On the Construction of Complete Sets of Geometric Invariants for Algebraic Curves

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We provide a solution to the important problem of constructing complete independent sets of Euclidean and affine invariants for algebraic curves. We first simplify algebraic curves through polynomial decompositions and then use some classical geometric results to construct functionally independent sets of invariants. The results presented here represent some new contributions to algebraic curve theory that can be used in many application areas, such as model-based vision, object recognition, graphics, geometric modeling, and CAD. © 2000 Academic Press

1. INTRODUCTION

Object identification and recognition are fundamental problems in many disciplines. When objects are defined by boundary curves, they can be represented either explicitly, using parametric equations, or implicitly, using polynomial equations defined by planar variables. Such equations, which define algebraic curves, have been studied extensively since the 17th century by many different individuals. Indeed, a good deal of 19th century algebraic geometry focused on the study of invariants for algebraic curves that are preserved under linear transformations. Such invariants were generally nonlinear algebraic functions of the coefficients of the curves. As noted in [1], invariant theory was formalized by George Boole in 1841, and subsequently developed by several notable mathematicians, including Arthur Cayley, James Sylvester, George Salmon, Paul Gordan, David Hilbert, and Emmy Noether.

Invariant theory for algebraic curves remains an active area of investigation [12], and with the advent of the modern digital computer, it is now...
possible to manipulate algebraic curves of arbitrary degree with relative ease, thus providing far more insight into their properties and behavior than was ever possible before. Also, implicit representations for 2D curves and 3D surfaces are becoming more useful in a variety of application areas; i.e., as noted in [5], “algebraic surfaces have many mathematical and computational advantages compared to the parametric elements now used, and are emerging as an active forefront research area.” Moreover, [3] states that “the use of parametric surfaces remains dominant in computer graphics and geometric modeling. Implicit surfaces are receiving increased attention, however, especially with respect to their natural representation of solid objects and their innate blend properties.” Furthermore, [4] remarks that “in computer aided geometric design (CAGD), surfaces are dealt with in several forms: parametric, explicit and implicit equations. The most common representation in the commercial software and research fields are parametric equations. Nevertheless, in the last few years, implicit representations are being used more frequently in CAGD, allowing a better treatment of several problems. As one example, the point classification problem is easily solved with the implicit representation: it consists of a simple evaluation of the implicit functions. Furthermore, the implicit representation offers surfaces of desired smoothness with the lowest possible degree. Finally, when we restrict ourselves to polynomial functions, the implicit representation is more general than the parametric representation.”

In this paper, which embellishes [16], we will employ a new mathematical decomposition of algebraic curves defined by implicit polynomial equations to develop complete sets of geometric invariants for such curves under both Euclidean and affine transformations. These invariants can be used for object identification and recognition in numerous applications, such as indexing into databases where objects can be segmented from the background, for printed character and font recognition, and in certain vision-based control applications, to approximate the translational and rotational velocities of moving objects [18].

Section 2 presents some background material that is used to formally develop a unique decomposition of algebraic curves in Section 3. Section 4 then defines the notions of intersection points, conic factor centers, and the distance between points defined by complex coordinates. An ordering of the lines defined by our decomposition is also given. Euclidean and affine transformations of equivalent curves are then introduced in Section 5. An appropriate correspondence between two sets of related points is also presented. In Section 6, a counting argument is used to determine the number of functionally independent invariants under various transformation groups. The main result of the paper is then presented in Section 7, where some classical geometric results are applied to our unique line decomposition to produce a new constructive procedure for determining complete sets of ge-
ometric invariants for algebraic curves. Some examples serve to illustrate our findings, and we conclude with some final observations in Section 8.

2. SOME PRELIMINARY OBSERVATIONS

Algebraic curves are defined implicitly by equations of the form \( f(x, y) = 0 \), where \( f(x, y) \) is a polynomial in the variables \( x, y \), i.e.,

\[
f(x, y) = \sum_{i+j} a_{ij} x^i y^j, \quad \text{where} \quad 0 \leq i + j \leq n \quad (n \text{ is finite}) \quad \text{and the coefficients} \quad a_{ij} \quad \text{are real numbers}.\]

Alternatively, the intersection of an explicit surface \( z = f(x, y) \) with the \( z = 0 \) plane yields an algebraic curve if \( f(x, y) \) is a polynomial. Since the field of real numbers is not “algebraically closed,” it is often useful and illuminating to extend this definition to the complex field [8].

2.1. Some Definitions

**Definition 1.** The *degree* of a polynomial \( f(x, y) \) is the maximal value of \( i + j \) for which \( a_{ij} \neq 0 \).

**Definition 2.** The *zero set* of a polynomial \( f(x, y) \) is defined to be the set

\[
Z(f) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}
\]

Since the zero set (and the degree) are unchanged when we multiply \( f \) by a non-zero scalar, it follows that:

**Definition 3.** A real algebraic curve is a non-zero real polynomial \( f \), up to multiplication by a non-zero scalar.

Algebraic curves of degree 1, 2, 3, 4, . . . are called lines, conics, cubics, quartics, . . ., etc.

In general, an algebraic curve of degree \( n \) can be defined by the implicit polynomial equation

\[
f_n(x, y) = \sum_{i+j} a_{ij} x^i y^j = 0
\]

where each binary form \( h_i(x, y) \) is a homogeneous polynomial of degree \( r \) in the variables \( x \) and \( y \). The number of terms in each \( h_i(x, y) \) is \( r + 1 \), so that the IP equation defined by (2.4) has one constant term, two terms of the
first degree, three terms of the second degree, etc., up to and including \( n + 1 \) terms of the (highest) \( n \)th degree, for a total of \( (n + 1)(n + 2)/2 \) coefficients. Since the above equation can be multiplied by a non-zero constant without changing the zero set, an algebraic curve defined by \( f_n(x, y) = 0 \) has \( (n + 1)(n + 2)/2 - 1 = n(n + 3)/2 \) independent coefficients or degrees of freedom (DOF). A monic polynomial \( f_n(x, y) = 0 \) will be defined by the condition that \( a_{n0} = 1 \) in (2.4).

