Nonlinear Distributional Geometry

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February 2001

Abstract

Colombeau's construction of generalized functions (in its special variant) is extended to a theory of generalized sections of vector bundles. As particular cases, generalized tensor analysis and exterior algebra are studied. A pointvalue characterization for generalized functions on manifolds is derived, several algebraic characterizations of spaces of generalized sections are established and consistency properties with respect to linear distributional geometry are derived. An application to nonsmooth mechanics indicates the additional flexibility offered by this approach compared to the purely distributional picture.

Mathematics Subject Classification (2000): Primary: 46F30; secondary: 46T30, 46F10, 37J05.
Keywords: Algebras of generalized functions, Colombeau algebras, generalized sections of vector bundles, distributional geometry.

1 Introduction

After their introduction in [6, 7] the main applications of Colombeau's new generalized functions lay in the field of linear and nonlinear partial differential equations involving singular coefficients or data (cf. [27], [13] and the literature cited therein for a survey). Over the past few years, however, the theory has found a growing number of applications in a more geometric context, most notably in general relativity (cf. e.g., [5], [30], [3],[24], as well as [31] for a survey). This shift of focus has necessitated a certain restructuring of the fundamental building blocks of the theory in order to adapt to the additional requirement of diffeomorphism invariance. Only recently ([12, 14]) this task has been completed

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for the scalar case. To be precise, this restructuring took place in the framework of the so-called *full* Colombeau algebra, distinguished by the existence of a canonical embedding of the space of Schwartz distributions into the algebra.

Already at a very early stage of development, the so-called special (or simplified) variant of Colombeau's algebras was introduced (cf. e.g. [2]) which, while no longer allowing for a canonical embedding of the space of distributions, due to its simpler basic structure (elements are basically equivalence classes of nets of smooth functions), allows for a particularly elegant and flexible way of modeling singularities in a nonlinear context. In particular, diffeomorphism invariance of the basic building blocks of the construction is automatically satisfied. These structural advantages of the special algebra have led to an increasing number of applications to geometric problems (cf. e.g. [10], [8], [23], [24]). The aim of the present paper is to initiate a systematic development of global analysis in the special version of Colombeau's construction.

In the remainder of this introduction we fix some notation concerning special Colombeau algebras and differential geometry. Section 2 gives a quick overview of distributional geometry, introducing those constructions that later on will furnish our main objects of reference for the limiting behavior of the corresponding Colombeau objects. In section 3 we introduce several equivalent definitions of as well as some basic operations on the special algebra $\mathcal{G}(X)$ on a manifold X. We then derive a pointvalue characterization of elements of $\mathcal{G}(X)$. Generalized sections of vector bundles as well as some algebraic characterizations of the resulting spaces are introduced in section 5. Important special cases of these general constructions are worked out in sections 6 and 7. In particular, section 7 provides an application of these concepts to nonsmooth mechanics.

Denoting by Ω an open subset of $\mathbb{R}^n$ we set (with $I = (0,1]$)

$$\begin{aligned}
\mathcal{E}(\Omega) &:= (C^\infty(\Omega))^I \\
\mathcal{E}_M(\Omega) &:= \{(u_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n \; \exists N \in \mathbb{N} \\
&\qquad \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\} \\
\mathcal{N}(\Omega) &:= \{(u_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \forall m \in \mathbb{N} : \\
&\qquad \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}.
\end{aligned}$$

The (special) Colombeau algebra on Ω is defined as the quotient space

$$\mathcal{G}(\Omega) := \mathcal{E}_M(\Omega) \, / \, \mathcal{N}(\Omega) \, .$$

Since we will only be considering this type of algebras we will omit the term "special" henceforth.

Elements of $\mathcal{G}(\Omega)$ will be denoted by capital letters, representatives by small letters, i.e., $\mathcal{G}(\Omega) \ni U = \mathrm{cl}[(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}(\Omega)$. $\mathcal{G}(\_)$ is a fine sheaf of differential algebras containing the smooth functions on Ω as a faithful subalgebra embedded simply by $\sigma(f) = \mathrm{cl}[(f)_\varepsilon]$. To embed non-smooth distributions we first have to fix a mollifier $\rho \in \mathcal{S}(\mathbb{R}^n)$ with unit integral satisfying the moment conditions $\int \rho(x)\, x^\alpha \, dx \; = \; 0 \, \forall |\alpha| \geq 1$. Setting $\rho_\varepsilon(x) = (1/\varepsilon)\rho(x/\varepsilon)$, compactly supported distributions are embedded by $\iota_0(w) = ((w * \rho_\varepsilon|_\Omega)_\varepsilon + \mathcal{N}(\Omega)$. Using partitions of unity and suitable cut-off functions one may explicitly construct an embedding $\iota : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega)$ which naturally induces a unique sheaf morphism (of complex vector spaces) $\hat{\iota} : \mathcal{D}' \to \mathcal{G}$ extending ι_0, commuting with partial derivatives and its restriction to $\mathcal{C}^\infty$ being a sheaf morphism of algebras. Note that $\hat{\iota}$ depends on the choice of the mollifier ρ, a fact which reflects a fundamental property of nonlinear modeling: In general, nonlinear properties of a singular object depend on the regularization. This opens a gate to a wide range of applications. Additional input on the regularization from, say, a physical model may enter the mathematical theory via this interface, leading to a sensible description of the problem at hand. A "macroscopic" description of calculations in $\mathcal{G}$ can often be effected through the concept of association: U, $V \in \mathcal{G}(\Omega)$ are called associated if $u_\varepsilon - v_\varepsilon \to 0$ in $\mathcal{D}'(\Omega)$, U is called associated to $w \in \mathcal{D}'(\Omega)$ if $u_\varepsilon \to w$ in $\mathcal{D}'(\Omega)$. Clearly these notions do not depend on the particular representatives and the first one gives rise to a linear quotient space of $\mathcal{G}(\Omega)$. Finally, we note that inserting $x \in \Omega$ into $U \in \mathcal{G}(\Omega)$ componentwise yields a well defined element of the ring of generalized numbers $\mathcal{K}$ (corresponding to $\mathbb{K} = \mathbb{R}$ resp. $\mathbb{C}$), defined as the set of moderate nets of numbers ($(r_\varepsilon)_\varepsilon \in \mathbb{K}^I$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some N) modulo negligible nets ($|r_\varepsilon| = O(\varepsilon^m)$ for each m).

Notations from differential geometry will basically be chosen in accordance with [1], [20]. Throughout this paper, X will denote a separable, smooth Hausdorff manifold of dimension n. For any vector bundle $E \to X$, by $\Gamma^k(X, E)$ (resp. $\Gamma^k_c(X, E)$) ($0 \le k \le \infty$) we denote the $\mathcal{C}^k(X)$-module of (compactly supported) $\mathcal{C}^k$-sections in E and frequently drop the superscript if $k = \infty$. In particular, by $\mathfrak{X}(X)$ resp. $\Omega^k(X)$ we denote the space of smooth vector fields resp. k-forms on X. Generally, for $M_1, \ldots, M_k$, M_0 modules over a commutative ring R, $L_R(M_1, \ldots, M_k; M_0)$ denotes the R-module of R-k-linear maps from $M_1 \times \ldots \times M_k$ into M_0. Since we will be considering tensor products with respect to different rings R, the notation $M_1 \otimes_R M_2$ will be used. By $\mathcal{P}(X, E)$ we denote the space of linear differential operators $\Gamma(X, E) \to \Gamma(X, E)$. For $E = X \times \mathbb{R}$ we write $\mathcal{P}(X)$ for $\mathcal{P}(X, E)$.

2 Distributional geometry

We shortly recall the basic facts of distributional geometry, i.e., of the theory of distribution valued sections of vector bundles. Writing $\mathrm{Vol}^q(X)$ for the q-volume bundle of X the space of $\mathcal{C}^k$-q-densities is denoted by $\Gamma^k(X, \mathrm{Vol}^q(X))$. The space $\mathcal{D}'^{(k)}(X, E \otimes \mathrm{Vol}^q(X))$ of E-valued distributions of order k and density character q is defined as the dual of the (LF)-space $\Gamma^k_c(X, E^* \otimes \mathrm{Vol}^{1-q})$, where E^* denotes the vector bundle dual to E, i.e.,

$$\mathcal{D}'^{(k)}(X, E \otimes \mathrm{Vol}^q(X)) := [\Gamma^k_c(X, E^* \otimes \mathrm{Vol}^{1-q}(X))]'. \qquad (1)$$

Locally integrable sections of $E \otimes \mathrm{Vol}^q$ may be naturally embedded into $D'^{(k)}(X, E \otimes \mathrm{Vol}^q(X))$ hence serve as regular objects in the respective distributional spaces. Analogous to the theory on open sets of Euclidean space the space of smooth regular objects, i.e., $\Gamma^\infty(X, E \otimes \mathrm{Vol}^q(X))$ is sequentially dense in $D'^{(k)}(X, E \otimes \mathrm{Vol}^q(X))$.

We explicitly mention the following special cases of (1): for $E = X \times \mathbb{C}$, $k = \infty$, $q = 0$ resp. $q = 1$ we obtain $\mathcal{D}'(X)$ resp. $\mathcal{D}'_d(X)$, the space of distributions resp. distributional densities on X. Similarly, taking E the tensor bundle $T^r_s(X)$, $k = \infty$ and $q = 0$ resp. $q = 1$ gives the spaces $\mathcal{D}'^r_s(X)$ of tensor distributions resp. $\mathcal{D}'^r_{ds}(X)$ of tensor distribution densities.

E-valued distributions of density character q may be written as classical sections of E with distributional coefficient "functions", more precisely

$$\mathcal{D}'(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X, E \otimes \mathrm{Vol}^q(X)) \cong \mathcal{D}'(X, E \otimes \mathrm{Vol}^q(X)).$$

For X an oriented manifold whose orientation is induced by a fixed nowhere vanishing $\theta \in \Omega^n(X)$, a rich theory of distributional geometry was introduced by Marsden in [26]. The basic idea underlying his approach is that of continuous extension of classical operations to spaces of currents: Since X is oriented we may identify one-densities and smooth n-forms and we set

$$\Omega^k(X)' := \mathcal{D}'(X, \Lambda^k T^*X \otimes \mathrm{Vol}(X))$$

Using the above identification it follows that $\Omega^k(X)'$ is the dual of $\Omega^{n-k}_c(X)$, the space of compactly supported $n-k$-forms (and *not* the dual of $\Omega^k(X)$ as might be suggested by this notation). Also, $\mathcal{D}'(X) \cong \Omega^0(X)' \cong \mathcal{D}'_d(X)$ and $\Omega^k(X)'$ is precisely the space of odd k-currents on X in the sense of de Rham ([9]). Marsden calls elements of $\Omega^k(X)'$ *generalized k-forms* but we prefer here the term *distributional k-forms* since the term "generalized" will be reserved for Colombeau objects in this work. Embedding of regular objects into distributional k-forms is effected by the map

$$\begin{array}{rcl} j : \Omega^k(X) & \to & \Omega^k(X)' \\ j(\omega)(\tau) & = & \int \omega \wedge \tau \end{array} \tag{2}$$

It then follows that $\Omega^k(X)'$ is the weak sequential closure of $j(\Omega^k(X))$ (in fact, Marsden *defines* $\Omega^k(X)'$ as this closure). Let us exemplify the method of continuously extending classical operations from smooth to distributional forms by considering the Lie derivative with respect to a smooth vector field ξ. By Stokes' theorem, for $\omega \in \Omega^k(X)$, $\tau \in \Omega^{n-k}_c(X)$ we have $j(L_\xi\omega)(\tau) = -\omega(L_\xi\tau)$. Hence setting $L_\xi\omega(\tau) := -\omega(L_\xi\tau)$ for $\omega \in \Omega^k(X)'$ gives the unique continuous extension of L_ξ to $\Omega^k(X)'$. By the same strategy, operations like exterior differentiation d and insertion i_ξ can be extended to distributional forms while preserving classical relations like $L_\xi = i_\xi \circ d + d \circ i_\xi$. Finally, we note that in this setting, $\mathcal{D}'^r_s(X)$ can be identified with the space of $\mathcal{C}^\infty$-multilinear maps $t : \Omega^1(X)^r \times \mathfrak{X}(X)^s \to \mathcal{D}'(X)$.

