \le n$ and (2.19), we apply Lemma 6.1 to see (8.7). $\square$

On the other hand,

$$w_k(x, 0) = 0, \quad x \in \Omega,\ 1 \le k \le n. \tag{8.11}$$

For the proof we have to verify $g(x) = 0$, $x \in \Omega$ from (8.6), (8.7) and (8.11). Through Proposition 5.1, we know that the Carleman estimate is applicable to (8.6), but in our case we have to overcome a difficulty that the matrices $C_k = C_k(x, t)$ in (8.6) are not invertible. Thus for recovering the invertibility, we simultaneously consider n solutions w_1, ..., w_n. First we write (8.6) as

$$\begin{pmatrix} Pw_1 \\ \vdots \\ Pw_n \end{pmatrix} - \Pi(x)Q^{-1}(x)S(x) \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & C_n \end{pmatrix} \begin{pmatrix} g \\ g \\ \vdots \\ g \end{pmatrix} = 0$$

$$\text{in } \Omega \times (-T, T). \tag{8.12}$$

Then we will determine $(n + 1) \times (n + 1)$ matrices A_{ij}, $1 \le i, j \le n$ such that

$$\left\{ \begin{array}{l} A_{11}(x, t) + \cdots + A_{1n}(x, t) = C_1(x, t) \\ A_{21}(x, t) + \cdots + A_{2n}(x, t) = C_2(x, t) \\ \quad\vdots \\ A_{n1}(x, t) + \cdots + A_{nn}(x, t) = C_n(x, t), \quad x \in \Omega,\ -T < t < T, \end{array} \right\} \tag{8.13}$$

and

$$|\det(A_{ij}(x,0))_{1\le i,j\le n}| \ge r_0 > 0, \quad x \in \overline{\Omega} \tag{8.14}$$

where $r_0 > 0$ is independent of $x \in \overline{\Omega}$. In fact, we can prove

LEMMA 8.1. *Under the assumptions (2.17) and (2.18) there exist* $(n+1) \times (n+1)$ *matrices* A_{ij}, $1 \le i,j \le n$ *satisfying (8.13), (8.14) and* $A_{ij}, \frac{\partial A_{ij}}{\partial t} \in C^2(\overline{\Omega} \times [-T,T])$.

PROOF OF LEMMA 8.1. In view of (2.18), without loss of generality, we may assume

$$|r_1^{(1)}(x,0)| \ge r_0 > 0, \qquad x \in \overline{\Omega}. \tag{8.15}$$

Let A_{ij}, $1 \le i \le n$, $1 \le j \le n-1$ be chosen. We set

$$A_{in}(x,t) = C_i(x,t) - \sum_{j=1}^{n-1} A_{ij}(x,t), \quad x \in \overline{\Omega},\ -T \le t \le T,\ 1 \le i \le n.$$

Then (8.13) is satisfied. Therefore it is sufficient to choose A_{ij}, $1 \le i \le n$, $1 \le j \le n-1$ in order that (8.14) is true. We have

$$\begin{aligned}
&\left|\det\begin{pmatrix} A_{11}(x,0) & \cdots & A_{1n}(x,0) \\ \vdots & & \vdots \\ A_{n1}(x,0) & \cdots & A_{nn}(x,0)\end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix} C_1(x,0) & A_{11}(x,0) & \cdots & A_{1n-1}(x,0) \\ \vdots & \vdots & \cdots & \vdots \\ C_n(x,0) & A_{n1}(x,0) & \cdots & A_{nn-1}(x,0)\end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix} r_0^{(1)} & r_1^{(1)} & \cdots & r_n^{(1)} & & & \\ \vdots & \vdots & & \vdots & \mathbf{b}_1 & \cdots & \mathbf{b}_{n^2-1} \\ r_0^{(n)} & r_1^{(n)} & \cdots & r_n^{(n)} & & & \\ r_1^{(1)} & 0 & \cdots & 0 & & & \\ \vdots & \vdots & & \vdots & \mathbf{a}_1 & \cdots & \mathbf{a}_{n^2-1} \\ r_n^{(n)} & 0 & \cdots & 0 & & & \end{pmatrix}(x,0)\right|
\end{aligned} \tag{8.16}$$

by exchanging rows. Here

$$\mathbf{b}_1, ..., \mathbf{b}_{n^2-1} \quad : n\text{-dimensional column vectors dependent on } x$$

$$\mathbf{a}_1, ..., \mathbf{a}_{n^2-1} \quad : n^2\text{-dimensional column vectors dependent on } x$$

and an $(n^2+n) \times (n^2+n)$ matrix

$$\begin{pmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_{n^2-1} \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n^2-1} \end{pmatrix}$$

is given from

$$\begin{pmatrix} A_{11}(x,0) & \cdots & A_{1n-1}(x,0) \\ \vdots & & \vdots \\ A_{n1}(x,0) & \cdots & A_{nn-1}(x,0) \end{pmatrix}$$

by exchanging rows suitably. Therefore for (8.14) it is sufficient to choose $\mathbf{b}_1(x,t)$, ..., $\mathbf{b}_{n^2-1}(x,t)$, $\mathbf{a}_1(x,t)$, ..., $\mathbf{a}_{n^2-1}(x,t)$. Let $\mathbf{a}_i, \mathbf{b}_i \in C^2(\overline{\Omega} \times [-T,T])$ satisfy $\mathbf{a}_i', \mathbf{b}_i' \in C^2(\overline{\Omega} \times [-T,T])$ for $1 \le i \le n^2-1$ and

$$\mathbf{b}_1(x,0) = \cdots = \mathbf{b}_{n^2-1}(x,0) = O_{n,1},$$

$$(\mathbf{a}_1(x,0), \cdots, \mathbf{a}_{n^2-1}(x,0)) = \begin{pmatrix} O_{1,n^2-1} \\ I_{n^2-1} \end{pmatrix}.$$