2.2. Asymptotes

We next note that an asymptote to the curve \( f_n(x, y) = 0 \) is a tangent line \( y = mx + b \) which intersects the curve at infinity \([7]\). Asymptotes can be determined by substituting \( y = mx + b \) for \( y \) in (2.4) and writing the highest degree terms first, so that

\[
f_n(x, y) = \{a_{n0} + a_{n-1,1}m + \cdots + a_{0,n}m^n\}x^n + \left[\{a_{n-1,0} + a_{n-2,1}m + \cdots + a_{0,n-1}m^{n-1}\}\right. \\
+ b[a_{n-1,1} + 2a_{n-2,2}m + 3a_{n-3,3}m^2 + \cdots + na_{0,n}m^{n-1}]\}x^{n-1} + \cdots \]

\[
H_n(m, b) \quad \cdot \quad \cdot
\]

The line \( y = mx + b \) will be an asymptote if both \( H_n(m) = 0 \) and \( H_{n-1}(m, b) = 0 \), since the curve will then have two roots at infinity, corresponding to two intersections with the asymptote there. Any line parallel to an asymptote \( y = mx + b \), such as \( y = mx \), will intersect the curve once at infinity, since (only) \( H_n(m) = 0 \). Each real root \( m_i \) of \( H_n(m) \) will then imply a corresponding real asymptote defined by the line \( y = m_i x + b_i \), with \( b_i = -H_{n-1,0}(m_i)/H_{n-1,1}(m_i) \).

In general, \( H_n(m) = 0 \) will have \( n \) roots, which may be complex (conjugates) or real. \( H_n(m) = 0 \) will have at least one real root when \( n \) is odd, so that odd curves are unbounded. Therefore, to represent 2D closed and bounded curves, IPs of even degree must be used. For a bounded (even) IP curve, \( H_n(m) = 0 \) will have exactly \( p = n/2 \) pairs of complex conjugate roots.

3. A UNIQUE DECOMPOSITION THEOREM

In this section, we will simplify the analysis and representation of algebraic curves through a unique decomposition theorem for algebraic curves \([15, 17, 18]\).
3.1. A Unique Leading Form Factorization

**Lemma 1.** Any homogeneous form \( h_r(x, y) \) can be factored uniquely over the complex field.

**Proof** The substitution of \( mx \) for \( y \) in \( h_r(x, y) \) implies that

\[
h_r(x, y = mx) = x^r a_0 (m - m_{r1})(m - m_{r2}) \cdots (m - m_{rr}),
\]

for possibly complex (conjugate) roots \( m_{ri} \), or that

\[
h_r(x, y) = a_0 (y - m_{r1}x)(y - m_{r2}x) \cdots (y - m_{rr}x).
\]

Therefore, a (monic) leading form \( h_n(x, y) \) can be uniquely factored as the product of \( n \) lines; i.e.,

\[
h_n(x, y) = \prod_{i=1}^{n} (x - (1/m_{ni})y) = \prod_{i=1}^{n} (x + l_{ni}y), \tag{3.1}
\]

where \( l_{ni} \overset{\text{def}}{=} -1/m_{ni} \). We will restrict our development here to the generic case of simple (non-repeated) roots.

3.2. A Partial Fraction Expansion

Since \( h_{n-1}(x, y)/h_n(x, y) \) is a strictly proper rational function, its partial fraction expansion implies that

\[
\frac{h_{n-1}(x, y)}{h_n(x, y)} = \frac{k_{n1}}{x + l_{n1}y} + \frac{k_{n2}}{x + l_{n2}y} + \cdots + \frac{k_{nn}}{x + l_{nn}y}, \tag{3.2}
\]

where \( k_{n1}, k_{n2}, \ldots, k_{nn} \) are \( n \) unique scalars that can be determined if we multiply both sides of (3.2) by \((x + l_{nj}y)\) and then set \( x = -l_{nj}y \); i.e.,

\[
k_{nj} = \left[ \frac{h_{n-1}(x, y)}{h_n(x, y)}(x + l_{nj}y) \right]_{x = -l_{nj}y} = \frac{h_{n-1}(-l_{j}, 1)}{\prod_{i \neq j}(l_{ni} - l_{nj})}, \tag{3.3}
\]

where the last expression is a direct consequence of the fact that the numerator and the denominator of \((h_{n-1}(x, y)/h_n(x, y))(x + l_{nj}y)\) are homogeneous polynomials after cancellation(s), so that the substitutions \( x = -l_j \) and \( y = 1 \) can be used instead of \( x = -l_jy \).
In light of (2.4) and (3.2), any monic

\[ f_n(x, y) = h_n(x, y) \left( 1 + \frac{h_{n-1}(x, y)}{h_n(x, y)} \right) + \sum_{i=0}^{n-2} h_i(x, y) \]

\[ = h_n(x, y) + h_n(x, y) \left\{ \frac{k_{n1}}{x + l_{n1}y} + \frac{k_{n2}}{x + l_{n2}y} + \cdots + \frac{k_{nn}}{x + l_{nn}y} \right\} \]

\[ + \sum_{i=0}^{n-2} h_i(x, y) = 0, \quad (3.4) \]

or

\[ f_n(x, y) = h_n(x, y) + \frac{h_n(x, y)}{\prod_{i=1}^{n}(x + l_{ni}y)} \]

\[ \times \left\{ k_{n1}(x + l_{n1}y)(x + l_{n2}y) \cdots (x + l_{nn}y) \right. \]

\[ + k_{n2}(x + l_{n2}y)(x + l_{n3}y) \cdots (x + l_{nn}y) + \cdots \]

\[ + k_{nn}(x + l_{n1}y)(x + l_{n2}y) \cdots (x + l_{nn-1}y) \right\} \]

\[ + \sum_{i=0}^{n-2} h_i(x, y) = 0. \quad (3.5) \]

Since \( h_n(x, y) = \prod_{i=1}^{n}(x + l_{ni}y) \), (2.4) and (3.5) imply that

\[ h_{n-1}(x, y) = k_{n1}(x + l_{n2}y)(x + l_{n3}y) \cdots (x + l_{nn}y) \]

\[ + k_{n2}(x + l_{n1}y)(x + l_{n3}y) \cdots (x + l_{nn}y) + \cdots + k_{nn}(x + l_{n1}y)(x + l_{n2}y) \cdots (x + l_{nn-1}y) \]

\[ = \sum_{j=1}^{n} \left[ k_{nj} \prod_{i\neq j}^{n}(x + l_{ni}y) \right] \quad (3.6) \]