3 Basic properties, pointvalue characterization

Lemma 1 *Set $\mathcal{E}(X) := (\mathcal{C}^\infty(X))^I$. The following spaces of nets are equal*

(i) $\{(u_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}(X) |\ \forall K \subset\subset X,\ \forall P \in \mathcal{P}(X)\ \exists N \in \mathbb{N} :$
$\sup_{p\in K} |Pu_\varepsilon(p)| = O(\varepsilon^{-N})\}$

(ii) $\{(u_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}(X) |\ \forall K \subset\subset X,\ \forall k \in \mathbb{N}_0\ \exists N \in \mathbb{N}\ \forall \xi_1, \ldots, \xi_k \in \mathfrak{X}(X) :$
$\sup_{p\in K} |L_{\xi_1} \ldots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{-N})\}$

(iii) $\{(u_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}(X) |$ *for each chart* $(V, \psi) :\ (u_\varepsilon \circ \psi^{-1})_\varepsilon \in \mathcal{E}_M(\psi(V))\}$

Proof. Since every iterated Lie derivative is an element of $\mathcal{P}(X)$ we have (i) $\subseteq$ (ii). (ii) $\subseteq$ (iii) is immediate from the local form of $L_{\xi_1} \ldots L_{\xi_k}$. Finally, (iii) $\subseteq$ (i) follows from Peetre's theorem. □

We denote by $\mathcal{E}_M(X)$ the set defined above and call it the space of *moderate* nets on X. Definition (i) was suggested in [10], (iii) is from [2]. (ii) is mentioned explicitly since the operation of taking Lie derivatives plays a central role in the theory (in the full version of the construction, a canonical embedding of $\mathcal{D}'$ commuting with Lie derivatives has been given in [14]). Replacing $\exists N$ by $\forall m$, and ε^{-N} by ε^m in (i) and (ii) as well as $\mathcal{E}_M(\psi(V))$ by $\mathcal{N}(\psi(V))$ in (iii) we obtain equivalent definitions of the space $\mathcal{N}(X)$ of *negligible* nets on X. Applying [12], Th. 13.1 locally, we arrive at the following characterization of $\mathcal{N}(X)$ as a subspace of $\mathcal{E}_M(X)$:

$$\mathcal{N}(X) = \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X) \mid \forall K \subset\subset X\, \forall m \in \mathbb{N} \sup_{x\in K} |u_\varepsilon| = O(\varepsilon^m)\} \tag{3}$$

Thus for elements of $\mathcal{E}_M(X)$ to belong to $\mathcal{N}(X)$ it suffices to require the $\mathcal{N}$-estimates to hold for the function itself, without taking into account any derivatives. The *Colombeau algebra of generalized functions on the manifold* X is defined as the quotient

$$\mathcal{G}(X) := \mathcal{E}_M(X) / \mathcal{N}(X) .$$

Again, elements in $\mathcal{G}(X)$ are denoted by capital letters, i.e., $U = \mathrm{cl}[(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}(X)$. Analogous to the case of open sets in Euclidean space, $\mathcal{E}_M(X)$ is a differential algebra (w.r.t. Lie derivatives) with componentwise operations and $\mathcal{N}(X)$ is a differential ideal in it. Moreover, $\mathcal{E}_M(X)$ and $\mathcal{N}(X)$ are invariant under the action of any $P \in \mathcal{P}(X)$. Thus we obtain

Definition 1 *Let* $U \in \mathcal{G}(X)$ *and* $P \in \mathcal{P}(X)$*. Then*

$$PU := \mathrm{cl}[(Pu_\varepsilon)_\varepsilon]$$

is a well defined element of $\mathcal{G}(X)$

This applies, in particular, to the Lie derivative $L_\xi U$ of U with respect to a smooth vector field $\xi \in \mathfrak{X}(X)$. It follows that $\mathcal{G}(X)$ is a differential $\mathbb{K}$-algebra.

It is now immediate that a generalized function U on X allows for the following local description via the assignment $\mathcal{G}(X) \ni U \mapsto (U_\alpha)_{\alpha\in A}$ with $U_\alpha := U \circ \psi_\alpha^{-1} \in \mathcal{G}(\psi_\alpha(V_\alpha))$ (with $\{(V_\alpha, \psi_\alpha) \mid \alpha \in A\}$ an atlas of X). We call U_α the *local expression* of U with respect to the chart (V_α, ψ_α). Thus we have

Proposition 1 *$\mathcal{G}(X)$ can be identified with the set of all families $(U_\alpha)_\alpha$ of generalized functions $U_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))$ satisfying the following transformation law*

$$U_\alpha|_{\psi_\alpha(V_\alpha\cap V_\beta)} = U_\beta|_{\psi_\beta(V_\alpha\cap V_\beta)} \circ \psi_\beta \circ \psi_\alpha^{-1}$$

for all $\alpha, \beta \in A$ with $V_\alpha \cap V_\beta \neq \emptyset$. □

It follows that $\mathcal{G}(\_)$ is a fine sheaf of $\mathbb{K}$-algebras on X. In fact, in [10], $\mathcal{G}$ is *defined* directly as a quotient sheaf of the sheaves of moderate modulo negligible sections.

An important feature distinguishing Colombeau generalized functions on open subsets Ω of $\mathbb{R}^n$ from spaces of distributions is the availability of a point-value characterization of elements of $\mathcal{G}(\Omega)$ ([28]). This characterization allows a direct generalization of results from classical analysis to Colombeau algebras thereby enabling a consistent treatment of a variety of geometric and analytic problems (see e.g. [17], [23]). Our aim in the remainder of this section is to derive a pointvalue characterization of Colombeau generalized functions also in the global context.

To begin with we shortly recall the basic definitions from [28]. Let $\Omega \subseteq \mathbb{R}^n$ open and set $\Omega_M = \{(x_\varepsilon)_\varepsilon \in \Omega^I \mid \exists N > 0\ \exists \eta > 0\ |x_\varepsilon| \leq \varepsilon^{-N}\ (0 < \varepsilon < \eta)\}$. Two elements $(x_\varepsilon)_\varepsilon$, $(y_\varepsilon)_\varepsilon$ are called equivalent ($(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon$) if $|x_\varepsilon - y_\varepsilon| = O(\varepsilon^m)$ for each $m > 0$. $\widetilde{\Omega}_c$, the set of compactly supported generalized points is then defined as the set of equivalence classes $\tilde{x}$ with respect to $\sim$ such that for one (hence every) representative $(x_\varepsilon)_\varepsilon$ of $\tilde{x}$ there exists a compact set containing x_ε for ε small. Then for any $U \in \mathcal{G}(\Omega)$ and $\tilde{x} \in \widetilde{\Omega}_c$, $U(\tilde{x}) := \mathrm{cl}[(u_\varepsilon(x_\varepsilon))_\varepsilon]$ is a well defined generalized number ([28], Prop. 2.3) and $U = 0$ iff $U(\tilde{x}) = 0$ for each $\tilde{x}$ in $\widetilde{\Omega}_c$ ([28], Th. 2.4).

In order to transfer these notions to the manifold-setting we will make use of an auxiliary Riemannian metric h on X. Of course we will then have to show that the constructions to follow are in fact independent of the chosen h.

We call a net $(p_\varepsilon)_\varepsilon \in X^I$ *compactly supported* if there exist $K \subset\subset X$ and $\eta > 0$ such that $p_\varepsilon \in K$ for $\varepsilon < \eta$. Denoting by d_h the Riemannian distance induced by h on X, two nets $(p_\varepsilon)_\varepsilon$, $(q_\varepsilon)_\varepsilon$ are called equivalent ($(p_\varepsilon)_\varepsilon \sim (q_\varepsilon)_\varepsilon$) if $d_h(p_\varepsilon, q_\varepsilon) = O(\varepsilon^m)$ for each $m > 0$. The equivalence classes with respect to this relation are called *compactly supported generalized points* on X. The set of compactly supported generalized points on X will be denoted by $\widetilde{X}_c$.

The fact that $\widetilde{X}_c$ does not depend on the auxiliary metric h follows immediately from the following lemma:

Lemma 2 *Let h_i be Riemannian metrics inducing the Riemannian distances d_i on X ($i = 1, 2$). Then for $K, K' \subset\subset X$ there exists $C > 0$ such that $d_2(p, q) \leq C d_1(p, q)$ for all $p \in K$, $q \in K'$.*

Proof. Assume to the contrary that there exist sequences p_m in K and q_m in K' such that $d_2(p_m, q_m) > m d_1(p_m, q_m)$. By choosing suitable subsequences we may additionally suppose that both p_m and q_m converge to some p. Let U be a relatively compact neighborhood of p. Then denoting by $B^i_r(q)$ the d_i-ball of radius r around q it follows that there exist $r_0 > 0$ and $\alpha > 0$ such that $B^1_r(q) \subseteq B^2_{\alpha r}(q)$ for all $q \in U$ and all $r < r_0$ (cf. e.g. [14], Lemma 3.4). But then for $m > \alpha$ sufficiently large we arrive at the contradiction $d_2(p_m, q_m) \leq \alpha d_1(p_m, q_m)$. □

Lemma 3 *Suppose that $(p_\varepsilon)_\varepsilon$, $(q_\varepsilon)_\varepsilon \in X^I$ are compactly supported in some chart (U_α, ψ_α) such that U_α is relatively compact and geodesically convex with respect to a Riemannian metric h on X. Then*

$$(p_\varepsilon)_\varepsilon \sim (q_\varepsilon)_\varepsilon \Leftrightarrow |\psi_\alpha(p_\varepsilon) - \psi_\alpha(q_\varepsilon)| = O(\varepsilon^m)\ \forall\, m > 0\,.$$