Here $O_{m,n}$ and I_m denote the $m \times n$ zero matrix and the $m \times m$ identity matrix respectively. This choice gives our desired A_{ij}. In fact, from (8.16) by exchanging columns, we see

$$\begin{aligned}
&\left|\det \begin{pmatrix} A_{11}(x,0) & \cdots & A_{1n}(x,0) \\ \vdots & \cdots & \vdots \\ A_{n1}(x,0) & \cdots & A_{nn}(x,0) \end{pmatrix}\right| \\
= &\left|\det \begin{pmatrix} r_1^{(1)} & \cdots & r_n^{(1)} & r_0^{(1)} & & & \\ \vdots & \vdots & \vdots & \vdots & & O_{n,n^2-1} & \\ r_1^{(n)} & \cdots & r_n^{(n)} & r_0^{(n)} & & & \\ 0 & \cdots & 0 & r_1^{(1)} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & * & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & * & 0 & \cdots & 1 \end{pmatrix} (x,0)\right| \\
= &\left|\det \begin{pmatrix} r_1^{(1)}(x,0) & \cdots & r_n^{(1)}(x,0) \\ \vdots & & \vdots \\ r_1^{(n)}(x,0) & \cdots & r_n^{(n)}(x,0) \end{pmatrix}\right| \times |r_1^{(1)}(x,0)|,
\end{aligned}$$

which never vanishes for $x \in \overline{\Omega}$ by (8.15) and (2.17). Thus the proof of Lemma 8.1 is complete. □

For completing the proof of Theorem 2 along the line of [3], [10], [17], we rewrite (8.12) in terms of the matrix $(A_{ij})_{1 \le i,j \le n}$ which is invertible at $t = 0$. In view of (8.13), we can rewrite (8.12) as

$$\begin{pmatrix} (Pw_1)(x,t) \\ \vdots \\ (Pw_n)(x,t) \end{pmatrix} - \Pi(x) Q^{-1}(x) S(x) A(x,t) \begin{pmatrix} g(x) \\ \vdots \\ g(x) \end{pmatrix} = 0, \quad x \in \Omega,\ -T < t < T, \tag{8.17}$$

where we set

$$A(x,t) = \begin{pmatrix} A_{11}(x,t) & \cdots & A_{1n}(x,t) \\ \vdots & \vdots & \vdots \\ A_{n1}(x,t) & \cdots & A_{nn}(x,t) \end{pmatrix}, \quad x \in \Omega,\ -T < t < T. \tag{8.18}$$

The condition (8.14) implies

$$|\det A(x,0)| \ge r_0 > 0, \quad x \in \overline{\Omega}. \tag{8.19}$$

Consequently we choose a small $\delta \in (0,T)$ such that

$$|\det A(x,t)| \ge r_0 > 0, \quad x \in \overline{\Omega},\ |t| < \delta. \tag{8.20}$$

Then it follows from (8.17) that

$$A^{-1} S^{-1} Q \Pi^{-1} \begin{pmatrix} Pw_1 \\ \vdots \\ Pw_n \end{pmatrix} - \begin{pmatrix} g \\ \vdots \\ g \end{pmatrix} = 0$$

in $\Omega \times (-\delta, \delta)$. Since g is independent of t, we differentiate the both sides in t, and we obtain

$$\frac{\partial}{\partial t} \begin{pmatrix} Pw_1 \\ \vdots \\ Pw_n \end{pmatrix} - H(x,t) \begin{pmatrix} Pw_1 \\ \vdots \\ Pw_n \end{pmatrix} = 0 \quad \text{in } \Omega \times (-\delta, \delta) \tag{8.21}$$

where

$$H(x,t) = -\Pi(x)Q^{-1}(x)S(x)A(x,t)(A^{-1}(x,t))'S^{-1}(x)Q(x)\Pi^{-1}(x), \quad x \in \Omega,\ |t| < \delta. \tag{8.22}$$

We easily see

$$H \in C^2(\overline{\Omega} \times [-\delta, \delta]). \tag{8.23}$$

Similarly to (7.5), defining an ordinary differential operator N in t by

$$N \begin{pmatrix} v_1(x,t) \\ \vdots \\ v_n(x,t) \end{pmatrix} = \begin{pmatrix} v_1'(x,t) \\ \vdots \\ v_n'(x,t) \end{pmatrix} - H(x,t) \begin{pmatrix} v_1(x,t) \\ \vdots \\ v_n(x,t) \end{pmatrix}, \quad x \in \Omega,\ |t| < \delta, \tag{8.24}$$

for functions $v_k(x,t) \in \mathbb{R}^n$, we obtain

$$NP \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = 0 \quad \text{in } \Omega \times (-\delta, \delta). \tag{8.25}$$

Second Step. Now we can proceed similarly to the second - fourth steps in the proof of Theorem 1. We will here show a sketch. Let us recall (7.7), (7.8), let $\epsilon > 0$ be sufficiently small and the cut-off function $\chi = \chi(x,t)$ be defined by (7.9). We set

$$W(x,t) = \begin{pmatrix} w_1(x,t) \\ \vdots \\ w_n(x,t) \end{pmatrix}, \quad V(x,t) = \chi(x,t)W(x,t), \quad (x,t) \in \phi_{c(\epsilon)}. \tag{8.26}$$

Then by (8.25) we have

$$\begin{aligned} PNV &= PN(\chi W) = NP(\chi W) + [P,N]V \\ &= [NP,\chi]W + \chi NPW + [P,N]V \\ &= [NP,\chi]W + [P,N]V, \quad (x,t) \in \phi_{c(\epsilon)}. \end{aligned} \tag{8.27}$$

By noting $\Delta(Kv) = (\Delta K)v + K\Delta v + 2\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial K_{ij}}{\partial x_k}\frac{\partial v_j}{\partial x_k}\right)^T_{1\le i\le m}$ for an $m\times m$ matrix $K = (K_{ij})_{1\le i,j\le m}$ and an m-dimensional vector $v = (v_1, ..., v_m)^T$, in terms of the regularity (8.1), direct calculations show

$$
\begin{aligned}
&[P,N]V = (H\Pi - \Pi H)V'' \\
&+\sum_{i=1}^{n+1} a_i(x,t)\frac{\partial V}{\partial x_i}(x,t) + a_0(x,t)V(x,t) \quad \text{in } \phi_{c(\epsilon)}
\end{aligned}
\tag{8.28}
$$

with some $a_i \in L^\infty(\phi_{c(\epsilon)})$, $0\le i\le n+1$. Here we note that $H\Pi \ne \Pi H$ in general for the matrices H and Π.