Equations (3.1) and (3.6) together then imply that the product

\[ \prod_{i=1}^{n} [x + l_{ni}y + k_{ni}] = \prod_{i=1}^{n}(x + l_{ni}y) + \sum_{j=1}^{n} \left[ k_{nj} \prod_{i\neq j}^{n}(x + l_{ni}y) \right] + r_{n-2}(x, y), \]

\[ \frac{h_n(x, y)}{h_n(x, y)} = \frac{h_{n-1}(x, y)}{h_{n-1}(x, y)} \quad (3.7) \]

for some “remainder” polynomial \( r_{n-2}(x, y) \) of degree \( n - 2 \).
3.3. Line Factors

Since each line factor \( x + l_{ni}y + k_{ni} \) can be written as the (vector) dot product

\[
\begin{bmatrix}
1 & l_{ni} & k_{ni}
\end{bmatrix} X = \begin{bmatrix} x & y & 1 \end{bmatrix} L_{ni},
\]

(3.4) and (3.7) imply that any monic

\[
f_n(x, y) = \prod_{i=1}^{n} L_{ni}^T X + f_{n-2}(x, y)
\]

for the \( n - 2 \) degree polynomial

\[
f_{n-2}(x, y) = \sum_{i=0}^{n-2} h_i(x, y) - r_{n-2}(x, y).
\]

3.4. Conic Factors

If \( l_{ni} \) and/or \( k_{ni} \) are complex numbers, with complex conjugates defined by \( l_{ni}^* \) and \( k_{ni}^* \), respectively, then \( x + l_{ni}^* y + k_{ni}^* = X^T L_{ni}^* \) also will appear as a line factor in (3.8). Any two such complex conjugate line factors will imply a corresponding real, degenerate conic factor

\[
C_{ni}(x, y) \overset{\text{def}}{=} X^T L_{ni}^* L_{ni} X = x^2 + (l_{ni} + l_{ni}^*)xy + l_{ni}l_{ni}^* y^2 + (k_{ni} + k_{ni}^*)x
\]

\[
+ (l_{ni}k_{ni}^* + l_{ni}^* k_{ni})y + k_{ni}k_{ni}^*.
\]

Therefore, a maximum of \( 2p \leq n \) complex (conjugate) values for \( l_{ni} \) or \( k_{ni} \) will imply that \( \Pi_n(x, y) \) in (3.8) can be expressed by the unique, real conic-line product

\[
\Pi_n(x, y) = \prod_{k=1}^{p} C_{nk}(x, y) \prod_{j=1}^{n-2p} L_{nj}^T X.
\]

We next note that if \( \gamma_{n-2} \) is the coefficient of \( x^{n-2} \) in the \( f_{n-2}(x, y) \) defined by (3.8), then a \( \Pi_{n-2}(x, y) \) can be defined for the monic \( f_{n-2}(x, y)/\gamma_{n-2} \), as above, so that

\[
f_{n-2}(x, y) = \gamma_{n-2}[\Pi_{n-2}(x, y) + f_{n-4}(x, y)]
\]

Subsequently defining \( \gamma_{n-4} \) as the coefficient of \( f_{n-4}(x, y) \), etc., we obtain the following theorem.
3.5. The Theorem

Theorem 3.11. A non-degenerate (monic) $f_n(x, y)$ can be uniquely expressed as a finite sum of real and complex line products or real conic-line products, namely

$$f_n(x, y) = \Pi_n(x, y) + \gamma_{n-2}[\Pi_{n-2}(x, y) + \gamma_{n-4}[\Pi_{n-4}(x, y) + \cdots]].$$  \hspace{1cm} (3.12)

For example, a (monic) conic, cubic, and quartic IP can be (line) decomposed as

\begin{align*}
f_2(x, y) &= L_1(x, y)L_2(x, y) + \alpha = 0, \\
f_3(x, y) &= L_1(x, y)L_2(x, y)L_3(x, y) + \alpha L_4(x, y) = 0,
\end{align*}

and

\begin{align*}
f_4(x, y) &= L_1(x, y)L_2(x, y)L_3(x, y)L_4(x, y) + \alpha L_5(x, y)L_6(x, y) + \beta = 0,
\end{align*}

respectively, where $\alpha$ and $\beta$ are real scalars. Note that each line factor in these (and higher degree decompositions) has two DOF. Therefore, by including the multiplicative scalars, we verify that a monic IP curve has $n(n+3)/2$ independent coefficients or DOF.

4. SOME GEOMETRIC PRELIMINARIES

4.1. Non-Visual Intersection Points

The intersection point $d_p = \{x_p, y_p\}$ of any two non-parallel line factors of (3.12), such as $L_{ij}^T X = x + l_{ij}y + k_{ij}$ and $L_{qr}^T X = x + l_{qr}y + k_{qr}$, can be defined by the matrix/vector relation

\begin{align*}
\begin{bmatrix}
1 & l_{ij} & k_{ij} \\
1 & l_{qr} & k_{qr}
\end{bmatrix}
\begin{bmatrix}
x_p \\
y_p \\
1
\end{bmatrix}
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\implies
\begin{bmatrix}
x_p \\
y_p
\end{bmatrix}
&= \begin{bmatrix}
l_{ij}k_{qr} - l_{qr}k_{ij} \\
k_{ij} - k_{qr}
\end{bmatrix} \div (l_{qr} - l_{ij}),
\end{align*}

which also can be represented as a vector cross-product [2].
Moreover, the center $d_c = \{x_c, y_c\}$ of the zero set of any conic factor of (3.9) can be defined by the matrix/vector relation [14]

$$\begin{bmatrix} 2 & l_{ni} + l_n^* & k_{ni} + k_n^* \\ l_{ni} + l_n^* & 2l_{ni}l_n^* & l_n^*k_{ni} + l_nk_n^* \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ (4.2)

It might be noted that this (real) center corresponds to the intersection of the imaginary line factors $L_n^T X = x + l_{ni}y + k_{ni}$ and $X^T L_n^* = x + l_n^*y + k_n^*$, which define the conic factor, since

$$\begin{bmatrix} 1 & 1 \\ l_{ni} & l_n^* \end{bmatrix} \begin{bmatrix} 1 & l_{ni} & k_{ni} \\ 1 & l_n^* & k_n^* \end{bmatrix} = \begin{bmatrix} 2 & l_{ni} + l_n^* & k_{ni} + k_n^* \\ l_{ni} + l_n^* & 2l_{ni}l_n^* & l_n^*k_{ni} + l_nk_n^* \end{bmatrix}.$$

As noted in Section 2.2, the leading form factorization for closed-bounded curves will imply only complex conjugate roots. Therefore, in the special case of closed-bounded quartics, (3.8) and (3.9) will imply that

$$f_4(x, y) = \Pi_4(x, y) + f_2(x, y) = C_4(x, y)C_2(x, y) + f_2(x, y),$$ (4.3)

where we used the fact that $\Pi_4(x, y)$ can be factored as the product of two degenerate conics $C_4(x, y) = L_1(x, y)L_2(x, y) = L_1(x, y)L_1^*(x, y)$ and $C_2(x, y) = L_3(x, y)L_4(x, y) = L_3(x, y)L_3^*(x, y)$. Note that $f_2(x, y) = \alpha L_5(x, y)L_6(x, y) + \beta$ is an arbitrary conic.