Proof. ($\Rightarrow$) Let $\gamma_\varepsilon : [\alpha_\varepsilon, \beta_\varepsilon] \to U_\alpha$ be the unique geodesic in U_α joining p_ε and q_ε. Then

$$d_h(p_\varepsilon, q_\varepsilon) = \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \|\dot\gamma_\varepsilon(s)\|_h\, ds = O(\varepsilon^m)\ \forall\, m > 0\,.$$

Since U_α is relatively compact there exists $C > 0$ such that $|\xi| \leq C(\|\psi_\alpha^*\xi\|_h)|_p$ for all $p \in U_\alpha$ and all $\xi \in \mathbb{R}^n$. Thus

$$|\psi_\alpha(p_\varepsilon) - \psi_\alpha(q_\varepsilon)| \leq \int_{\alpha_\varepsilon}^{\beta_\varepsilon} |(\psi_\alpha \circ \gamma_\varepsilon)'(s)|\, ds \leq C \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \|\dot\gamma_\varepsilon(s)\|_h\, ds = O(\varepsilon^m)$$

($\Leftarrow$) Let $K \subset\subset U_\alpha$ such that $p_\varepsilon, q_\varepsilon \in K$ for ε small. Using a cut-off function supported in $\psi_\alpha(U_\alpha)$ and equal to 1 in a neighborhood of $\psi_\alpha(K)$ we may extend the pullback under ψ_α of the Euclidian metric on $\psi_\alpha(U_\alpha)$ to a Riemannian metric g on X. Then $d_g(p_\varepsilon, q_\varepsilon) \leq |\psi_\alpha(p_\varepsilon) - \psi_\alpha(q_\varepsilon)| = O(\varepsilon^m)$, so the claim follows from Lemma 2. □

Proposition 2 *Let $U \in \mathcal{G}(X)$ and $\tilde p \in \widetilde{X}_c$. Then*

$$U(\tilde p) := \mathrm{cl}[(u_\varepsilon(p_\varepsilon))_\varepsilon]$$

is a well defined element of $\mathcal{K}$.

Proof. Since $(p_\varepsilon)_\varepsilon$ is compactly supported it is clear that $(u_\varepsilon(p_\varepsilon))$ is moderate resp. negligible if $(u_\varepsilon)_\varepsilon$ is. Suppose now that $(p_\varepsilon)_\varepsilon \sim (q_\varepsilon)_\varepsilon$ and choose $K \subset\subset X$ such that $p_\varepsilon, q_\varepsilon \in K$ for ε small. We have to show that $(u_\varepsilon(p_\varepsilon) - u_\varepsilon(q_\varepsilon))_\varepsilon \in \mathcal{N}$. To this end we choose some auxiliary Riemannian metric h and cover K by finitely many geodesically convex charts. Then K can be written as the union of finitely many compact sets K_i each contained in a single chart. Also, for ε sufficiently small both p_ε and q_ε will lie in the same K_i (with i depending on ε). Thus the claim follows from Lemma 1 and Lemma 3 by applying the mean value theorem in each chart as in [28], Prop. 2.3. □

Theorem 1 *Let $U \in \mathcal{G}(X)$. Then $U = 0$ in $\mathcal{G}(X)$ iff $U(\tilde p) = 0$ in $\mathcal{K}$ for all $\tilde p \in \widetilde{X}_c$.*

Proof. Necessity is immediate from Proposition 2. Conversely, fix some Riemannian metric h and cover X by relatively compact geodesically convex charts (U_α, ψ_α). Let $\tilde{x} \in \psi_\alpha(U_\alpha)_c$. Then by Lemma 3 $\tilde{p} := \mathrm{cl}[(\psi_\alpha^{-1}(x_\varepsilon))_\varepsilon]$ is a well defined element of $\widetilde{X}_c$. By assumption $u_\varepsilon(p_\varepsilon) = u_\varepsilon \circ \psi_\alpha^{-1}(x_\varepsilon)$ is a negligible net in $\mathbb{K}$. Thus by [28], Th. 2.4, $U \circ \psi_\alpha^{-1} = 0$ in $\mathcal{G}(\psi_\alpha(V_\alpha))$ for all α, so $U = 0$ by Proposition 1. □

4 Compatibility with distributional geometry, embeddings, and association

As in [10] we call $U \in \mathcal{G}(X)$ *associated* to 0, $U \approx 0$, if $\int_X u_\varepsilon \mu \to 0$ $(\varepsilon \to 0)$ for all compactly supported one densities $\mu \in \Gamma_c^\infty(\mathrm{Vol}(X))$ and one (hence every) representative $(u_\varepsilon)_\varepsilon$ of U. Clearly, $\approx$ induces an equivalence relation on $\mathcal{G}(X)$ giving rise to a linear quotient space. If $\int_X u_\varepsilon \mu \to w(\mu)$ for some $w \in \mathcal{D}'(X)$ then w is called the *distributional shadow* or (*macroscopic aspect*) of U and we write $U \approx w$. In terms of the local description established in Proposition 1 we have

$$U \approx 0 \Leftrightarrow U_\alpha \approx 0 \text{ in } \mathcal{G}(\psi_\alpha(V_\alpha)) \quad \forall \alpha \tag{4}$$

From this it follows that $U_1 \approx U_2$ implies $PU_1 \approx PU_2$ for each $P \in \mathcal{P}(X)$.

By [18], 6.3.4, any $w \in \mathcal{D}'(X)$ can be identified with a family $(w_\alpha)_{\alpha \in A}$, where $w_\alpha \in \mathcal{D}'(\psi_\alpha(V_\alpha))$ satisfies the transformation law

$$w_\beta = (\psi_\alpha \circ \psi_\beta^{-1})^*(w_\alpha)\,.$$

Here f^*w denotes the pullback of a distribution w under the diffeomorphism f. In particular, $w_\alpha = (\psi_\alpha^{-1})^*(w|_{V_\alpha})$. Again a straightforward calculation gives

$$U \approx w \Leftrightarrow U_\alpha \approx w_\alpha \text{ in } \mathcal{G}(\psi_\alpha(V_\alpha)) \quad \forall \alpha \tag{5}$$

Association relations will be our main tool in establishing compatibility with linear distributional geometry later on. Before we proceed with this analysis, however, let us address the problem of embedding $\mathcal{C}^\infty(X)$ and $\mathcal{D}'(X)$ into $\mathcal{G}(X)$. As in the case of open subsets of $\mathbb{R}^n$, $\mathcal{C}^\infty(X)$ is embedded into $\mathcal{G}(X)$ via the "constant" embedding $\sigma : \mathcal{C}^\infty(X) \hookrightarrow \mathcal{G}(X)$, $f \to \mathrm{cl}[(f)_\varepsilon]$.

Turning now to the interrelations between $\mathcal{D}'(X)$ and $\mathcal{G}(X)$ let us first clarify what we can expect at all from such an embedding. The method of choice for open subsets of $\mathbb{R}^n$, i.e., convolution with a mollifier ρ as in Section 1 is manifestly not diffeomorphism invariant, as is demonstrated by the following simple

Example 1 Consider the diffeomorphism $\mu(x) = 2x$ on $\mathbb{R}$ and set $w = \delta \in \mathcal{D}'(\mathbb{R})$. Then $\mu^*\delta = \frac{1}{2}\delta$ and we have

$$\begin{aligned} ((\iota \circ \mu^*)\,\delta)_\varepsilon &= \iota(\frac{1}{2}\,\delta)_\varepsilon = \frac{1}{2}\,\rho_\varepsilon \\ ((\mu^* \circ \iota)\,\delta)_\varepsilon &= \mu^*\rho_\varepsilon = \rho_\varepsilon(2\,.)\,. \end{aligned}$$

From this we see that $((\iota \circ \mu^* - \mu^* \circ \iota)\delta)_\varepsilon = \frac{1}{2}\rho_\varepsilon(x) - \rho_\varepsilon(2x)$ is not in the ideal $\mathcal{N}(\mathbb{R})$. However, it is evident that $(\iota \circ \mu^* - \mu^* \circ \iota)\delta \approx 0$. In fact diffeomorphism invariance does hold on the level of association (cf.[2], Th. 9.1.2).

Finally, as was shown in [10], Remark 3, there can be no embedding of $\mathcal{D}'(X)$ into $\mathcal{G}(X)$ that commutes with differentiation in all local coordinates. The fact that a canonical embedding commuting with Lie derivatives was constructed in [14] for the full Colombeau algebra rests heavily on the dependence of representatives on an additional parameter $\phi \in \mathcal{D}(X)$ (and on the ensuing modified definition of Lie derivatives of such representatives). Therefore we cannot expect an embedding providing this property in the setting of the special Colombeau algebra on manifolds.

On the positive side, the existence of injective sheaf morphisms $\iota : \mathcal{D}' \hookrightarrow \mathcal{G}$ coinciding with σ on $\mathcal{C}^\infty$ and satisfying $\iota(w) \approx w$ for each $w \in \mathcal{D}'(X)$ has been proved by de Roever and Damsma [10] using de Rham-regularizations (cf. [9], §15). In view of the above restrictions these properties of the embedding seem optimal (unless one is willing to furnish X with additional structure).

In the following construction[1] we give an embedding which, while also providing a sheaf morphism possessing these optimal properties, is considerably simpler than the construction in [10], Th. 1.

Theorem 2 *Let $\mathcal{A} = (\psi_\alpha, V_\alpha)_\alpha$ be an atlas of X and let $\{\chi_j : j \in \mathbb{N}\}$ a smooth partition of unity subordinate to $(V_\alpha)_\alpha$. Let supp$(\chi_j) \subseteq V_{\alpha_j}$ for $j \in \mathbb{N}$ and choose for every $j \in \mathbb{N}$ some $\zeta_j \in \mathcal{D}(V_{\alpha_j})$ such that $\zeta_j \equiv 1$ on supp(χ_j). Fix some mollifier $\rho \in \mathcal{S}(\mathbb{R}^n)$ with unit integral and $\int \rho(x)x^\alpha\,dx = 0$ for all $|\alpha| \geq 1$. The map*

$$\iota_{\mathcal{A}} : \mathcal{D}'(X) \to \mathcal{G}(X)$$
$$u \to \mathrm{cl}[(\sum_{j=1}^{\infty} \zeta_j \cdot (((\chi_j \circ \psi_{\alpha_j}^{-1})u_{\alpha_j}) * \rho_\varepsilon) \circ \psi_{\alpha_j})_\varepsilon]$$

is a linear embedding that coincides with σ on $\mathcal{C}^\infty(X)$. Moreover, for each $u \in \mathcal{D}'(X)$ we have $\iota_{\mathcal{A}}(u) \approx u$ and $\mathrm{supp}(u) = \mathrm{supp}(\iota_{\mathcal{A}}(u))$.