By (8.7) and the definition (7.9) of χ, we have $NV \in (H_0^2(\phi_{c(\epsilon)}))^{n+1}$. Apply Proposition 5.1, the Carleman estimate, to (8.27) by noting (8.28):

$$
\begin{aligned}
&\xi\sum_{|\alpha|\le 1}\|D^\alpha(NV)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} \\
\le& M\|(PNV)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} \\
\le& M\|([NP,\chi]W)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} \\
&+M\sum_{|\alpha|\le 1}\|(D^\alpha V)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} + M\|V''\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})}
\end{aligned}
\tag{8.29}
$$

for sufficiently large $\xi > 0$. Similarly to the verification of (7.31) and (7.32), we apply Lemma 7.1 for $\psi(x,t) = e^{\tau\phi(x,t)}$ to obtain

$$
\begin{aligned}
&\sum_{|\alpha|\le 1}\|(D^\alpha V)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} \\
\le& M\sum_{|\alpha|\le 1}\|D^\alpha(NV)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})}.
\end{aligned}
\tag{8.30}
$$

On the other hand, since $V' - HV = NV$ in $\phi_{c(\epsilon)}$, we have $V'' = HV' + H'V + (NV)'$ in $\phi_{c(\epsilon)}$, so that application of (8.30) yields

$$
\|V''\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} \le M\sum_{|\alpha|\le 1}\|D^\alpha(NV)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})}.
\tag{8.31}
$$

Therefore by (8.29) - (8.31), we obtain

$$
\begin{aligned}
&(\xi - 2M^2) \sum_{|\alpha|\le 1} \|D^\alpha(NV)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} \\
\le & M\|([NP,\chi]W)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} \\
= & M\|([NP,\chi]W)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(3\epsilon)})} \\
+ & M\|([NP,\chi]W)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)}\setminus\phi_{c(3\epsilon)})}
\end{aligned}
$$

for sufficiently large $\xi > 0$. By the definition (7.9) of χ, we have $\chi = 1$ in $\phi_{c(3\epsilon)}$ and $[NP, 1]W = 0$. Consequently we see

$$
\begin{aligned}
&\xi \sum_{|\alpha|\le 1} \|D^\alpha(NV)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(3\epsilon)})} \\
\le & M\|[NP,\chi]W\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)}\setminus\phi_{c(3\epsilon)})} \qquad (8.32)
\end{aligned}
$$

for sufficiently large $\xi > 0$. Since $\exp(\xi e^{\tau\phi}) \ge \exp(\xi e^{\tau c(3\epsilon)^2})$ in $\phi_{c(3\epsilon)}$ and $\exp(\xi e^{\tau\phi}) \le \exp(\xi e^{\tau c(3\epsilon)^2})$ in $\phi_{c(\epsilon)} \setminus \phi_{c(3\epsilon)}$, the estimate (8.32) implies

$$
\xi \sum_{|\alpha|\le 1} \|D^\alpha(NV)\|^2_{L^2(\phi_{c(3\epsilon)})} \le M\|[NP,\chi]W\|^2_{L^2(\phi_{c(\epsilon)}\setminus\phi_{c(3\epsilon)})}
$$

for sufficiently large $\xi > 0$. Hence we see

$$
(NV)(x,t) = 0, \qquad (x,t) \in \phi_{c(3\epsilon)} \qquad (8.33)
$$

by letting ξ tend to ∞. By (8.11) and (7.9), recalling (8.26), we have $V(x, 0) = 0$, $x \in \Omega$. Moreover by means of the uniqueness of solutions to the initial value problem for the ordinary differential equation $(NV)(x,t) = 0$, we see by (8.33) that $V(x,t) = 0$, $(x,t) \in \phi_{c(\epsilon)}$. Returning to the definition (8.26) of V, since $\epsilon > 0$ is arbitrarily small, we obtain $W(x,t) = 0$, $(x,t) \in \phi_{c(0)}$. Then we can directly follow the argument after Third Step in §7. In particular, the extension argument by (7.50) is similarly carried out. Hence we do not repeat the arguments here. Thus the proof of Theorem 2 is complete. □

§9. Proof of Theorem 3

Similarly to the proof of Theorem 2, we set

$$w(x,t) = Q^{-1}(x)\begin{pmatrix}\nabla^T y(x,t)\\ y(x,t)\end{pmatrix}, \quad x \in \Omega,\ -T < t < T, \tag{9.1}$$

$$q_0(x,t) = (\nabla^T(\gamma R))(x,t) = \nabla^T\left(\gamma\begin{pmatrix}r_1\\ \vdots\\ r_n\end{pmatrix}\right)(x,t),$$
$$q_k(x,t) = (\gamma r_k)(x,t), \quad x \in \Omega,\ -T < t < T, 1 \le k \le n \tag{9.2}$$

and

$$g(x) = \begin{pmatrix} f(x)\\ (\partial_{x_1} f)(x)\\ (\partial_{x_2} f)(x)\\ 0\\ \vdots\\ 0\end{pmatrix} \in \mathbb{R}^{n+1}, \quad x \in \Omega,\ -T < t < T. \tag{9.3}$$

By (2.26) and Lemma 4.1, we see

$$\begin{aligned} w \in & C^2([-T,T]; C^2(\overline{\Omega})^{n+1}) \cap C^3([-T,T]; C^1(\overline{\Omega})^{n+1})\\ & \cap C^4([-T,T]; C(\overline{\Omega})^{n+1}). \end{aligned} \tag{9.4}$$

We define an $(n+1)\times(n+1)$ matrix by

$$C(x,t) = \begin{pmatrix} q_0(x,t) & q_1(x,t) & q_2(x,t) & \\ q_1(x,t) & 0 & 0 & O_{3,n-2}\\ q_2(x,t) & 0 & 0 & \\ q_3(x,t) & 0 & 0 & \\ \vdots & \vdots & \vdots & I_{n-2}\\ q_n(x,t) & 0 & 0 & \end{pmatrix},$$
$$x \in \Omega,\ -T < t < T, \tag{9.5}$$

where $O_{3,n-2}$ denotes the $3\times(n-2)$ zero matrix. Then by (2.23) and (2.24), we see that

$$C, C', C'' \in C^2(\overline{\Omega}\times[-T,T]), \quad q_k, q_k', q_k'' \in C^2(\overline{\Omega}\times[-T,T]), \\ 0 \le k \le n. \tag{9.6}$$

We recall that Q, S and Π are defined by (4.7), (7.25) and (4.8) respectively and I_{n-2} is the identity matrix of order $n-2$.