### 4.2. Distance

The distance between two points with generally complex coordinates, $P_1 = \{a_1 + ib_1, c_1 + id_1\}$ and $P_2 = \{a_2 + ib_2, c_2 + id_2\}$, is defined by the (complex) inner product of their difference vector

$$(P_2 - P_1) = [(a_2 - a_1) + i(b_2 - b_1) \ (c_2 - c_1) + i(d_2 - d_1)]^T,$$

namely $(P_2 - P_1)\dagger(P_2 - P_1)$, where $\dagger$ denotes the conjugate transpose, whose length (the distance between the points) is

$$||P_2 - P_1|| \overset{\text{def}}{=} |P_2P_1|$$

$$= |P_2P_1| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2 + (d_2 - d_1)^2} \geq 0$$ (4.4)
Example 4.5. To illustrate our results thus far, consider the closed, bounded quartic curve depicted in Fig. 1, which is defined by the (monic) implicit polynomial equation

\[ f_4(x, y) = x^4 - x^3 y + 1.5x^2 y^2 - xy^3 + 0.5y^4 - x^3 - xy^2 + x^2 - 2xy + 2y^2 + y - 0.5 = 0 \]  \hspace{1cm} (4.6)

In light of (3.13) and (4.3), this particular curve has a decomposition defined by

\[ f_4(x, y) = C_{41}(x, y)C_{42}(x, y) + aL_5(x, y)L_6(x, y), \]

with \( C_{41}(x, y) = x^2 + y^2 = L_3(x, y)L_2(x, y) = L_3(x, y)L_2^*(x, y) = [x - iy][x + iy], \) \( C_{42}(x, y) = x^2 - xy + 0.5y^2 - x + 0.5 = L_3(x, y)L_4(x, y) = L_3(x, y)L_2^*(x, y) = [x - (0.5 - 0.5i)y - (0.5 + 0.5i)y - (0.5 - 0.5i)] [x - (0.5 + 0.5i)y - (0.5 - 0.5i)], \) and a degenerate (since \( \beta = 0 \)) \( f_2(x, y) = aL_5(x, y)L_6(x, y) = 0.5x^2 - 2xy + 1.5y^2 + y - 0.5 = 0.5[x - 3y + 1][x - y - 1]. \)

The intersection points for the first four lines are determined to be \( P_{12} = \{0, 0\}, \) \( P_{13} = P_{24} = \{(2 + i)/5, (1 - 2i)/5\}, \) \( P_{14} = P_{23} = \{-i, -1\}, \) and \( P_{34} = \{1, 1\}. \) Using (4.4), it follows that \( |P_{12}P_{13}| = \sqrt{10}/5, |P_{12}P_{14}| = \sqrt{2}, \) \( \ldots, |P_{24}P_{34}| = \sqrt{30}/5. \)
4.3. A Line Ordering

One can order the \( k \geq 3 \) lines in any set defined by some \( \Pi_q(x, y) \) of (3.12) by first determining all of the (real and complex) line intersections. For any set of \( k \) non-parallel lines, there will be \( \binom{k}{2} = \frac{k(k-1)}{2} \) such points, \( P_{ij} \). The first two lines in the set, \( L_1(x, y) = 0 \) and \( L_2(x, y) = 0 \), will then be defined (but not ordered) by the lowest real value of \( z_{ij} \equiv f_n(P_{ij}) \). \( L_3(x, y) = 0 \) (and possibly \( L_4(x, y) = 0 \)) will then be defined by the next lowest value, etc. Increasing values of \( f_n(x, y) \) at the intersection points will be used to order the lines. All of the real intersection points will be ordered first. The real values of any complex conjugate \( z_{ij} \) will be ordered next, with the imaginary values used to order the conjugates so that \( z = a - ib < z^* = a + ib \) for \( b > 0 \).

To illustrate this ordering process for the first four (\( = k \)) lines of Example 4.4, we first determine that \( z_{12} = -0.5 \), \( z_{13} = -0.74 - 0.32i = z_{24}^* \), \( z_{14} = -0.5 - 2i = z_{23}^* \), and \( z_{34} = 0.5 \). Therefore, in this particular case, our ordering procedure implies that

\[
\begin{align*}
z_{12} < z_{34} < z_{13} = z_{24}^* < z_{24} < z_{14} = z_{23}^* < z_{23},
\end{align*}
\]

which verifies the given line order \( \{L_1, L_2, L_3, L_4\} \), because if (say)

\[
z_{12} < z_{34} < z_{24} = z_{13}^* < z_{13} < \cdots,
\]

the appropriate ordering would be \( \{L_2, L_1, L_4, L_3\} \), and if

\[
z_{34} < z_{12} < z_{14} = z_{23}^* < z_{23} < \cdots,
\]

the appropriate ordering would be \( \{L_4, L_3, L_1, L_2\} \).

5. Geometric Transformations

It is of obvious interest to determine when two algebraic curves are equivalent under a variety of mathematical transformations, such as an affine transformation \( A \), which is defined by both a linear transformation \( M \) and a translation \( P \); i.e.,

\[
\begin{bmatrix}
x \\
y \\
\end{bmatrix}
= \begin{bmatrix}
m_1 & m_2 \\
m_3 & m_4 \\
\end{bmatrix}
\begin{bmatrix}
\tilde{x} \\
\tilde{y} \\
\end{bmatrix}
+ \begin{bmatrix}
p_x \\
p_y \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
m_1 & m_2 & p_x \\
m_3 & m_4 & p_y \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\tilde{x} \\
\tilde{y} \\
1 \\
\end{bmatrix}
\]

\begin{equation}
(5.1)
\end{equation}
The mathematical relationship defined by (5.1), abbreviated as $X \xrightarrow{A} A \hat{X}$, will be called a Euclidean transformation if $M$ is an orthogonal (rotation) matrix, so that $M^T M = M M^T = I$. Clearly, a Euclidean transformation, which has three independent parameters, is a special case of an affine transformation, which has six independent parameters.