Proof. In the proof we will for the sake of brevity replace α_j by j and set $\widetilde{V}_\alpha = \psi_\alpha(V_\alpha)$. It is obvious that

$$u_\varepsilon := \sum_{j=1}^{\infty} \zeta_j \cdot (((\chi_j \circ \psi_j^{-1})u_j) * \rho_\varepsilon) \circ \psi_j$$

is a smooth function on X. Our first task will therefore consist in verifying the $\mathcal{E}_M$-bounds for $(u_\varepsilon)_\varepsilon$. This means that we have to estimate $u_\varepsilon \circ \psi_\alpha^{-1}$ for arbitrary $\alpha \in A$. Let $K \subset\subset \widetilde{V}_\alpha$. Then $L_j = \psi_j(\mathrm{supp}(\zeta_j) \cap \psi_\alpha^{-1}(K))$ is a compact subset of $\widetilde{V}_j$. The fact that the $\mathcal{E}_M(\widetilde{V}_j)$-function $(((\chi_j \circ \psi_j^{-1})u_j) * \rho_\varepsilon)_\varepsilon$ satisfies the necessary bounds on L_j shows that $(u_\varepsilon \circ \psi_\alpha^{-1})_\varepsilon \in \mathcal{E}_M(\widetilde{V}_\alpha)$.

[1]suggested by M. Oberguggenberger

To prove injectivity of $\iota_{\mathcal{A}}$, we suppose that $(u_\varepsilon \circ \psi_\alpha^{-1})_\varepsilon \in \mathcal{N}(\widetilde{V}_\alpha)$ for all $\alpha \in A$. We have to show that $u_\alpha = 0$ in $\mathcal{D}'(\widetilde{V}_\alpha)$ for all α. Fix some $\alpha \in A$ and let $\varphi \in \mathcal{D}(\widetilde{V}_\alpha)$. The term $\langle u_\varepsilon \circ \psi_\alpha^{-1}, \varphi\rangle$ is a finite sum of expressions of the form

$$\begin{aligned}
&\int_{\widetilde{V}_\alpha} \zeta_j \circ \psi_\alpha^{-1}(x)(((\chi_j \circ \psi_j^{-1})u_j) * \rho_\varepsilon)(\psi_j \circ \psi_\alpha^{-1})(x)\varphi(x)\,dx \\
&= \int_{\psi_\alpha(V_j\cap V_\alpha)} \zeta_j \circ \psi_\alpha^{-1}(x)(((\chi_j \circ \psi_j^{-1})u_j) * \rho_\varepsilon)(\psi_j \circ \psi_\alpha^{-1})(x)\varphi(x)\,dx \\
&= \int_{\psi_j(V_j\cap V_\alpha)} \zeta_j \circ \psi_j^{-1}(y)(((\chi_j \circ \psi_j^{-1})u_j) * \rho_\varepsilon)(y)\varphi \circ \psi_\alpha \circ \psi_j^{-1}(y) \\
&\qquad |\det(D(\psi_\alpha \circ \psi_j^{-1}))(y)|\,dy.
\end{aligned}$$

For $\varepsilon \to 0$, this converges to

$$\begin{aligned}
&\langle \zeta_j \circ \psi_j^{-1} \cdot (\chi_j \circ \psi_j^{-1})u_j, \varphi \circ \psi_\alpha \circ \psi_j^{-1}|\det(D(\psi_\alpha \circ \psi_j^{-1}))|\rangle \\
&= \langle \zeta_j \circ \psi_\alpha^{-1} \circ \psi_\alpha \circ \psi_j^{-1} \cdot (\chi_j \circ \psi_\alpha^{-1} \circ \psi_\alpha \circ \psi_j^{-1})u_j, \\
&\quad \varphi \circ \psi_\alpha \circ \psi_j^{-1}|\det(D(\psi_\alpha \circ \psi_j^{-1}))|\rangle \\
&= \langle (\zeta_j \circ \psi_\alpha^{-1})(\chi_j \circ \psi_\alpha^{-1})(\psi_j \circ \psi_\alpha^{-1})^* u_j, \varphi\rangle \\
&= \langle (\zeta_j \circ \psi_\alpha^{-1})(\chi_j \circ \psi_\alpha^{-1})u_\alpha, \varphi\rangle.
\end{aligned}$$

Therefore, for $\varepsilon \to 0$ we have

$$\begin{aligned}
\langle u_\varepsilon \circ \psi_\alpha^{-1}, \varphi\rangle &\to \sum_{j=1}^\infty \langle (\zeta_j \circ \psi_\alpha^{-1})(\chi_j \circ \psi_\alpha^{-1})u_\alpha, \varphi\rangle \\
&= \sum_{j=1}^\infty \langle (\chi_j \circ \psi_\alpha^{-1})u_\alpha, \varphi\rangle = \langle u_\alpha, \varphi\rangle.
\end{aligned}$$

On the other hand, since $(u_\varepsilon)_\varepsilon \in \mathcal{N}(X)$, the above expression converges to 0, which establishes the injectivity of $\iota_{\mathcal{A}}$. Also, the above calculation shows that $\iota_{\mathcal{A}}(u) \approx u$ for each $u \in \mathcal{D}'(X)$.

Let $f \in \mathcal{C}^\infty(X)$. We claim that $U := \iota_{\mathcal{A}}(f) = \sigma(f)$. Considered as an element of $\mathcal{D}'(X)$, f is identified with $(f \circ \psi_\alpha^{-1})_\alpha)_\alpha$, so

$$u_\varepsilon = \sum_{j=1}^\infty \zeta_j \cdot (((\chi_j f) \circ \psi_j^{-1}) * \rho_\varepsilon) \circ \psi_j.$$

We have to show that $((u_\varepsilon - f) \circ \psi_\alpha^{-1})_\varepsilon \in \mathcal{N}(\widetilde{V}_\alpha)$ for all $\alpha \in A$. Now

$$f(x) = \sum_{j=1}^\infty \zeta_j(x)(\chi_j \cdot f)(x) = \sum_{j=1}^\infty \zeta_j(x)((\chi_j \cdot f) \circ \psi_j^{-1})(\psi_j(x)),$$

so

$$\begin{aligned}
&(u_\varepsilon - f) \circ \psi_\alpha^{-1} = \\
&\sum_{j=1}^\infty \zeta_j \circ \psi_\alpha^{-1} \underbrace{[(((\chi_j \cdot f) \circ \psi_j^{-1}) * \rho_\varepsilon) - (\chi_j \cdot f) \circ \psi_j^{-1}]}_{(*)} \circ \psi_j \circ \psi_\alpha^{-1}.
\end{aligned}$$

It therefore suffices to notice that each of the terms $(*)$ is in $\mathcal{N}(\widetilde{V}_j)$. But this follows by Taylor expansion as in the corresponding proof for open subsets of $\mathbb{R}^n$. Finally, preservation of supports is also deduced exactly as in the local case (cf. e.g. [22], 1.2.8). □

It immediately follows that $\iota_{\mathcal{A}}$ is a local operator, i.e., it indeed induces a sheaf morphism with the above properties. Nevertheless, just as the corresponding construction in [10] $\iota_{\mathcal{A}}$ is *nongeometric* in an essential way, i.e., it depends on the chosen atlas as well as on the functions ζ_j, χ_j, etc. For practical purposes however, this drawback is often compensated by the availability of regularization procedures adapted to the specific problem at hand that can be used to model the singularities directly in $\mathcal{G}(X)$ without the use of a distinguished embedding. The connection to the distributional picture is then effected by means of association procedures (cf. e.g. [30], [24]) whose basic properties we now continue to study.

To this end let us first discuss consistency properties with respect to classical products (in the sense of association). In the absence of a distinguished embedding ι we have to be slightly more cautious than in the case of $\mathbb{R}^n$. For example the following (naive) generalization of the statement that the product $\mathcal{C}^\infty \times \mathcal{D}' \to \mathcal{D}'$ is respected by association (more precisely $\iota(f)\iota(u) \approx \iota(fu)$ for all $f \in \mathcal{C}^\infty(\Omega)$, $u \in \mathcal{D}'(\Omega)$): "$U, V \in \mathcal{G}(X)$, $U \approx f \in \mathcal{C}^\infty$ and $V \approx w \in \mathcal{D}'(X)$ $\Rightarrow UV \approx fw$" is wrong in general. To see this take $\rho \in \mathcal{D}(\mathbb{R})$ with $\int \rho = 1$. Then $\mathrm{cl}[(\rho(\frac{x}{\varepsilon}))_\varepsilon] \approx 0$ and clearly $\mathrm{cl}[(\frac{1}{\varepsilon})\rho(\frac{x}{\varepsilon})_\varepsilon] \approx \delta$ but $\rho(\frac{x}{\varepsilon})\,(\frac{1}{\varepsilon})\rho(\frac{x}{\varepsilon}) \to \delta \int \rho^2$ in $\mathcal{D}'$. The reason for the validity of the corresponding $\mathbb{R}^n$-statement ultimately is that $f * \rho_\varepsilon \to f$ uniformly on compact sets already for a continuous function f, whereas $\rho(x/\varepsilon) \to 0$ only weakly. Therefore we introduce the following stronger equivalence relations on $\mathcal{G}(X)$.

Definition 2 *Let $U \in \mathcal{G}(X)$.*

*(i) U is called $\mathcal{C}^k$-*associated *to 0 ($0 \le k \le \infty$), $U \approx_k 0$, if for all $l \le k$, all $\xi_1, \dots, \xi_l \in \mathfrak{X}(X)$ and one (hence any) representative $(u_\varepsilon)_\varepsilon$*

$$L_{\xi_1} \dots L_{\xi_l}\, u_\varepsilon \;\to\; 0 \;\; \textit{uniformly on compact sets.}$$

*(ii) We say that U admits f as $\mathcal{C}^k$-*associated function*, $U \approx_k f$, if for all $l \le k$, all $\xi_1, \dots, \xi_l \in \mathfrak{X}(X)$ and one (hence any) representative*

$$L_{\xi_1} \dots L_{\xi_l}\,(u_\varepsilon - f) \;\to\; 0 \; \textit{uniformly on compact sets.}$$

Clearly if U is $\mathcal{C}^k$-associated to f then $f \in \mathcal{C}^k(X)$. Moreover, if U admits for a $\mathcal{C}^k$-associated function at all the latter is unique. Note also that the above notion of convergence may equivalently be expressed by saying that all $(u_{\alpha\,\varepsilon})_\varepsilon$ converge uniformly in all derivatives of order less or equal k (resp. in all derivatives if $k = \infty$) on compact sets. We are now prepared to state the following

Proposition 3 *Let $U, V \in \mathcal{G}(X)$.*

(i) If $V \approx w \in \mathcal{D}'(X)$, $f \in \mathcal{C}^\infty(X)$, and either (a) $U = \sigma(f)$ or (b) $U \approx_\infty f$, then $UV \approx fw$.

(ii) If $U \approx_k f$ and $V \approx_k g$ then $UV \approx_k fg$ $(f, g \in \mathcal{C}^k(X))$.

Proof. (i)(a) is clear since $\int f v_\varepsilon \mu = v_\varepsilon(f\mu) \to w(f\mu)$ for all compactly supported one-densities μ. To prove (i)(b) we use the fact that multiplication: $\mathcal{C}^\infty \times \mathcal{D}' \to \mathcal{D}'$ as a bilinear separately continuous map is jointly sequentially continuous since both factors are barrelled ([21], §42.2(3) and §40.1). (ii) follows from elementary analysis. □

Proposition 3 (i)(a) is the reconciliation of the respective $\mathcal{C}^\infty$-module structures of $\mathcal{D}'$ and $\mathcal{G}$ on the level of association. Next we introduce the notion of integration of generalized functions.