Then using the assumptions (2.23) and (2.24), and noting (9.4) and (9.6), in view of Proposition 4.1, we can obtain from (2.22)

$$\left\{\begin{aligned} &(Pw)(x,t) - \Pi(x)Q^{-1}(x)S(x)C(x,t)g(x) = 0,\\ &(Pw)'(x,t) - \Pi(x)Q^{-1}(x)S(x)C'(x,t)g(x) = 0,\\ &\qquad\qquad\qquad\qquad x\in\Omega,\ -T<t<T, \end{aligned}\right\} \tag{9.7}$$

$$w(x,0) = w'(x,0) = 0, \qquad x\in\Omega \tag{9.8}$$

and

$$w(x,t) = \frac{\partial w}{\partial \nu}(x,t) = 0, \quad x\in\partial\Omega,\ -T<t<T. \tag{9.9}$$

In fact, (9.9) can be derived similarly to (8.7).

Non-invertibility of C involves a similar difficulty to §8 for the application of the Carleman estimate, and we have to apply similar enlargement by taking time derivatives of (9.7) without changing non-homogeneous terms in the system (2.4). First we show

Lemma 9.1. *Under the assumption (2.23), we can choose $(n+1)\times(n+1)$ matrices A_1, A_2, A_3, A_4 such that*

$$A_k, A_k' \in C^2(\overline{\Omega}\times[-T,T]), \qquad 1\le k\le 4, \tag{9.10}$$

$$\left\{\begin{aligned} &A_1(x,t) + A_2(x,t) = C(x,t)\\ &A_3(x,t) + A_4(x,t) = C'(x,t), \quad x\in\Omega,\ -T\le t\le T \end{aligned}\right\} \tag{9.11}$$

and

$$\left|\det\begin{pmatrix} A_1(x,0) & A_2(x,0) \\ A_3(x,0) & A_4(x,0)\end{pmatrix}\right| \ge r_0 > 0, \quad x \in \overline{\Omega}, \tag{9.12}$$

where $r_0 > 0$ *is independent of* $x \in \overline{\Omega}$.

Proof of Lemma 9.1. Since

$$\left|\det\begin{pmatrix} q_1(x,0) & q_2(x,0) \\ q_1'(x,0) & q_2'(x,0)\end{pmatrix}\right| = \gamma^2(x)\left|\det\begin{pmatrix} r_1(x,0) & r_2(x,0) \\ r_1'(x,0) & r_2'(x,0)\end{pmatrix}\right|,$$

we have

$$|q_1'(x,0)q_2(x,0) - q_1(x,0)q_2'(x,0)| \ge \zeta_1 > 0, \quad x \in \overline{\Omega} \tag{9.13}$$

by (1.3), (4.11) and (2.25). Here $\zeta_1 > 0$ is independent of $x \in \overline{\Omega}$. Therefore we can choose a large constant $\sigma > 0$ such that

$$p_0(x) \equiv q_2(x,0) - \sigma\{q_1'(x,0)q_2(x,0) - q_1(x,0)q_2'(x,0)\} \ne 0, \quad x \in \overline{\Omega}. \tag{9.14}$$

Defining an $(n+1)\times(n+1)$ matrix $A_2(x,t)$ by

$$A_2(x,t) = \begin{pmatrix} 0 & q_1(x,t) & q_2(x,t) & \\ 1 & 0 & 0 & O_{3,n-2} \\ 0 & \sigma q_2(x,t) & -\sigma q_1(x,t) & \\ & O_{n-2,3} & & I_{n-2}\end{pmatrix}, \quad x \in \overline{\Omega}, \tag{9.15}$$

we see by (9.6) that $A_2, A_2' \in C^2(\overline{\Omega}\times[-T,T])$. We further set

$$\left\{\begin{aligned} & A_1(x,t) = C(x,t) - A_2(x,t) \\ & A_3(x,t) = C'(x,t) - I_{n+1} \\ & A_4(x,t) = I_{n+1}, \quad x \in \Omega,\ -T < t < T. \end{aligned}\right\} \tag{9.16}$$

Then these A_1, A_2, A_3, A_4 are desired matrices. In fact, the condition (9.10) and (9.11) are satisfied directly by (9.6) and (9.16). We will verify (9.12). It is sufficient to verify that for any fixed $x \in \overline{\Omega}$, the equations

$$\left\{\begin{aligned} & A_1(x,0)a + A_2(x,0)b = 0 \\ & A_3(x,0)a + A_4(x,0)b = 0, \quad a, b \in \mathbb{R}^{n+1} \end{aligned}\right\} \tag{9.17}$$

implies $a = b = 0$. We set

$$E(x,t) = C(x,t) - A_2(x,t)C'(x,t). \tag{9.18}$$

Direct computations yield

$$E(x,0) = \begin{pmatrix} (q_0 - q_1q_1' - q_2q_2')(x,0) & q_1(x,0) & q_2(x,0) & \\ (q_1 - q_0')(x,0) & -q_1'(x,0) & -q_2'(x,0) & O_{3,n-2} \\ p_0(x) & 0 & 0 & \\ (q_3 - q_3')(x,0) & 0 & 0 & \\ \vdots & \vdots & \vdots & I_{n-2} \\ (q_n - q_n')(x,0) & 0 & 0 & \end{pmatrix}$$

and

$$\det E(x,0) = p_0(x)(q_1'(x,0)q_2(x,0) - q_1(x,0)q_2'(x,0)), \quad x \in \overline{\Omega}.$$

Therefore by (9.13) and (9.14) we see that

$$E^{-1}(x,0) \text{ exists for all } x \in \overline{\Omega}. \tag{9.19}$$

Furthermore we note

$$A_2(x,0)^{-1} \text{ exists for all } x \in \overline{\Omega}. \tag{9.20}$$

In fact, we have

$$\det A_2(x,0) = \sigma(q_1(x,0)^2 + q_2(x,0)^2), \quad x \in \overline{\Omega},$$

and (9.13) implies that $q_1(x,0)^2 + q_2(x,0)^2$ never vanishes for $x \in \overline{\Omega}$.