### 5.1. Affine Equivalence and Related Points

In general, any two (nth degree) curves defined by a monic $f_n(x, y) = 0$ and a monic $\tilde{f}_n(\tilde{x}, \tilde{y}) = 0$ will be affine equivalent if for some scalar $s_n$,

$$f_n(x, y) = 0 \xrightarrow{A} f_n(m_1 \tilde{x} + m_2 \tilde{y} + p_x, m_3 \tilde{x} + m_4 \tilde{y} + p_y) \overset{\text{def}}{=} s_n \tilde{f}_n(\tilde{x}, \tilde{y}) = 0.$$  

(5.2)

Two corresponding related-points of the affine equivalent curves defined by $f_n(x, y) = 0$ and $\tilde{f}_n(\tilde{x}, \tilde{y}) = 0$, such as \{xᵢ, yᵢ\} and \{\tilde{xᵢ}, \tilde{yᵢ}\}, respectively, will be defined by the condition that

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 & p_x \\ m_3 & m_4 & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \\ 1 \end{bmatrix} \implies \{x_i, y_i\} \xrightarrow{A} \{\tilde{x}_i, \tilde{y}_i\}. \quad (5.3)$$

In light of (5.2), any two corresponding related-points will satisfy the relation

$$f_n(x_i, y_i) \overset{\text{def}}{=} z_i = s_n \tilde{f}_n(\tilde{x}_i, \tilde{y}_i) = s_n \tilde{z}_i. \quad (5.4)$$

In the case of affine transformations, bitangent points, inflection points, centroids, critical-points [17], and line factor intersections all represent related-points which can be determined from knowledge of the curves. It might be noted that bitangent points and centroids are quite difficult to determine from knowledge of an IP equation. Inflection points, which correspond to the intersections of a curve with its Hessian, can be determined accurately, but with some computational effort. However, line factor intersections can be determined rather easily and precisely from an IP equation.

### 5.2. The Correct Correspondence

To establish the correct correspondence between the points in two sets of $k$ corresponding real, distinct related-points, such as \{xᵢ, yᵢ\} and \{\tilde{xᵢ}, \tilde{yᵢ}\},
we next note that if \( f_n(x_i, y_i) \) \( \overset{\text{def}}{=} z_i \) and \( \tilde{f}_n(\tilde{x}_i, \tilde{y}_i) \) \( \overset{\text{def}}{=} \tilde{z}_i \), as in (5.4), then 
\[ z_i = s_n \tilde{z}_i, \]
and
\[ s_n = \frac{z_i}{\tilde{z}_i} = \frac{\sum_{i=1}^{k} z_i}{\sum_{i=1}^{k} \tilde{z}_i}. \] (5.5)

Therefore, we will always ”order” the related-points so that \( z_1 < z_2 < \cdots < z_k \), and
\[ \tilde{z}_1 < \tilde{z}_2 < \cdots < \tilde{z}_k \text{ if } s_n > 0, \quad \text{and} \quad \tilde{z}_1 > \tilde{z}_2 > \cdots > \tilde{z}_k \text{ if } s_n < 0 \]
(5.6)

5.3. Line Factor and Conic Factor Transformations

Under an affine transformation \( A \), every \( \Pi_q(x, y) \) in (3.12), namely
\[ 
\Pi_q(x, y) = \prod_{i=1}^{q} L_{qi}^T X \overset{A}{\mapsto} \prod_{i=1}^{q} L_{qi}^T A X 
\]
\[ = \prod_{i=1}^{q} (m_1 + l_q m_2) \tilde{L}_{qi}^T \tilde{X} = s_q \prod_{i=1}^{q} \tilde{L}_{qi}^T \tilde{X} \overset{\text{def}}{=} s_q \Pi_q(\tilde{x}, \tilde{y}) \] (5.7)
for a real scalar \( s_q = \prod_{i=1}^{q} l_q \), and \( q \) monic line factors \( \tilde{L}_{qi}^T \tilde{X} \), for \( q = n, n-2, n-4, \ldots \), which will imply real conic factors when they appear in complex conjugate pairs. Therefore, the implicit polynomial defined by (3.12), namely
\[ f_n(x, y) \overset{A}{\mapsto} s_n \tilde{f}_n(\tilde{x}, \tilde{y}) = s_n \tilde{\Pi}_n(\tilde{x}, \tilde{y}) + \gamma_{n-2} s_n \tilde{\Pi}_{n-2}(\tilde{x}, \tilde{y}) + \gamma_{n-4} s_n \tilde{\Pi}_{n-4}(\tilde{x}, \tilde{y}) + \cdots, \] (5.8)
which defines a unique monic polynomial that is affine equivalent to \( f_n(x, y) \),
\[
\check{f}_n(\check{x}, \check{y}) = \check{f}_n(\check{x}, \check{y}) + \frac{\gamma_{n-2}}{s_{n-2}}[\check{f}_{n-2}(\check{x}, \check{y}) + \frac{\gamma_{n-4}}{s_{n-4}}[\check{f}_{n-4}(\check{x}, \check{y}) + \cdots]].
\]

(5.9)

Each \(\Pi_q(x, y)\) of \(f_n(x, y)\), and each corresponding \(\check{\Pi}_q(\check{x}, \check{y})\) of an affine equivalent \(\check{f}_n(\check{x}, \check{y})\), clearly will have the same number of conic factors and line factors. Moreover, in light of (5.7), all of these factors will map to one another under affine transformations. Therefore, \(f_n(x, y)\) and \(\check{f}_n(\check{x}, \check{y})\) will have the same number of corresponding related-points, as defined by the centers of their corresponding conic factors and/or all possible intersections of their corresponding line factors.

The related-points defined by each \(\Pi_q(x, y)\) will map to the same number of corresponding related-points defined by each affine equivalent \(\check{\Pi}_q(\check{x}, \check{y})\), and we generally will employ some number of these related-points to determine whether or not two known IP curves are affine equivalent. This number need not equal the total number of corresponding related-points defined by decompositions of \(f_n(x, y)\) and \(\check{f}_n(\check{x}, \check{y})\). For example, in light of (5.3), any three corresponding related-points of two affine equivalent curves will directly and uniquely define \(A\) via the relation

\[
\begin{bmatrix}
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
1 & 1 & 1 \\
\end{bmatrix} \underbrace{=}_{\text{def}} \begin{bmatrix}
\check{x}_1 & \check{x}_2 & \check{x}_3 \\
\check{y}_1 & \check{y}_2 & \check{y}_3 \\
1 & 1 & 1 \\
\end{bmatrix} \underbrace{=}_{\text{def}} \begin{bmatrix}
T \\
\check{T} \\
T \\
\end{bmatrix} \implies A = T \check{T}^{-1}.
\]

6. FUNCTIONALLY INDEPENDENT INVARIANTS

6.1. Counting Argument

If there is a configuration space \(S\), on which a group \(G\) acts, the number \(I\) of functionally independent primitive scalar invariants is \(I \geq \dim(S) - \dim(G)\). The details of this development are given in [12, Chap. 1].