Definition 3 *Let $U \in \mathcal{G}(X)$ and $\mu \in \Gamma^\infty(Vol(X))$. Then we define the* integral of U with respect to μ *over the relatively compact Lebesgue measurable set $M \subset\subset X$ by*

$$\int_M U\mu \,=\, \mathrm{cl}[(\int_M u_\varepsilon \mu)_\varepsilon]\,.$$

For $U\mu$ compactly supported we set $\int_M U\mu := \int_K U\mu$ where K is any compact set containing $\mathrm{supp}(U\mu)$ in its interior. It is easily seen that this definition is independent of the chosen K. Also, we have $\int_{\mathbb{R}} \delta(x)\,dx = 1$. We close this section by showing that the Lie derivative respects associated distributions.

Proposition 4 *Let X be orientable and $U \approx w$. Then $L_\xi U \approx L_\xi w$.*

Orientability is supposed in order to be able to identify one-densities with n-forms, where a Lie derivative is defined. Moreover, Stokes' theorem is used in the following

Proof. Let $\nu \in \Omega^n_c(X)$ then

$$\int (L_\xi u_\varepsilon)\nu = -\int u_\varepsilon(L_\xi \nu) \to -w(L_\xi \nu) = L_\xi w(\nu)$$

□

5 Generalized sections of vector bundles

For a section $s \in \Gamma(X, E)$ we call $s^i_\alpha := \Psi^i_\alpha \circ s \circ \psi_\alpha^{-1}$ its i-th component with respect to the vector bundle chart (V_α, Ψ_α) $(i = 1, \ldots, N$, where N is the dimension of the fibers).

Definition 4 *Let $E \to X$ be a vector bundle, and again $I = (0, 1]$.*

$$\begin{aligned}
\mathcal{E}(X,E) &:= (\Gamma(X,E))^I \\
\mathcal{E}_M(X,E) &:= \{(s_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}(X,E) :\ \forall\alpha, \forall i = 1,\ldots,N : \\
&\quad\ (s^i_{\alpha\,\varepsilon})_\varepsilon := (\Psi^i_\alpha \circ s_\varepsilon \circ \psi_\alpha^{-1})_\varepsilon \in \mathcal{E}_M(\psi_\alpha(V_\alpha))\} \\
\mathcal{N}(X,E) &:= \{(s_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}(X,E) :\ \forall\alpha, \forall i = 1,\ldots,N : \\
&\quad\ (s^i_{\alpha\,\varepsilon})_\varepsilon \in \mathcal{N}(\psi_\alpha(V_\alpha))\}
\end{aligned}$$

First note that although the composition $f \circ U$ of a generalized function U with a smooth function f generally need not be moderate the notions of moderateness and negligibility as defined above are preserved under the change of bundle charts due to the (fiberwise) linearity of the transition functions. In particular, these notions do not depend on the chosen atlas. In fact, using Peetre's theorem we obtain the following global description of moderate resp. negligible sections:

$$\begin{aligned} \mathcal{E}_M(X,E) &= \{(s_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}(X,E) : \ \forall P \in \mathcal{P}(X,E)\, \forall K \subset\subset X\, \exists N \in \mathbb{N} : \\ &\quad \sup_{p\in K} \|P u_\varepsilon(p)\| = O(\varepsilon^{-N})\} \\ \mathcal{N}(X,E) &= \{(s_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}(X,E) : \ \forall P \in \mathcal{P}(X,E)\, \forall K \subset\subset X\, \forall m \in \mathbb{N} : \\ &\quad \sup_{p\in K} \|P u_\varepsilon(p)\| = O(\varepsilon^{m})\} \end{aligned}$$

Here $\| \ \|$ denotes the norm induced on the fibers of E by any Riemannian metric. Similar to (3), [12], Th. 13.1 yields a characterization of $\mathcal{N}(X,E)$ as a subspace of $\mathcal{E}_M(X,E)$ that imposes the above growth restrictions on representatives only with respect to differential operators of order 0. In order to define generalized sections of the bundle $E \to X$ we need the following

Proposition 5 *With operations defined componentwise (i.e., for each* ε*),* $\mathcal{E}_M(X, E)$ *is a* $\mathcal{G}(X)$*-module with* $\mathcal{N}(X,E)$ *a submodule in it.*

Proof. We need to establish the following statements (a) $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$, $(s_\varepsilon)_\varepsilon \in \mathcal{E}_M(X,E) \Rightarrow (u_\varepsilon s_\varepsilon)_\varepsilon \in \mathcal{E}_M(X,E)$, (b) $(u_\varepsilon)_\varepsilon \in \mathcal{N}(X)$, $(s_\varepsilon)_\varepsilon \in \mathcal{E}_M(X,E) \Rightarrow (u_\varepsilon s_\varepsilon)_\varepsilon \in \mathcal{N}(X,E)$ and (c) $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$, $(s_\varepsilon)_\varepsilon \in \mathcal{N}(X,E) \Rightarrow (u_\varepsilon s_\varepsilon)_\varepsilon \in \mathcal{N}(X,E)$, which easily follow from the local description in Proposition 1 and the definitions above. $\Box$

Now we are in the position to define.

Definition 5 *The* $\mathcal{G}(X)$*-module of generalized sections of* $E \to X$ *is defined as the quotient*

$$\mathcal{G}(X,E) := \mathcal{E}_M(X,E) \,/\, \mathcal{N}(X,E)\,.$$

As usual we denote generalized objects by capital letters, e.g., $S = \mathrm{cl}[(s_\varepsilon)_\varepsilon]$. By the very definition of $\mathcal{G}(X,E)$ we may describe a generalized section S by a family $(S_\alpha)_\alpha = ((S^i_\alpha)_\alpha)_{i=1}^N$, where S_α is called the *local expression* of S. Its *components* $S^i_\alpha := \Psi^i_\alpha \circ S \circ \psi_\alpha^{-1} \in \mathcal{G}(\psi_\alpha(V_\alpha))$ $(i = 1, \ldots, N)$ satisfy

$$S^i_\alpha(x) = (\boldsymbol{\psi}_{\alpha\beta})^i_j(\psi_\beta \circ \psi_\alpha^{-1}(x))\, S^j_\beta(\psi_\beta \circ \psi_\alpha^{-1}(x)) \tag{6}$$

for all $x \in \psi_\alpha(V_\alpha \cap V_\beta)$, where $\boldsymbol{\psi}_{\alpha\beta}$ denotes the transition functions of the bundle. Hence formally generalized sections of $E \to X$ are locally simply given by "ordinary" sections with generalized "coefficients." We shall see shortly that this property in fact also holds globally (cf. Theorem 4 below).

As before smooth sections may be embedded into $\mathcal{G}(X,E)$ by the "constant" embedding now denoted by Σ, i.e., $\Sigma(s) = \mathrm{cl}[(s)_\varepsilon]$. Since $\mathcal{C}^\infty(X)$ is a subring of $\mathcal{G}(X)$, $\mathcal{G}(X,E)$ can also be viewed as a $\mathcal{C}^\infty(X)$-module and the two respective module structures on the space of generalized sections are compatible in the sense of the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{C}^\infty(X) \times \Gamma(X,E) & \xrightarrow{\sigma\times\Sigma} & \mathcal{G}(X) \times \mathcal{G}(X,E) \\
\Big\downarrow \cdot & & \Big\downarrow \cdot \\
\Gamma(X,E) & \xrightarrow{\Sigma} & \mathcal{G}(X,E)
\end{array}$$

The most important structural properties of $\mathcal{G}(X,E)$ are subsumed in the following results.

Theorem 3 *$\mathcal{G}(\_,E)$ is a fine sheaf of $\mathcal{G}(\_)$-modules.*

Proof. This is a straightforward generalization of the $\mathbb{R}^n$-case. □

Theorem 4 *The following chain of $\mathcal{C}^\infty(X)$-module isomorphisms holds:*

$$\mathcal{G}(X,E) \cong \mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,E) \cong L_{\mathcal{C}^\infty(X)}(\Gamma(X,E^*),\mathcal{G}(X))$$

Proof. $\Gamma(X,E)$ is projective and finitely generated ([11], 2.23, Cor.), $\Gamma(X,E^*) \cong \Gamma(X,E)^*$ ([11], 2.24, Rem.), and, consequently, $\Gamma(X,E)^{**} \cong \Gamma(X,E)$ (Here $\Gamma(X,E)^*$ denotes the dual $\mathcal{C}^\infty(X)$-module of $\Gamma(X,E)$). Hence $\mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,E) \cong L_{\mathcal{C}^\infty(X)}(\Gamma(X,E^*),\mathcal{G}(X))$ follows from [4], Ch. II, §4, 2.

Since both $\mathcal{G}(\_,E)$ and $L_{\mathcal{C}^\infty(\_)}(\Gamma(\_,E^*),\mathcal{G}(\_))$ are sheaves of $\mathcal{C}^\infty(\_)$-modules (cf. e.g. [19], (2.2.4)) and the isomorphy of the second and third module in the above chain of course also holds locally, in order to finish the proof it suffices to show that $\mathcal{G}(U,E) \cong \mathcal{G}(U) \otimes_{\mathcal{C}^\infty(U)} \Gamma(U,E)$ for any trivializing open set $U \subseteq X$. But for such a U we have $\mathcal{G}(U,E) \cong \mathcal{G}(U)^N$ and $\Gamma(U,E) \cong \mathcal{C}^\infty(U)^N$, so the claim follows. □

Remark 1 Endowing $\mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,E)$ with the canonical $\mathcal{G}(X)$-module structure induced by $u_1 \cdot (u_2 \otimes \xi) = (u_1 u_2) \otimes \xi$, $(u_1, u_2 \in \mathcal{G}(X), \xi \in \Gamma(X,E))$ it follows immediately that the $\mathcal{C}^\infty(X)$-module isomorphism $\mathcal{G}(X,E) \cong \mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,E)$ is in fact also a $\mathcal{G}(X)$-module isomorphism.