Therefore from (9.17) we derive $A_2(x,0)^{-1}A_1(x,0)a + b = A_3(x,0)a + b = 0$, namely, $(A_2(x,0)^{-1}A_1(x,0) - A_3(x,0))a = 0$. On the other hand, we see by (9.16)

$$\begin{aligned} &A_2(x,0)^{-1}A_1(x,0) - A_3(x,0) \\ =&A_2(x,0)^{-1}(C(x,0) - A_2(x,0)) - (C'(x,0) - I_{n+1}) \\ =&A_2(x,0)^{-1}(C(x,0) - A_2(x,0)C'(x,0)) = A_2(x,0)^{-1}E(x,0). \end{aligned}$$

By (9.19) we obtain $a = 0$ and so $b = 0$. Thus the proof of Lemma 9.1 is complete. $\square$

In view of Lemma 9.1, we can rewrite (9.7) as

$$\begin{pmatrix} Pw(x,t) \\ (Pw)'(x,t) \end{pmatrix} - \Pi Q^{-1} S \begin{pmatrix} A_1 & A_2 \\ A_3 & I_{n+1} \end{pmatrix} \begin{pmatrix} g(x) \\ g(x) \end{pmatrix} = 0, \quad x \in \Omega,\ -T < t < T. \tag{9.21}$$

By (9.12), (9.19) and (9.20), there exists $\delta > 0$ such that

$$\left|\det \begin{pmatrix} A_1(x,t) & A_2(x,t) \\ A_3(x,t) & I_{n+1} \end{pmatrix}\right|, \quad |\det E(x,t)|, \quad |\det A_2(x,t)| \geq \frac{r_0}{2}, \quad x \in \overline{\Omega},\ |t| \leq \delta. \tag{9.22}$$

We set

$$\begin{pmatrix} X_1(x,t) & X_2(x,t) \\ X_3(x,t) & X_4(x,t) \end{pmatrix} \begin{pmatrix} A_1(x,t) & A_2(x,t) \\ A_3(x,t) & A_4(x,t) \end{pmatrix} = \begin{pmatrix} I_{n+1} & 0 \\ 0 & I_{n+1} \end{pmatrix}, \quad x \in \overline{\Omega},\ |t| \leq \delta, \tag{9.23}$$

where X_k, $1 \leq k \leq 4$ are $(n+1) \times (n+1)$ matrices. In terms of (9.6), (9.10), (9.13) and (9.18), direct calculations show that

$$X_1(x,t) = E^{-1}(x,t), \quad X_2(x,t) = -E^{-1}(x,t)A_2(x,t), \quad x \in \overline{\Omega},\ |t| < \delta, \tag{9.24}$$

and

$$E, E', X_1, X_2, X_1', X_2' \in C^2(\overline{\Omega} \times [-\delta, \delta]). \tag{9.25}$$

Therefore by (9.21) and (9.23) we obtain

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} S^{-1}Q\Pi^{-1}(Pw) \\ S^{-1}Q\Pi^{-1}(Pw)' \end{pmatrix} - \begin{pmatrix} g \\ g \end{pmatrix} = 0 \quad \text{in } \Omega \times (-\delta, \delta).$$

The first component gives

$$X_2S^{-1}Q\Pi^{-1}(Pw)' + X_1S^{-1}Q\Pi^{-1}(Pw) = g \quad \text{in } \Omega \times (-\delta, \delta).$$

Differentiating the both sides in t, we obtain

$$\begin{aligned}&(X_2S^{-1}Q\Pi^{-1})(Pw)'' + ((X_2S^{-1}Q\Pi^{-1})' + X_1S^{-1}Q\Pi^{-1})(Pw)'\\ &+(X_1S^{-1}Q\Pi^{-1})'(Pw) = 0 \qquad \text{in } \Omega \times (-\delta, \delta).\end{aligned}$$

Hence

$$\begin{aligned}(Pw)''(x,t)+&H_1(x,t)(Pw)'(x,t) + H_2(x,t)(Pw)(x,t) = 0,\\ &x \in \Omega,\ |t| < \delta,\end{aligned} \tag{9.26}$$

where

$$\begin{aligned}H_1(x,t) &= -\,\Pi Q^{-1}SA_2^{-1}E((X_2S^{-1}Q\Pi^{-1})' + X_1S^{-1}Q\Pi^{-1})(x,t)\\ H_2(x,t) &= -\,\Pi Q^{-1}SA_2^{-1}E(X_1S^{-1}Q\Pi^{-1})'(x,t), \quad x \in \Omega,\ |t| < \delta.\end{aligned} \tag{9.27}$$

Then

$$H_1, H_2 \in C^2(\overline{\Omega} \times [-\delta, \delta]). \tag{9.28}$$

We define a differential operator N of the second order by

$$\begin{aligned}(Nv)(x,t) = v''(x,t) + H_1(x,t)v'(x,t) + H_2(x,t)v(x,t),&\\ x \in \Omega,\ |t| < \delta.&\end{aligned} \tag{9.29}$$

Then we can rewrite (9.26) as

$$(NPw)(x,t) = 0, \qquad x \in \Omega,\ |t| < \delta. \tag{9.30}$$

Now we can follow the argument in Second Step of the proof of Theorem 1 in §7 with some modifications. Here we will mainly explain such modifications. Let us recall (7.7) - (7.9), (5.1) and (5.2). We set

$$v(x,t) = \begin{pmatrix} v_1(x,t) \\ \vdots \\ v_{n+1}(x,t) \end{pmatrix} = \chi(x,t)w(x,t) \quad \text{in } \phi_{c(\epsilon)}. \tag{9.31}$$

Then, noting (9.30) and $\Delta(H_2 v) = H_2\Delta v + (\Delta H_2)v + 2\left(\sum_{j=1}^{n+1}\sum_{k=1}^{n}\frac{\partial (H_2)_{ij}}{\partial x_k}\frac{\partial v_j}{\partial x_k}\right)^T_{1\le i\le n+1}$, etc., we have

$$\begin{aligned} &PNv = NPv + [P,N]v \\ &=[NP,\chi]w + \chi NPw + [P,N]v = [NP,\chi]w + [P,N]v \quad \text{in } \phi_{c(\epsilon)}. \end{aligned} \tag{9.32}$$