In general \(\dim(O) = \dim(G)\), where \(\dim(O)\) denotes the dimension of the orbit, in which case \(I = \dim(S) - \dim(O)\). Equality will hold in our subsequent discussions. For our purposes, \(\dim(S)\) is the DOF and \(\dim(O)\) is the number of independent transformation parameters. Using this counting argument we can determine the minimum number of functionally independent scalar invariants for plane algebraic curves under a variety of transformation groups.
As mentioned earlier, a monic algebraic curve of degree $n$ has $n(n + 3)/2$ DOF and therefore will have at least $n(n + 3)/2 - 3$ functionally independent invariants under the Euclidean group, $n(n + 3)/2 - 6$ functionally independent invariants under the affine group, and $n(n + 3)/2 - 8$ functionally independent invariants under the projective group, respectively. Note that the numbers 3, 6, and 8 refer to the number of independent parameters in each transformation group. For example, a conic has 5 DOF. Therefore, it has at least 2 Euclidean invariants, but no affine or projective invariants. A quartic curve has 14 DOF. Therefore, it has at least 11 Euclidean invariants, 8 affine invariants, and 6 projective invariants. A key question is: \textit{How can one obtain a functionally independent set of invariants?}

We will now give several geometric constructions related to our curve decomposition and answer this question for the Euclidean and affine cases. We will denote the number of functionally independent Euclidean invariants by $I_E$ and the corresponding number of affine invariants by $I_A$.

Clearly, first degree monic curves (lines), $f_1(x, y) = x + \alpha y + \beta = 0$, will have no Euclidean or affine invariants since they have only two DOF. Therefore, we first consider conics.

6.2. \textit{Conics} ($n = 2$)

A general (monic) conic, which has 5 DOF, can be represented as

$$f_2(x, y) = x^2 + \alpha xy + \beta y^2 + \gamma x + \varepsilon y + \delta = 0$$

A non-degenerate conic can be brought into its canonical frame using an appropriate rotation and translation, in which case its \textit{canonical equation} is [14]

$$x^2/a^2 \pm y^2/b^2 = 1,$$

with the half-axes lengths $a$ and $b$ invariant under any Euclidean transformation. Hence, any non-degenerate conic with five DOF has two Euclidean invariants. Under affine transformations these lengths will not be preserved and, therefore, conics have no affine invariants. Note that this result is consistent with the counting argument, since

$$I_E = \text{DOF} - \dim(G) = 5 - 3 = 2 \quad \text{and} \quad I_A = \text{DOF} - \dim(G) = 5 - 6 < 0$$

Alternatively, in light of Theorem 3.11, any conic can be written as

$$f_2(x, y) = L_1(x, y)L_2(x, y) + \alpha,$$

where $L_i(x, y) = x + l_iy + k_i$ and $\alpha$ is a constant. Clearly, we have $2 + 2 + 1 = 5$ DOF. Since the lines can be real or complex (conjugates), we have two different cases to analyze:
6.2.1. The Lines are Real

The scalar $\alpha$ and the angle $\theta$ between these two lines, as depicted in Fig. 2, will be invariant under any Euclidean transformation. The angle $\theta$ between any two real lines $L_1(x, y) = A_1x + B_1y + C_1 = 0$ and $L_2(x, y) = A_2x + B_2y + C_2 = 0$ is defined by the relation

$$\tan \theta = \frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}, \quad (6.1)$$

which follows from a well known formula in analytic geometry [14].

We will now present an alternative relation to (6.1) using a more powerful vector analysis approach that also can be extended to the complex case. We first observe that the angle between two lines (or two planes) is equal to the angle between their gradient vectors, since a gradient vector is perpendicular to a curve or surface. In the case of lines $L_1(x, y) = 0$ and $L_2(x, y) = 0$, $\nabla L_1 = [A_1, B_1]^T$ and $\nabla L_2 = [A_2, B_2]^T$. Using the inner product, it then follows that

$$\langle \nabla L_1, \nabla L_2 \rangle = \|\nabla L_1\|\|\nabla L_2\| \cos \theta \Rightarrow \cos \theta = \frac{\nabla L_1^T\nabla L_2}{\|\nabla L_1\|\|\nabla L_2\|}, \quad (6.2)$$

where

$$\langle \nabla L_1, \nabla L_2 \rangle = \nabla L_1^T\nabla L_2 = A_1A_2 + B_1B_2,$$

$$\|\nabla L_1\| = \sqrt{A_1^2 + B_1^2} \quad \text{and} \quad \|\nabla L_2\| = \sqrt{A_2^2 + B_2^2}.$$
It might be noted that
\[
\cos^2 \theta = \frac{(A_1A_2 + B_1B_2)^2}{(A_1^2 + B_1^2)(A_2^2 + B_2^2)},
\]
and using the trigonometric identity \(\tan^2 \theta = 1/\cos^2 \theta - 1\), we verify (6.1).

6.2.2. The Lines are Complex Conjugates

In this case, the intersection of the complex conjugate lines is a real point, which corresponds to the center of the degenerate conic defined by the product of these lines. By replacing the transpose \(T\) by the conjugate transpose \(\dagger\), (6.2) will define (a generally complex) angle \(\theta\) between any two arbitrary complex lines \(L_1(x, y) = 0\) and \(L_2(x, y) = 0\). In particular, since the (complex) inner product \(\langle \nabla L_1, \nabla L_2 \rangle \overset{def}{=} \nabla L_1^\dagger \nabla L_2\), which implies that \(\langle \nabla L_1, \nabla L_i \rangle = \nabla L_1^\dagger \nabla L_i = \|\nabla L_i\|^2\), it follows that
\[
\cos \theta = \frac{\nabla L_1^\dagger \nabla L_2}{\|\nabla L_1\|\|\nabla L_2\|}. \tag{6.3}
\]
A real angle \(\theta\) can be defined by using the modulus of (6.3). For example, one can verify that \(|\cos \theta| = 1\) for parallel complex lines, so that the angle between such lines is 0°.