Corollary 1 *Let $E_1,\dots,E_k$, F be vector bundles with base manifold X. Then the following isomorphism of $\mathcal{C}^\infty(X)$-modules holds:*

$$\mathcal{G}(X, L(E_1,\dots,E_k;F)) \cong L_{\mathcal{C}^\infty(X)}(\Gamma(X,E_1),\dots,\Gamma(X,E_k);\mathcal{G}(X,F))$$

Proof. By Theorem 4 the right hand side can be written as

$$\begin{aligned}
& L_{\mathcal{C}^\infty(X)}(\Gamma(X,E_1),\dots,\Gamma(X,E_k);\mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,F)) \\
\cong\ & L_{\mathcal{C}^\infty(X)}(\Gamma(X,E_1) \otimes_{\mathcal{C}^\infty(X)} \dots \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,E_k); \\
& \mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,F)) \\
\cong\ & \mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} L_{\mathcal{C}^\infty(X)}(\Gamma(X,E_1) \otimes_{\mathcal{C}^\infty(X)} \dots \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,E_k); \\
& \Gamma(X,F)) \\
\cong\ & \mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} L_{\mathcal{C}^\infty(X)}(\Gamma(X,E_1),\dots,\Gamma(X,E_k);\Gamma(X,F))
\end{aligned}$$

Here the second isomorphism holds by [4], Ch. II §4, 2., Prop. 2 since the module $\Gamma(X, E_1) \otimes_{\mathcal{C}^\infty(X)} \ldots \otimes_{\mathcal{C}^\infty(X)} \Gamma(X, E_k)$ is finitely generated and projective. Now

$$L_{\mathcal{C}^\infty(X)}(\Gamma(X, E_1) \ldots \Gamma(X, E_k); \Gamma(X, F)) \cong \Gamma(X, L(E_1, \ldots, E_k; F))$$

by [11], 2.24, Cor. 2, so the claim follows from Theorem 4. □

Theorem 5 *The $\mathcal{G}(X)$-module $\mathcal{G}(X, E)$ is finitely generated and projective.*

Proof. Choose a vector bundle F such that $E \oplus F = X \times \mathbb{R}^N$ for some $N \in \mathbb{N}$ ([11], 2.23). Then we have the following $\mathcal{G}(X)$-isomorphisms:

$$\mathcal{G}(X, E) \oplus_{\mathcal{G}(X)} \mathcal{G}(X, F) \cong \mathcal{G}(X, X \times \mathbb{R}^N) \cong \mathcal{G}(X)^N$$

It follows that the $\mathcal{G}(X)$-module $\mathcal{G}(X, E)$ is a direct summand in a finitely generated free $\mathcal{G}(X)$-module, hence is projective and finitely generated ([4], Ch. II, §2, 2., Cor. 1). □

We will study further properties of $\mathcal{G}(X, E)$ as a $\mathcal{G}(X)$-module after Lemma 4.

Concerning the question of embedding $\mathcal{D}'(X, E)$ into $\mathcal{G}(X, E)$ we note the following result

Theorem 6 *There exist injective sheaf morphisms $\mathcal{D}'(X, E) \hookrightarrow \mathcal{G}(X, E)$ coinciding with Σ on $\Gamma(X, E)$.*

Proof. Since $\Gamma(X, E)$ is projective, hence flat, from any embedding

$$\mathcal{C}^\infty(X) \hookrightarrow \mathcal{D}'(X) \hookrightarrow \mathcal{G}(X)$$

as described before Theorem 2 we obtain injections

$$\begin{aligned} \mathcal{C}^\infty(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X, E) &\hookrightarrow \mathcal{D}'(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X, E) \\ &\hookrightarrow \mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X, E), \end{aligned}$$

i.e.,

$$\Gamma(X, E) \hookrightarrow \mathcal{D}'(X, E) \hookrightarrow \mathcal{G}(X, E)$$

Clearly the sheaf morphism-properties are not lost in this process. □

The same restrictions as in the scalar case ("non-geometric" embedding) of course also apply here. Thus, analogously to the earlier cases we set up coupled calculus in order to obtain a convenient language for describing compatibility with the distributional setting. In the following definition, $(.|.)$ denotes the canonical vector bundle homomorphism

$$\begin{aligned} (.|.) &:= \mathrm{tr}_E \otimes \mathrm{id} \\ E \otimes E^* \otimes \mathrm{Vol}(X) &\to (X \times \mathbb{C}) \otimes \mathrm{Vol}(X) = \mathrm{Vol}(X) \end{aligned}$$

where tr_E is the vector bundle isomorphism induced by the pointwise action of $v^* \in E_p^*$ on $v \in E_p$.

Definition 6 *(i) A generalized section $S \in \mathcal{G}(X,E)$ is called* associated to 0, *$S \approx 0$, if for all $\mu \in \Gamma_c(X, E^* \otimes Vol(X))$ and one (hence any) representative $(s_\varepsilon)_\varepsilon$ of S*

$$\lim_{\varepsilon\to 0} \int\limits_X (s_\varepsilon|\mu) \,=\, 0\,.$$

(ii) Let $S \in \mathcal{G}(X,E)$ and $w \in \mathcal{D}'(X,E)$. We say that S admits w as associated distribution (with values in E) *and call w the* distributional shadow *(or* macroscopic aspect*) of S if for all $\mu \in \Gamma_c(X, E^* \otimes Vol(X))$ and one (hence any) representative*

$$\lim_{\varepsilon\to 0} \int\limits_X (s_\varepsilon|\mu) \,=\, w(\mu)\,,$$

where $w(\mu)$ denotes the distributional action of w on μ. In that case we use the notation $S \approx w$.

$S \approx T :\Leftrightarrow S - T \approx 0$ defines an equivalence relation giving rise to a linear quotient of $\mathcal{G}(X,E)$. If $S \approx T$ we call S and T *associated to each other.* In complete analogy to the scalar case, by localization we immediately have

Proposition 6 *(i) $S \approx 0$ in $\mathcal{G}(X,E) \Leftrightarrow S^i_\alpha \approx 0$ in $\mathcal{G}(\psi_\alpha(V_\alpha))$ $\forall \alpha, i = 1,\dots,N$*

(ii) $S \approx w \in \mathcal{D}'(X,E) \Leftrightarrow S^i_\alpha \approx w^i_\alpha$ in $\mathcal{G}(\psi_\alpha(V_\alpha))$ $\forall \alpha, i = 1,\dots,N$

□

Definition 7 *Let $S \in \mathcal{G}(X,E)$.*

(i) S is called $\mathcal{C}^k$-associated to 0 $(0 \le k \le \infty)$, $S \approx_k 0$, if for one (hence any) representative $(s_\varepsilon)_\varepsilon$ and $\forall \alpha$, $i = 1,\dots,N$ $s^i_{\alpha\,\varepsilon} \to 0$ uniformly on compact sets in all derivatives of order less or (if $k < \infty$) equal to k.

(ii) We say that S allows $t \in \Gamma^k(X,E)$ as a $\mathcal{C}^k$-associated section, $S \approx_k t$, if for one (hence any) representative $(s_\varepsilon)_\varepsilon$ and $\forall \alpha$, $i = 1,\dots,N$ $s^i_{\alpha\,\varepsilon} \to t^i_\alpha$ uniformly on compact sets in all derivatives of order less or (if $k < \infty$) equal to k.

As is the case with $\mathcal{G}(X)$ the different $\mathcal{C}^\infty$-module structures of $\mathcal{D}'(X,E)$ and $\mathcal{G}(X,E)$, respectively, may be reconciled at the level of association:

Proposition 7 *Let $U \in \mathcal{G}(X)$ and $S \in \mathcal{G}(X,E)$.*

(i) If $U \approx w \in \mathcal{D}'(X)$, $s \in \Gamma(X,E)$ and either (a) $S = \Sigma(s)$ or (b) $S \approx_\infty s$, then $U\,S \approx ws$.

(ii) If $S \approx s \in \mathcal{D}'(X,E)$, $f \in \mathcal{C}^\infty(X)$ and either (a) $U = \sigma(f)$ or (b) $U \approx_\infty f$, then $U\,S \approx fs$.

(iii) If $U \approx_k f$ and $S \approx_k s$ then $U\,S \approx_k fs$ $(f \in \mathcal{C}^k(X)$, $s \in \Gamma^k(X,E))$.

Proof. Simply apply Proposition 3 componentwise. □

6 Generalized tensor analysis

In the case where $E \to X$ is some tensor bundle $T^r_s(X)$ over the manifold X we shall use the notation $\mathcal{G}^r_s(X)$ for $\mathcal{G}(X, T^r_s(X))$ and similarly for $\mathcal{E}$, $\mathcal{E}_M$ and $\mathcal{N}$. The space of smooth tensor fields will be denoted by $\mathcal{T}^r_s(X)$. One of the main goals in our analysis of this particular case of generalized sections of vector bundles is to demonstrate the relative ease with which arguments from classical analysis can be carried over to the generalized functions setting. Our first result gives several algebraic characterizations of $\mathcal{G}^r_s(X)$.

Theorem 7 *(i) As $\mathcal{G}(X)$-module, $\mathcal{G}^r_s(X) \cong L_{\mathcal{G}(X)}(\mathcal{G}^0_1(X)^r, \mathcal{G}^1_0(X)^s; \mathcal{G}(X))$.*

(ii) As $\mathcal{C}^\infty(X)$-module, $\mathcal{G}^r_s(X) \cong L_{\mathcal{C}^\infty(X)}(\Omega^1(X)^r, \mathfrak{X}(X)^s; \mathcal{G}(X))$.

(iii) As $\mathcal{C}^\infty(X)$-module and also as $\mathcal{G}(X)$-module, $\mathcal{G}^r_s(X) \cong \mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{T}^r_s(X)$.

To simplify notations we will set $r = 1 = s$ in the proof. We first establish the following localization result.

Lemma 4 *Let $T \in L_{\mathcal{G}(X)}(\mathcal{G}^0_1(X), \mathcal{G}^1_0(X); \mathcal{G}(X))$, $A \in \mathcal{G}^0_1(X)$ and $\Xi \in \mathcal{G}^1_0(X)$ with $\Xi|_U = 0$ for some open $U \subseteq X$. Then $T(A, \Xi)|_U = 0$.*

Proof. Since U can be written as the union of a collection of open sets $(U_p)_{p\in U}$ such that each $\overline{U}_p \subset\subset V_\alpha$ for some chart V_α and due to the sheaf property of $\mathcal{G}(X)$ we may assume without loss of generality that $\overline{U} \subset\subset V_\alpha$ and write $\Xi|_{V_\alpha} = \Xi^i \partial_i$ with $\Xi^i \in \mathcal{G}(V_\alpha)$ vanishing on U. Let now f be a bump function on $\overline{U}$ (i.e., $f \in \mathcal{D}(V_\alpha)$, $f|_{\overline{U}} = 1$) then (using summation convention)

$$\begin{aligned} T(A,\Xi)|_U &= f^2|_U\, T(A,\Xi)|_U \;=\; f^2\, T(A,\Xi)|_U \\ &= T(A, f\Xi^i f\partial_i)|_U \;=\; f\Xi^i\, T(A, f\partial_i)|_U \\ &= f\Xi^i|_U\, T(A, f\partial_i)|_U \;=\; 0\,, \end{aligned}$$

where we did not distinguish notationally between f and $\sigma(f)$. □

From this result it follows that for any $V \subseteq X$ open, $A \in \mathcal{G}^0_1(V)$ and $\Xi \in \mathcal{G}^1_0(V)$ we may unambiguously define $T|_V(A, \Xi)$.