Moreover by direct calculations we see that

$$\begin{aligned} &[P,N]v = (\Pi H_1 - H_1\Pi)v''' + (2\Pi H_1' + \Pi H_2 - H_2\Pi)v'' \\ &+(\Pi H_1'' + 2\Pi H_2' - \Delta H_1 + D_1 H_1)v' \\ &-2\left(\sum_{j=1}^{n+1}\sum_{k=1}^{n}\frac{\partial (H_1)_{ij}}{\partial x_k}\frac{\partial v_j'}{\partial x_k}\right)^T_{1\le i\le n+1} - 2\left(\sum_{j=1}^{n+1}\sum_{k=1}^{n}\frac{\partial (H_2)_{ij}}{\partial x_k}\frac{\partial v_j}{\partial x_k}\right)^T_{1\le i\le n+1} \\ &+(\Pi H_2'' - \Delta H_2 + D_1 H_2)v \quad \text{in } \Omega\times(-\delta,\delta). \end{aligned}$$

That is, in view of (9.28), we can write it as

$$\begin{aligned} [P,N]v(x,t) = &\sum_{k=0}^{3} a_k(x,t)\frac{\partial^k v}{\partial t^k}(x,t) + \sum_{j=1}^{n} b_j(x,t)\frac{\partial v'}{\partial x_j}(x,t) \\ &+\sum_{j=1}^{n} c_j(x,t)\frac{\partial v}{\partial x_j}(x,t), \end{aligned} \tag{9.33}$$

where $a_k, b_j, c_j \in C(\overline{\Omega}\times[-\delta,\delta])$, $0\le k\le 3$, $1\le j\le n$. Furthermore similarly to (7.16), we can prove that

$$Nv \in H_0^2(\phi_{c(\epsilon)})^{n+1}.$$

Without loss of generality, in addition to $T > \frac{\sup_{x\in\Omega}|x|}{\sqrt{\theta}}$, we can assume that $T - \frac{\sup_{x\in\Omega}|x|}{\sqrt{\theta}}$ is sufficicently small such that Proposition 5.1 is applicable.

Thus we apply Proposition 5.1 to (9.32):

$$
\begin{aligned}
&\xi \sum_{|\alpha|\le 1} \|D^\alpha(Nv)\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})}\\
\le& M\|[NP,\chi]w\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} + M\|[P,N]v\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})}\\
\le& M\|[NP,\chi]w\exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} + M\sum_{i=0}^{3}\left\|\frac{\partial^i v}{\partial t^i}\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}\\
+& M\sum_{j=1}^{n}\left\|\frac{\partial v}{\partial x_j}\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})} + M\sum_{j=1}^{n}\left\|\frac{\partial^2 v}{\partial x_j\partial t}\exp(\xi e^{\tau\phi})\right\|^2_{L^2(\phi_{c(\epsilon)})}
\end{aligned}
\tag{9.34}
$$

for sufficiently large $\xi > 0$. Since $v(x,0) = v'(x,0) = 0$, $x \in \overline{\Omega}$, by the initial condition in (2.22) and the definition (7.9) of χ, in view of (9.28), we can construct an $(n+1)\times(n+1)$ matrix function $K = K(x,t,s) \in C^2(\overline{\Omega}\times[-\delta,\delta]^2)$ which solves (9.29) by

$$
v(x,t) = \int_0^t K(x,t,s)(Nv)(x,s)ds, \quad (x,t)\in\phi_{c(\epsilon)}. \tag{9.35}
$$

Therefore in $\phi_{c(\epsilon)}$, we have

$$
\frac{\partial v}{\partial t}(x,t) = K(x,t,t)(Nv)(x,t) + \int_0^t \frac{\partial K}{\partial t}(x,t,s)(Nv)(x,s)ds,
$$

$$
\begin{aligned}
\frac{\partial^2 v}{\partial t^2}(x,t) =& \frac{\partial K(x,t,t)}{\partial t}(Nv)(x,t) + K(x,t,t)(Nv)'(x,t)\\
&+\frac{\partial K}{\partial t}(x,t,t)(Nv)(x,t) + \int_0^t \frac{\partial^2 K}{\partial t^2}(x,t,s)(Nv)(x,s)ds,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 v}{\partial t\partial x_j}(x,t) =& \frac{\partial K}{\partial x_j}(x,t,t)(Nv)(x,t) + K(x,t,t)\frac{\partial(Nv)}{\partial x_j}(x,t)\\
&+\int_0^t \frac{\partial^2 K}{\partial t\partial x_j}(x,t,s)(Nv)(x,s)ds + \int_0^t \frac{\partial K}{\partial t}(x,t,s)\frac{\partial(Nv)}{\partial x_j}(x,s)ds,
\end{aligned}
$$

so that Lemma 7.1 implies

$$\left\| \left(\frac{\partial^i v}{\partial t^i}\right) \exp(\xi e^{\tau\phi}) \right\|_{L^2(\phi_{c(\epsilon)})}, \quad \left\| \left(\frac{\partial^2 v}{\partial t \partial x_j}\right) \exp(\xi e^{\tau\phi}) \right\|_{L^2(\phi_{c(\epsilon)})}$$

$$\leq M \sum_{|\alpha|\leq 1} \left\| D^\alpha (Nv) \exp(\xi e^{\tau\phi}) \right\|_{L^2(\phi_{c(\epsilon)})}, \quad i = 0, 1, 2,\ 1 \leq j \leq n. \tag{9.36}$$

Moreover in view of (9.29), we have

$$v'''(x,t) = -H_1(x,t)v''(x,t) - H_1'(x,t)v'(x,t) - H_2(x,t)v'(x,t)$$
$$-H_2'(x,t)v(x,t) + (Nv)'(x,t), \quad (x,t) \in \phi_{c(\epsilon)},$$

so that from (9.36) we obtain

$$\|v''' \exp(\xi e^{\tau\phi})\|_{L^2(\phi_{c(\epsilon)})} \leq M \sum_{|\alpha|\leq 1} \|D^\alpha (Nv) \exp(\xi e^{\tau\phi})\|_{L^2(\phi_{c(\epsilon)})}. \tag{9.37}$$