7. THE MAIN RESULT

7.1. Some Classical Geometric Results

Consider an algebraic curve \(f_n(x, y) = 0\) and a line \(L_i(x, y) = 0\) which intersects the curve at \(n\) distinct points, \(I_{1i}, I_{2i}, \ldots, I_{ni}\), as depicted in Fig. 3. If points \(P_{ij}\) and \(P_{ik}\) on the line satisfy the relations \(f_n(P_{ij}) \neq 0\) and \(f_n(P_{ik}) \neq 0\), it follows that [1]¹
\[
\frac{|f_n(P_{ij})|}{|f_n(P_{ik})|} = \frac{\Pi_{q=1}^n |I_{iq} - P_{ij}|}{\Pi_{q=1}^n |I_{iq} - P_{ik}|} = \frac{\Pi_{q=1}^n |I_{iq}P_{ij}|}{\Pi_{q=1}^n |I_{iq}P_{ik}|}. \tag{7.4}
\]
Therefore, for any ordered sequence of lines, \(L_1(x, y) = 0, L_2(x, y) = 0,\ldots, L_r(x, y) = 0\),
\[
\frac{|f_n(P_{1r})|}{|f_n(P_{12})|} \cdots \frac{|f_n(P_{r,r-1})|}{|f_n(P_{r1})|} = \frac{\Pi_{q=1}^n |I_{1r}P_{1r}|}{\Pi_{q=1}^n |I_{1r}P_{12}|} \cdots \frac{\Pi_{q=1}^n |I_{r1}P_{r-1}|}{\Pi_{q=1}^n |I_{r1}P_{r1}|} = 1. \tag{7.5}
\]

¹The author of [1] attributes (7.4) and (7.5) to Hilton [11] and (quite possibly) the ideas of Newton.
We next note that if \( r = n \) in (7.5), and if the algebraic curve \( f_n(x, y) = 0 \) is replaced by an additional line \( L_{n+1}(x, y) = 0 \), so that \( I_{iq} = P_{i,n+1} \) for \( i = 1, 2, \ldots, n \), then (7.5) will imply Menelaus’ theorem for “n-gons” defined by \( (n) \) ordered non-parallel line intersections \( \{P_{2n}, P_{12}, P_{23}, \ldots, P_{n-1,n}\} \), namely

\[
\frac{|P_{1n}P_{1,n+1}|}{|P_{1,n+1}P_{12}|} \cdot \frac{|P_{12}P_{2,n+1}|}{|P_{2,n+1}P_{23}|} \cdot \ldots \cdot \frac{|P_{n-1,n}P_{n,n+1}|}{|P_{n,n+1}P_{1n}|} = 1.
\]  

(7.6)

Figure 4 illustrates this theorem when \( n = 3 \) and

\[
\frac{|P_{13}P_{14}|}{|P_{14}P_{12}|} \cdot \frac{|P_{12}P_{24}|}{|P_{24}P_{23}|} \cdot \frac{|P_{23}P_{34}|}{|P_{34}P_{13}|} = 1.
\]  

(7.7)

### 7.2. Complete Ordered Invariant Sets

Since all line intersections map to one another under affine transformations, it follows [4] that all of the line segments (distances) defined in (7.4), (7.5), and (7.6) are Euclidean invariants and all of the distance ratios are affine invariants. When comparing two different curves, \( f_n(x, y) = 0 \) and \( f_n(x, y) = 0 \), which may be equivalent, corresponding sets of lines and their intersections should first be determined for both curves and ordered as in Section 4.3, with a reverse ordering applied to \( f_n(x, y) = 0 \) whenever \( s_n = \sum z_{ij} - \sum \tilde{z}_{ij} < 0 \), as in (4.6). In this way, the invariants also can be ordered. Although (7.4), (7.5), and (7.6) imply many different sets of invariants, we now focus on the determination of complete independent sets of invariants.
7.3. Cubics

In the special case of cubics, which have nine DOF, the counting argument will imply at least \(9 - 3 = 6\) independent Euclidean invariants and \(9 - 6 = 3\) affine invariants. Our sum of line products for cubics is defined by

\[ f_3(x, y) = L_1(x, y)L_2(x, y)L_3(x, y) + \alpha L_4(x, y) = 0, \]

where \(L_4(x, y) = 0\), and at least one of the first three lines, will be real. Whether the other two lines are real or complex conjugates is irrelevant, however, because (in either case) if \(L_4(x, y) = 0\) is chosen as the transversal that intersects the base triangle defined by the first three ordered lines, (7.7) will imply five independent distance and two independent affine invariants. The scalar \(\alpha\) will define an additional Euclidean and affine invariant, thus implying the six Euclidean and three affine invariants noted above.

7.4. Quartics

In the case of quartic curves, which have 14 DOF, our line product decomposition will imply that

\[ f_4(x, y) = L_1(x, y)L_2(x, y)L_3(x, y)L_4(x, y) + \alpha L_5(x, y)L_6(x, y) + \beta = 0, \]

with \(\alpha\) and \(\beta\) representing two independent scalar (Euclidean and affine) invariants. In this case, the counting argument will imply at least \(14 - 3 - 2 = 9\) additional, independent Euclidean invariants and \(14 - 6 - 2 = 6\)
additional affine invariants. If \( L_4(x, y) = 0 \), \( L_5(x, y) = 0 \) and \( L_6(x, y) = 0 \) are defined as transversals of the triangle defined by the first three lines, the first of these transversals, \( L_4(x, y) = 0 \), would imply five Euclidean distance and two affine ratio invariants relative to the base triangle. The other two lines, \( L_5(x, y) = 0 \) and \( L_6(x, y) = 0 \), would each define two more independent Euclidean distance invariants, relative to the base triangle, and two independent ratio invariants, hence a total of nine Euclidean and six affine invariants, as noted above.