Proof of the theorem. (i) Let $T = \mathrm{cl}[(t_\varepsilon)_\varepsilon] \in \mathcal{G}^1_1(X)$, $A = \mathrm{cl}[(a_\varepsilon)_\varepsilon] \in \mathcal{G}^0_1(X)$ and $\Xi = \mathrm{cl}[(\xi_\varepsilon)_\varepsilon] \in \mathcal{G}^1_0(X)$. Using classical contraction we define componentwise the following map

$$\tilde{T} :\, (a_\varepsilon, \xi_\varepsilon) \mapsto f_\varepsilon := t_\varepsilon(a_\varepsilon, \xi_\varepsilon)\,.$$

From the local description it is easy to see that $F = \mathrm{cl}[(f_\varepsilon)_\varepsilon] \in \mathcal{G}(X)$, $\tilde{T} : \mathcal{G}^0_1(X) \times \mathcal{G}^1_0(X) \to \mathcal{G}(X)$ is well-defined and $\mathcal{G}(X)$-bilinear, so $\tilde{T} \in L_{\mathcal{G}(X)}(\mathcal{G}^0_1(X), \mathcal{G}^1_0(X); \mathcal{G}(X))$. Moreover, the assignment $T \mapsto \tilde{T}$ is also $\mathcal{G}(X)$-linear, so it only remains to show that the latter is an isomorphism.

To prove injectivity assume $\tilde{T} = 0$, that is $(t_\varepsilon(a_\varepsilon, \xi_\varepsilon))_\varepsilon \in \mathcal{N}(X)$ for all $A = \mathrm{cl}[(a_\varepsilon)_\varepsilon] \in \mathcal{G}^0_1(X)$ and all $\Xi = \mathrm{cl}[(\xi_\varepsilon)_\varepsilon] \in \mathcal{G}^1_0(X)$. To show that $T = 0 \in \mathcal{G}^1_1(X)$ it suffices to work locally. Choose $K \subset\subset V_\alpha$ and $A \in \mathcal{G}^0_1(X)$, $\Xi \in$

$\mathcal{G}^1_0(X)$ whose compact supports are contained in V_α and such that $A = \Sigma(dx^i)$, $\Xi = \Sigma(\partial_j)$ on an open neighborhood U of K in V_α $(1 \leq i,j \leq n)$. Then $\mathcal{N}(U) \ni (t_\varepsilon(a_\varepsilon,\xi_\varepsilon)|_U)_\varepsilon = (t_\alpha{}^i_{j\,\varepsilon}|_U)_\varepsilon$. Since i,j were arbitrary we are done.

To show surjectivity choose $\tilde{T} \in L_{\mathcal{G}(X)}(\mathcal{G}^0_1(X), \mathcal{G}^1_0(X); \mathcal{G}(X))$. By the remark following Lemma 4, for any chart (V_α, ψ_α) with coordinates x^i we may define

$$T_\alpha{}^i_j \,=\, \tilde{T}|_{V_\alpha}(dx^i, \partial_j) \circ \psi_\alpha^{-1} \,\in \mathcal{G}(\psi_\alpha(V_\alpha))\,,$$

Since $\tilde{T}$ is globally defined the $(T_\alpha)_\alpha$ form a coherent family. Hence by the sheaf property of $\mathcal{G}^1_1(X)$ there exists a unique $T \in \mathcal{G}^1_1(X)$ represented by the family $(T_\alpha)_\alpha$ and by construction $\tilde{T}$ is the image of T.

(ii) follows from Corollary 1 (alternatively, it can be proved analogously to (i)). Finally, (iii) is immediate from Theorem 4 and Remark 1. □

Theorem 7 (iii) was suggested as a *definition* for the space of Colombeau tensor fields in [15], Ch. 2. The proof of Theorem 7 (i) is easily adapted to yield the following result on spaces of generalized sections:

Proposition 8 *Let $E_1, \ldots E_k$, F be vector bundles with base manifold X. Then the following isomorphism of $\mathcal{G}(X)$-modules holds:*

$$\mathcal{G}(X, L(E_1, \ldots, E_k; F)) \cong L_{\mathcal{G}(X)}(\mathcal{G}(X,E_1), \ldots, \mathcal{G}(X,E_k); \mathcal{G}(X,F))$$

□

(An alternative proof of Proposition 8 can be given along the lines of [11], 2.24.) Hence

$$L_{\mathcal{G}(X)}(\mathcal{G}(X,E), \mathcal{G}(X)) \cong \mathcal{G}(X, E^*)\,. \tag{7}$$

It follows that the $\mathcal{G}(X)$-module $\mathcal{G}(X,E)$ is reflexive. Also, we note that the proof of [11], Ch. II, Prop. XIV can directly be adapted to establish:

Proposition 9 *Let E, F be vector bundles with base manifold X. Then the following isomorphism of $\mathcal{G}(X)$-modules holds:*

$$\mathcal{G}(X,E) \otimes_{\mathcal{G}(X)} \mathcal{G}(X,F) \cong \mathcal{G}(X, E \otimes F) \tag{8}$$

□

In particular, from (7), Proposition 9 and Theorem 5 we conclude:

$$\begin{aligned}
&L_{\mathcal{G}(X)}(\mathcal{G}(X,E_1), \ldots, \mathcal{G}(X,E_k); \mathcal{G}(X,F)) \cong \\
&L_{\mathcal{G}(X)}(\mathcal{G}(X,E_1) \otimes_{\mathcal{G}(X)} \ldots \otimes_{\mathcal{G}(X)} \mathcal{G}(X,E_k); \mathcal{G}(X)) \otimes_{\mathcal{G}(X)} \mathcal{G}(X,F) \cong \\
&L_{\mathcal{G}(X)}(L_{\mathcal{G}(X)}(\mathcal{G}(X,E_1^*) \otimes_{\mathcal{G}(X)} \ldots \otimes_{\mathcal{G}(X)} \mathcal{G}(X,E_k^*); \mathcal{G}(X)); \mathcal{G}(X)) \\
&\otimes_{\mathcal{G}(X)} \mathcal{G}(X,F) \cong \\
&L_{\mathcal{G}(X)}(\mathcal{G}(X,E_1), \mathcal{G}(X)) \otimes_{\mathcal{G}(X)} \ldots \otimes_{\mathcal{G}(X)} L_{\mathcal{G}(X)}(\mathcal{G}(X,E_k), \mathcal{G}(X)) \\
&\otimes_{\mathcal{G}(X)} \mathcal{G}(X,F)
\end{aligned}$$

(using [4], Ch. II, §4, 2., Prop. 2, 4., Prop. 4, Cor. 1, and 2., Rem. (2)).

Returning now to the special case of tensor bundles, given a generalized tensor field $T \in \mathcal{G}^r_s(X)$ we shall call the n^{r+s} generalized functions on V_α defined by

$$T^{\alpha}{}^{\,i_1\dots i_r}_{\,j_1\dots j_s} := T|_{V_\alpha}(dx^{i_1},\dots,dx^{i_r},\partial_{j_1},\dots,\partial_{j_s})$$

its *components* with respect to the chart (V_α, ψ_α). We shall use abstract index notation (cf. [29], Chap. 2) whenever convenient and write $T^{a_1\dots a_r}_{b_1\dots b_s} \in \mathcal{G}^r_s(X)$. To clearly distinguish between the notions of abstract and concrete indices we reserve the letters a, b, c, d, e, f for the previous one and $i, j, k, l, \dots$ for the latter one. Hence we shall denote the components of $\Xi^a \in \mathcal{G}^1_0(X)$ and $A_a \in \mathcal{G}^0_1(X)$ by $\Xi^{\alpha\, i}$ and $A^\alpha{}_i$ respectively. Similarly the components of a representative $(t^{a_1\dots a_r}_{b_1\dots b_s\ \varepsilon})_\varepsilon \in \mathcal{E}_M{}^r_s(X)$ of $T^{a_1\dots a_r}_{b_1\dots b_s} \in \mathcal{G}^r_s(X)$ will be denoted by $(t^{\alpha}{}^{\,i_1\dots i_r}_{\,j_1\dots j_s\ \varepsilon})_\varepsilon$.

The spaces of moderate respectively negligible nets of tensor fields may be characterized invariantly by the Lie derivative (similar to the scalar case, cf. Lemma 1 (ii)).

Proposition 10

$$\begin{aligned}
\mathcal{E}_M{}^r_s(X) &= \{(t_\varepsilon)_{\varepsilon\in I} \in (\mathcal{E})^r_s(X) : \forall K \subset\subset X,\ \forall k \in \mathbb{N}_0\ \exists N \in \mathbb{N}\ \forall \xi_1,\dots,\xi_k \\
&\qquad \in \mathcal{T}^1_0(X) : \sup_{p\in K} ||L_{\xi_1}\dots L_{\xi_k}\, t_\varepsilon(p)|| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\} \\
\mathcal{N}^r_s(X) &= \{(t_\varepsilon)_{\varepsilon\in I} \in (\mathcal{E})^r_s(X) : \forall K \subset\subset X,\ \forall k, m \in \mathbb{N}_0\ \forall \xi_1,\dots,\xi_k \\
&\qquad \in \mathcal{T}^1_0(X) : \sup_{p\in K} ||L_{\xi_1}\dots L_{\xi_k}\, t_\varepsilon(p)|| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}
\end{aligned}$$

where $||.||$ denotes the norm induced on $\mathcal{T}^r_s(X)$ by any Riemannian metric on X.

Definition 8 *Let $S \in \mathcal{G}^r_s(X)$ and $T \in \mathcal{G}^{r'}_{s'}(X)$. We define the* tensor product *$S \otimes T \in \mathcal{G}^{r+r'}_{s+s'}(X)$ of S and T by*

$$S \otimes T := \mathrm{cl}[(s_\varepsilon \otimes t_\varepsilon)_\varepsilon]\,.$$

Using the local description it is easily checked that the tensor product is well defined. Moreover it is $\mathcal{G}(X)$-bilinear, associative and by a straightforward generalization of Proposition 3 displays the following consistency properties with respect to the classical resp. distributional tensor product.

Proposition 11 *Let $S \in \mathcal{G}^r_s(X)$ and $T \in \mathcal{G}^{r'}_{s'}(X)$.*

(i) If $T \approx w \in \mathcal{D}'^{r'}_{s'}(X)$, $s \in \mathcal{T}^r_s(X)$ and either (a) $S = \Sigma(s)$ or (b) $S \approx_\infty s$ then $S \otimes T \approx s \otimes w$ in $\mathcal{G}^{r+r'}_{s+s'}(X)$.

(ii) If $S \approx_k s$ and $T \approx_k t$ then $S \otimes T \approx_k s \otimes t$ in $\mathcal{G}^{r+r'}_{s+s'}(X)$ ($s \in \Gamma^k(X, T^r_s(X))$, $t \in \Gamma^k(X, T^{r'}_{s'}(X))$).

□

We may now easily generalize the following notions of classical tensor calculus.