Consequently substitution of (9.36) and (9.37) in (9.34) yields

$$\xi \sum_{|\alpha|\leq 1} \|D^\alpha (Nv) \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})}$$
$$\leq M \|[NP, \chi] w \exp(\xi e^{\tau\phi})\|^2_{L^2(\phi_{c(\epsilon)})} \tag{9.38}$$

for sufficiently large $\xi > 0$. Now similarly to the proof of Theorem 1, noting $\chi = 1$ in $\phi_{c(3\epsilon)}$, and letting ξ tend to ∞ in (9.38), we see that $Nv = 0$ in $\phi_{c(3\epsilon)}$. Then in view of (9.35), it follows that $v = 0$ in $\phi_{c(3\epsilon)}$, namely, $w = 0$ in $\phi_{c(3\epsilon)}$. Therefore $g = 0$ and $y = 0$ in $\phi_{c(3\epsilon)}$ by (9.1). Since $\epsilon > 0$ is arbitrarily small, we obtain

$$\left\{ \begin{array}{l} g(x) = 0, \quad x \in \Omega \quad \text{with } \sqrt{\omega^2 - \theta\delta^2} \leq |x| \leq \omega \\ y(x,t) = 0, \quad (x,t) \in \phi_{c(0)}. \end{array} \right\} \tag{9.39}$$

Then we can follow the argument after Third Step of the proof of Theorem 1 in §7. For convenience, we will supply a sketch of the proof. We recall that $c_1(\epsilon)$ and $\kappa = \kappa(t)$ are given by (7.48) and (7.49), and we set

$$\left\{ \begin{array}{r} \widetilde{A_k}(x,t) = A_k(x,0) + \kappa(t)(A_k(x,t) - A_k(x,0)), \\ \widetilde{E}(x,t) = E(x,0) + \kappa(t)(E(x,t) - E(x,0)), \\ (x,t) \in \overline{\phi_{c_1(\epsilon)}},\ 1 \leq k \leq 4. \end{array} \right\} \tag{9.40}$$

First we verify

$$\left|\det\begin{pmatrix}\widetilde{A_1}(x,t) & \widetilde{A_2}(x,t)\\ \widetilde{A_3}(x,t) & \widetilde{A_4}(x,t)\end{pmatrix}\right|, \quad |\det \widetilde{E}(x,t)|,$$

$$|\det \widetilde{A_2}(x,t)| \ne 0, \quad (x,t) \in \overline{\phi_{c_1(\epsilon)}}. \tag{9.41}$$

If (9.41) is verified, then we can repeat the arguments after Fourth Step of the proof of Theorem 1 in §7. In fact, by (9.39) and (9.40), we easily have

$$\begin{aligned}&\Pi(x)Q^{-1}(x)S(x)\begin{pmatrix}\widetilde{A_1}(x,t) & \widetilde{A_2}(x,t)\\ \widetilde{A_3}(x,t) & \widetilde{A_4}(x,t)\end{pmatrix}\begin{pmatrix}g(x)\\ g(x)\end{pmatrix}\\ =&\begin{cases}0, \quad (x,t) \in \phi_{c_1(\epsilon)} \cap \{(x,t); \sqrt{\omega^2-\theta\delta^2} \le |x| \le \omega\}\\ \Pi(x)Q^{-1}(x)S(x)\begin{pmatrix}A_1(x,t) & A_2(x,t)\\ A_3(x,t) & I_n(x,t)\end{pmatrix}\begin{pmatrix}g(x)\\ g(x)\end{pmatrix},\\ (x,t) \in \phi_{c_1(\epsilon)} \cap \{(x,t); |x| < \sqrt{\omega^2-\theta\delta^2}\},\end{cases}\end{aligned}$$

so that in $\phi_{c_1(\epsilon)}$ we can repeat the previous argument.

Now we return to the proof of (9.41). Here we will show only (9.41) for $\widetilde{A_2}$, because the positivity of the rest two determinants are seen similarly. We set

$$A_2(x,t) = (a^{(2)}_{ij}(x,t))_{1\le i,j\le n+1}, \quad \widetilde{A_2}(x,t) = (\widetilde{a^{(2)}_{ij}}(x,t))_{1\le i,j\le n+1}.$$

Then for $1 \le i,j \le n+1$ and $|t| < \delta$, we have

$$\begin{aligned}&|a^{(2)}_{ij}(x,t) - \widetilde{a^{(2)}_{ij}}(x,t)| = |(1-\kappa(t))(a^{(2)}_{ij}(x,t) - \widetilde{a^{(2)}_{ij}}(x,0))|\\ \le&\delta\|(a^{(2)}_{ij})'\|_{L^\infty(\Omega\times(-T,T))}.\end{aligned}$$

Since the determinant: $A = (a_{ij})_{1\le i,j\le n+1} \longmapsto \det A$ is continuous in a_{ij}, $1 \le i,j \le n+1$, in terms of (9.22), we can in advance choose $\delta > 0$ so small that

$$|\det A_2(x,t)| \ge \frac{r_0}{2}, \quad |\det A_2(x,t) - \det \widetilde{A_2}(x,t)| \le \frac{r_0}{4}, \quad x \in \overline{\Omega},\ |t| \le \delta.$$

Then we obtain

$$|\det \widetilde{A_2}(x,t)| \ge \frac{r_0}{4}, \quad x \in \overline{\Omega},\ |t| \le \delta. \tag{9.42}$$

On the other hand, by (9.20) and the definition (7.49) of κ, we see

$$|\det \widetilde{A_2}(x,t)| = |\det A_2(x,0)| \ge \frac{r_0}{2}, \quad x \in \overline{\Omega},\ |t| > \delta. \tag{9.43}$$

The inequalities (9.42) and (9.43) yield

$$|\det \widetilde{A_2}(x,t)| \ge \frac{r_0}{4}, \quad (x,t) \in \overline{\phi_{c_1(\epsilon)}}.$$

Consequently we see (9.41) for $\widetilde{A_2}(x,t)$.