To illustrate this observation, one can verify that if \( L_4(x, y) = 0 \) is defined as a transversal that cuts the first three lines in Example 4.5 at \( P_{14}, P_{24}, \) and \( P_{34} \), analogous to the situation depicted in Fig. 4, then

\[
\begin{vmatrix}
|P_{13}P_{14}| & |P_{12}P_{24}| & |P_{23}P_{34}|
|P_{14}P_{12}| & |P_{24}P_{25}| & |P_{34}P_{13}|
\end{vmatrix}
= \frac{4\sqrt{5}/5}{\sqrt{2}} \frac{\sqrt{10}/5}{4\sqrt{5}/5} \frac{\sqrt{6}}{\sqrt{30}/5} = (1.265)(0.354)(2.236) = 1. 
\tag{7.8}
\]

In this case, the five (ordered) distances, \( 4\sqrt{5}/5, \sqrt{2}, \sqrt{10}/5, \sqrt{6}, \) and \( \sqrt{30}/5 \) will be independent Euclidean invariants, and the 2 (ordered) ratios \( 4\sqrt{5}/5/\sqrt{2} = 1.265 \) and \( \sqrt{10}/4\sqrt{5} = 0.354 \) will be independent affine invariants. Each of the lines \( L_5(x, y) = 0 \) and \( L_6(x, y) = 0 \) will introduce two additional independent Euclidean invariants, such as \( |P_{14}P_{12}| \) and \( |P_{12}P_{24}| \), and two independent affine invariants, such as \( |P_{13}P_{14}|/|P_{14}P_{12}| \) and \( |P_{12}P_{2k}|/|P_{2k}P_{23}| \), for \( k = 5 \) and 6.

Example 7.9. To illustrate the preceding, consider the curve depicted in Fig. 1 which undergoes an affine transformation defined by

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 2 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]

The resulting (affine equivalent) curve is defined by

\[
s_n f_n(\bar{x}, \bar{y}) = f_\delta(\bar{x}, \bar{y}) = \bar{x}^4 + 2\bar{x}^3\bar{y} + 6\bar{x}^2\bar{y}^2 + 2\bar{x}\bar{y}^3 + 5\bar{y}^4 - 2\bar{x}^3 - 10\bar{x}\bar{y}^2 + 4\bar{y}^3
\]

\[+ 2.5\bar{x}^2 - 9\bar{x}\bar{y} + 8.5\bar{y}^2 - 4\bar{x} + 10\bar{y} + 3 = 0 \implies s_n = 1,
\]

in light of (5.8), and is depicted in Fig. 5.

Equations (3.13) and (4.3) next imply that this particular curve has a real conic line decomposition defined by

\[
\tilde{f}_\delta(\bar{x}, \bar{y}) = \tilde{C}_{41}(\bar{x}, \bar{y})\tilde{C}_{42}(\bar{x}, \bar{y}) + \tilde{a}\tilde{L}_5(\bar{x}, \bar{y})\tilde{L}_6(\bar{x}, \bar{y}),
\]
with \( C_{41}(\tilde{x}, \tilde{y}) = L_1(\tilde{x}, \tilde{y})L_2(\tilde{x}, \tilde{y}) = L_1(\tilde{x}, \tilde{y})L_3(\tilde{x}, \tilde{y}) = [\tilde{x} + (1 - 2i)\tilde{y} - i][\tilde{x} + (1 + 2i)\tilde{y} + i] \), \( C_{42}(\tilde{x}, \tilde{y}) = L_3(\tilde{x}, \tilde{y})L_4(\tilde{x}, \tilde{y}) = L_3(\tilde{x}, \tilde{y})L_2(\tilde{x}, \tilde{y})^* = [\tilde{x} + i\tilde{y} - 1][\tilde{x} - i\tilde{y} - 1] \), and a degenerate \( f_2(\tilde{x}, \tilde{y}) = aL_3(\tilde{x}, \tilde{y})L_6(\tilde{x}, \tilde{y}) = 0.5[\tilde{x} - 5\tilde{y} - 2][\tilde{x} - \tilde{y} - 2] \).

The intersection points for the first four lines are determined to be \( P_{12} \approx 0.5, -0.5 \), \( P_{13} \approx 0.8 + 0.4i, -0.4 - 0.2i \), \( P_{24} \approx P_{1}, P_{14} \approx 1 - i, -1 \approx P_{13} \), and \( P_{24} \approx 1, 0 \), with (since \( s_n = 1 \) in this case) \( z_{12} = z_{12} = -0.5, z_{13} = z_{13} = -0.74 - 0.32i = z_{24}, z_{14} = z_{14} = -0.5 - 2i = z_{23} \) and \( z_{34} = z_{34} = 0.5 \).

If \( L_4(\tilde{x}, \tilde{y}) \) is now defined as a transversal that cuts the first three lines, as in our earlier example, it follows that

\[
\frac{|P_{13}P_{14}|}{|P_{14}P_{12}|} \cdot \frac{|P_{23}P_{34}|}{|P_{24}P_{23}|} = \frac{\sqrt{2} \cdot \sqrt{3}}{\sqrt{1} \cdot \sqrt{4}} = \frac{\sqrt{2}}{\sqrt{2}} = 1
\]

the same (ordered) ratio invariants as in (7.8).

### 7.5. Higher Degree Curves

By using the first three lines in our line decomposition of \( f_n(x, y) = 0 \) to define a base triangle that is subsequently intersected by all of the remaining lines, we can obtain complete sets of Euclidean and affine invariants. For example, consider the sextic case when

\[
f_6 = L_1L_2L_3L_4L_5L_6 + \alpha L_7L_8L_9L_{10} + \beta L_{11}L_{12} + \gamma
\]
The first transversal $L_4$ will imply 5 Euclidean distance and 2 affine ratio invariants relative to our base triangle. Each of the remaining (8) lines will imply 2 additional Euclidean and affine invariants which, together with the (3) scalars, will define a total of 24 Euclidean and 21 affine invariants, as predicted by the counting argument.

7.6. Angular Invariants

We finally remark that (6.3) can be used to define the angles between any two lines defined by a decomposition of an algebraic curve. An ordered set of such angles would define another set of geometric Euclidean invariants in addition to the distance invariants defined by (7.6). In light of this observation, and the preceding results, it should be clear that it is now relatively straightforward to define complete sets of geometric invariants for algebraic curves of virtually any degree $n$. Moreover, any such set(s) of invariants could be ordered appropriately for classification and comparison purposes.

8. CONCLUDING REMARKS

We have now presented a constructive procedure for obtaining a complete, functionally independent set of geometric invariants using a new algebraic curve decomposition. Our particular treatment provides significant new insight about invariants. Unlike the more classical algebraic invariants [6, 9, 12, 13], most of the invariants obtained here have a natural interpretation in terms of lengths (distances) and angles. Therefore, one can apply several different measures when comparing invariants for equivalent curves.

REFERENCES