Thus the proof of Theorem 3 is complete. □

§10. Proofs of Theorems 4 and 5

Proof of Theorem 4. We set

$$y_k(x,t) = \widetilde{u_k}(x,t) - u_k(x,t), \quad x \in \overline{\Omega},\ -T \le t \le T,\ 1 \le k \le n$$

and

$$f(x) = \rho(x) - \widetilde{\rho}(x), \quad R_k(x,t) = u_k''(x,t), \quad x \in \overline{\Omega},\ -T \le t \le T,\ 1 \le k \le n.$$

Then from (3.2), (3.3) and (3.8), we obtain

$$\left\{\begin{array}{l} \widetilde{\rho}(x) y_k''(x,t) = (\mathcal{L}y_k)(x,t) + f(x)R_k(x,t), \quad x \in \Omega,\ -T < t < T, \\ y_k(x,0) = 0, \qquad x \in \Omega, \\ y_k(x,t) = \sigma(y_k)(x,t)\nu(x) = 0, \quad x \in \partial\Omega,\ -T < t < T, \end{array}\right\} \tag{10.1}$$

for $1 \le k \le n$. Since $y_k \in C^4(\overline{\Omega} \times [-T,T])^n$, by (3.2) we have

$$\begin{aligned} R_k(x,0) = u_k''(x,0) &= \frac{1}{\rho(x)}(\mathcal{L}u_k)(x,0) \\ &= \frac{1}{\rho(x)}(\mathcal{L}a_k)(x), \quad x \in \overline{\Omega},\ 1 \le k \le n. \end{aligned} \tag{10.2}$$

Therefore the conditions (3.5) and (3.6) imply the corresponding (2.17) and (2.18) in Theorem 2. In view of $\widetilde{\rho} \in \mathcal{U}$, we can apply Theorem 2 and the proof of Theorem 4 is complete. □

Proof of Theorem 5. Setting

$$y(x,t) = \widetilde{u}(x,t) - u(x,t), \quad f(x) = \rho(x) - \widetilde{\rho}(x),$$
$$R(x,t) = u''(x,t), \quad x \in \overline{\Omega},\ -T \le t \le T,$$

from (3.9) and (3.10) we derive

$$(10.3) \quad \left\{ \begin{array}{ll} \widetilde{\rho}(x)y''(x,t) = (\mathcal{L}y)(x,t) + f(x)R(x,t), & x \in \Omega,\ -T < t < T, \\ y(x,0) = y'(x,0) = 0, & x \in \Omega, \\ y(x,t) = \sigma(y)(x,t)\nu(x) = 0, & x \in \partial\Omega,\ -T < t < T. \end{array} \right\}$$

Similarly to (10.2), since $y \in C^5(\overline{\Omega} \times [-T,T])^n$ from (3.9) we see

$$R(x,0) = \frac{1}{\rho(x)}(\mathcal{L}a)(x), \quad x \in \overline{\Omega}$$

and

$$R'(x,0) = u'''(x,0) = \frac{\partial}{\partial t}\left(\frac{1}{\rho(x)}(\mathcal{L}u)(x,t)\right)\Bigg|_{t=0} = \frac{1}{\rho(x)}(\mathcal{L}b)(x), \qquad x \in \overline{\Omega}.$$

Therefore in the system (10.3) the condition (3.12) means (2.25). Since $f = \rho - \widetilde{\rho}$ satisfies (2.23), we apply Theorem 3. Thus the proof of Theorem 5 is complete. □

Appendix. Proof of Lemma 7.1

By the definition (5.2) of ϕ_c, we can take a domain $\mathcal{D} \subset \Omega$ such that

$$\phi_{c(\epsilon)} = \{(x,t) \in \mathbb{R}^{n+1}; x \in \mathcal{D},\ |t| < l(x)\},$$

where we set

$$l(x) = \frac{1}{\sqrt{\theta}}\left(|x|^2 - c(\epsilon)^2\right)^{\frac{1}{2}}.$$

Therefore by Schwarz's inequality and change of orders of integrations in s and t, we have

$$
\begin{aligned}
&\int_{\phi_{c(\epsilon)}\cap\{t\geq 0\}} e^{2\xi\psi(x,t)} \left|\int_0^t |p(x,s)|ds\right|^2 dxdt \\
=&\int_{\mathcal{D}} \left(\int_0^{l(x)} e^{2\xi\psi(x,t)} \left(\int_0^t |p(x,s)|ds\right)^2 dt\right) dx \\
\leq&\int_{\mathcal{D}} \left(\int_0^{l(x)} te^{2\xi\psi(x,t)} \left(\int_0^t |p(x,s)|^2 ds\right) dt\right) dx \\
=&\int_{\mathcal{D}} \left(\int_0^{l(x)} |p(x,s)|^2 \left(\int_s^{l(x)} te^{2\xi\psi(x,t)} dt\right) ds\right) dx \\
\leq&\int_{\mathcal{D}} l(x) \left(\int_0^{l(x)} |p(x,s)|^2 \left(\int_s^{l(x)} e^{2\xi\psi(x,t)} dt\right) ds\right) dx.
\end{aligned}
$$

Since $\psi'(x,t) \leq 0$ for $t \geq 0$, we obtain

$$
\begin{aligned}
&\int_{\phi_{c(\epsilon)}\cap\{t\geq 0\}} e^{2\xi\psi(x,t)} \left|\int_0^t |p(x,s)|ds\right|^2 dxdt \\
\leq&\int_{\mathcal{D}} l(x) \left(\int_0^{l(x)} (l(x)-s)|p(x,s)|^2 e^{2\xi\psi(x,s)} ds\right) dx \\
\leq& l(x)^2 \int_{\mathcal{D}} \left(\int_0^{l(x)} |p(x,s)|^2 e^{2\xi\psi(x,s)} ds\right) dx \\
=& l(x)^2 \int_{\phi_{c(\epsilon)}\cap\{t\geq 0\}} |p(x,s)|^2 e^{2\xi\psi(x,s)} dxds.
\end{aligned}
$$

Similarly we can prove

$$
\begin{aligned}
&\int_{\phi_{c(\epsilon)}\cap\{t<0\}} e^{2\xi\psi(x,t)} \left|\int_0^t |p(x,s)|ds\right|^2 dxdt \\
\leq& l(x)^2 \int_{\phi_{c(\epsilon)}\cap\{t<0\}} |p(x,s)|^2 e^{2\xi\psi(x,s)} dxds.
\end{aligned}
$$

The proof of Lemma 7.1 is complete. □

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