Definition 9 *(i) Let $T^{a_1\dots a_r}_{b_1\dots b_s} \in \mathcal{G}^r_s(X)$. We define the contraction of $T^{a_1\dots a_r}_{b_1\dots b_s}$ by*

$$T^{a_1\dots i\dots a_r}_{b_1\dots i\dots b_s} := \mathrm{cl}[(t^{a_1\dots i\dots a_r}_{b_1\dots i\dots b_s\ \varepsilon})_\varepsilon] \in \mathcal{G}^{r-1}_{s-1}(X)\,.$$

(ii) For any smooth vector field ξ on X the Lie derivative of $T \in \mathcal{G}^r_s(X)$ with respect ot ξ is given by

$$L_\xi T \,:=\, \mathrm{cl}[(L_\xi t_\varepsilon)_\varepsilon]\,.$$

(iii) Finally, we define the universal generalized tensor algebra over X by

$$\hat{G}(X) \,:=\, \bigoplus_{r,s} \mathcal{G}^r_s(X)\,.$$

The Lie derivative displays the following consistency property with respect to its distributional counterpart

Proposition 12 *Let X be orientable and $T \approx t$ in $\mathcal{G}^r_s(X)$. Then $L_\xi T \approx L_\xi t$.* □

Next we introduce the generalized Lie derivative, i.e., the Lie derivative with respect to a generalized vector field. We note that an analogous definition (i.e., Lie derivative of a distributional tensor field with respect to a distributional vector field) is impossible in the purely distributional setting (cf. [26], §5).

Definition 10 *Let $\Xi \in \mathcal{G}^1_0(X)$ and $T \in \mathcal{G}^r_s(X)$. We define the* generalized Lie derivative *of T with respect to Ξ by*

$$L_\Xi(T) \,:=\, \mathrm{cl}[(L_{\xi_\varepsilon}(t_\varepsilon))_\varepsilon]\,.$$

In case $U \in \mathcal{G}(X)$ we also use the notation $\Xi(U)$ for $L_\Xi U$.

The well-definedness of $L_\Xi(T)$ is an easy consequence of the local description. Literally all classical (algebraic) properties of the Lie derivative carry over since they hold componentwise. In particular, for generalized vector fields Ξ, H we have $L_\Xi H = [\Xi, H] := \mathrm{cl}[\,([\xi_\varepsilon, \eta_\varepsilon])_\varepsilon]$ and for all generalized functions U we have: $[U\Xi, H] = U[\Xi, H] - H(U)\Xi$. Moreover, we immediately get the following consistency properties.

Proposition 13 *Let $\Xi \in \mathcal{G}^1_0(X)$ and $T \in \mathcal{G}^r_s(X)$*

(i) If $\Xi = \Sigma(\xi)$ for some $\xi \in \mathcal{T}^1_0(X)$ then $L_\Xi(T) = L_\xi(T)$.

(ii) If $\Xi \approx_\infty \xi \in \mathcal{T}^1_0(X)$ and $T \approx t \in \mathcal{D}'^r_s(X)$ or conversely, if $\Xi \approx \xi \in \mathcal{D}'^1_0(X)$ and $T \approx_\infty t \in \mathcal{T}^r_s(X)$ then $L_\Xi(T) \approx L_\xi t$.

(iii) If $\Xi \approx_k \xi$ and $T \approx_{k+1} t$ then $L_\Xi(T) \approx_k L_\xi t$ ($\xi \in \Gamma^k(X, TX)$, $t \in \Gamma^{k+1}(X, T^r_s(X))$).

□

For a generalized vector field Ξ the map $L_\Xi \equiv \Xi : \mathcal{G}(X) \to \mathcal{G}(X)$ is clearly $\mathbb{R}$-linear (in fact even $\mathcal{R}$-linear) and obeys the Leibniz rule, hence is a derivation on $\mathcal{G}(X)$. Moreover any derivation on the algebra of generalized function arises this way.

Theorem 8 *$\mathcal{G}^1_0(X)$ is ($\mathbb{R}$-linearly) isomorphic to $Der(\mathcal{G}(X))$.*

Proof. It suffices to show that for any derivation θ on $\mathcal{G}(X)$ we may construct a unique generalized vector field Ξ such that $\theta(U) = \Xi(U)$ for all $U \in \mathcal{G}(X)$. We start by showing that θ is a local operator, i.e., that $U = 0$ on $V (\subseteq X)$ open implies $\theta(U)|_V = 0$. To this end choose any open $W \subset\subset V$ and a function $f \in \mathcal{D}(V)$ equal to 1 on W. Then $U = (1-f)U$ and

$$\theta(U)|_W \;=\; \theta(1-f)U|_W + (1-f)\theta(U)|_W \;=\; 0 \;\in \mathcal{G}(W)$$

Since $\mathcal{G}$ is a sheaf, $\theta(U)|_V = 0$. Now let (V_α, ψ_α) be a chart in X, $x = \psi_\alpha(p)$ and $U \in \mathcal{G}(X)$. Then for y in a neighborhood of x

$$\begin{aligned}(U \circ \psi_\alpha^{-1})(y) &= (U \circ \psi_\alpha^{-1})(x) + \int_0^1 \frac{d}{dt}(U \circ \psi_\alpha^{-1})(x + t(y-x))\,dt \\ &= (U \circ \psi_\alpha^{-1})(x) + \sum_{i=1}^n (y^i - x^i) \int_0^1 D_i(U \circ \psi_\alpha^{-1})(x + t(y-x))\,dt\,.\end{aligned}$$

Hence in a neighborhood of p ($q = \psi_\alpha^{-1}(y)$), $U(q) = U(p) + \sum_{i=1}^n (\psi_\alpha^i(q) - \psi_\alpha^i(p))\, g_i(q)$, where g_i is given by the integral above whence, in particular, $g_i(p) = \frac{\partial}{\partial x^i}(U \circ \psi_\alpha^{-1})|_x$. Consequently

$$(\theta(U))(p) \;=\; \sum_{i=1}^n \partial_i U(p)\, \theta(\psi_\alpha^i)(p)$$

and we define Ξ locally to be given by $\Xi_\alpha^i = \theta(\psi_\alpha^i)$ (this is well defined by the first part of the proof). It is easily checked that this indeed defines a coherent family in the sense of (6). □

7 Exterior Algebra, Hamiltonian Mechanics

In this section we are going to study generalized sections of the bundle $\Lambda^k T^* X$, i.e., generalized k-forms, thereby setting the stage for nonsmooth Hamiltonian mechanics.

To simplify notations we set $\mathcal{G}^{\wedge^k}(X) := \mathcal{G}(X, \Lambda^k T^* X)$ and similar for the spaces of moderate resp. negligible nets of k-forms. If X is oriented (with its orientation induced by θ) it follows from the local description of generalized sections that $\Sigma(\omega) \approx j(\omega)$ for all $\omega \in \Omega^k(X)$, where j is the embedding of regular objects into the space of distributional k-forms from [26] (see (2)). The basic operations of exterior algebra are carried over to our setting by componentwise definitions.

Definition 11 *Let $A = \mathrm{cl}[(\alpha_\varepsilon)_\varepsilon] \in \mathcal{G}^{\wedge^k}(X)$, $B = \mathrm{cl}[(\beta_\varepsilon)_\varepsilon] \in \mathcal{G}^{\wedge^l}(X)$ and $\Xi = \mathrm{cl}[(\xi_\varepsilon)_\varepsilon] \in \mathcal{G}^1_0(X)$. We define the exterior derivative, the wedge product and the insertion operator, respectively, by:*

(i) $dA := \mathrm{cl}[(d\alpha_\varepsilon)_\varepsilon] \in \mathcal{G}^{\wedge^{k+1}}(X)$

(ii) $A \wedge B := \mathrm{cl}[(\alpha_\varepsilon \wedge \beta_\varepsilon)_\varepsilon] \in \mathcal{G}^{\wedge^{k+l}}(X)$

(iii) $i_\Xi A := \mathrm{cl}[(i_{\xi_\varepsilon}\alpha_\varepsilon)_\varepsilon] \in \mathcal{G}^{\wedge^{k-1}}(X)$

Of course all the classical relations remain valid in our framework where (in contrast to the distributional setting) in every multilinear operation all factors may be generalized; in particular for $A \in \mathcal{G}^{\wedge^k}(X)$ and $\Xi, \Xi_1, \dots \Xi_k \in \mathcal{G}^1_0(X)$ we have $(\iota_\Xi A)(\Xi_2, \dots, \Xi_k) = A(\Xi, \Xi_2, \dots, \Xi_k)$ and $L_\Xi = d \circ i_\Xi + i_\Xi \circ d$.

A generalized k-form A is called closed if $dA = 0$ and exact if there exists $B \in \mathcal{G}^{\wedge^{k-1}}(X)$ with $dB = A$. Clearly every exact generalized k-form is closed. The converse—as in the smooth case—holds locally:

Theorem 9 *(Poincaré Lemma)*
Let $A \in \mathcal{G}^{\wedge^k}(X)$ closed. Then for each $p \in X$ there exists a neighborhood U of p and $B \in \mathcal{G}^{\wedge^{k-1}}(X)$ such that

$$A|_U = dB|_U\,.$$

Proof. Since it suffices to work in a local chart we may suppose that $U \subseteq \mathbb{R}^n$ is a ball around zero. Let $(\alpha_\varepsilon)_\varepsilon$ denote a representative of A. Then $d\alpha_\varepsilon = n_\varepsilon \in \mathcal{N}^{\wedge^{k+1}}(U)$. Analogous to the classical proof (cf. e.g.[1], 2.4.17) we define an operator $H : \Omega^k(U) \to \Omega^{k-1}(U)$ by

$$H\omega(x)(v_1, \dots, v_{k-1}) = \int_0^1 t^{k-1}\omega(tx)(x, v_1, \dots, v_{k-1})\, dt,$$

where $v_1, \dots, v_{k-1} \in \mathbb{R}^n$. Then $d \circ H + H \circ d = \mathrm{id}$, so $\alpha_\varepsilon = Hd\alpha_\varepsilon + dH\alpha_\varepsilon$ for each $\varepsilon > 0$. It is immediate from the explicit form of H that $H(\mathcal{E}_M^{\wedge^k})(U) \subseteq \mathcal{E}_M^{\wedge^{k-1}}(U)$ and $H(\mathcal{N}^{\wedge^k}(U)) \subseteq \mathcal{N}^{\wedge^{k-1}}(U)$. Thus $H\alpha_\varepsilon \in \mathcal{E}_M^{\wedge^{k-1}}(U)$, $H(d\alpha_\varepsilon) \in \mathcal{N}^{\wedge^k}(U)$ and, consequently, $A = d(HA)$ in $\mathcal{G}^{\wedge^k}(U)$. □

In what follows we suppose X to be oriented. Analogous to Definition 3, for $K \subset\subset X$, $A \in \mathcal{G}^{\wedge^n}(X)$ we define the integral of $A := \mathrm{cl}[(\alpha_\varepsilon)_\varepsilon]$ over K by

$$\int_K A := \mathrm{cl}[(\int_K \alpha_\varepsilon)_\varepsilon].$$

For A compactly supported we set $\int_X A = \int_L A$ where L is any compact neighborhood of supp(A). This notion of integration is compatible with the one introduced by Marsden for compactly supported distributional n-forms (cf. [26], 2.6). More precisely, let $\alpha \in \Omega^n_c(X)'$ and $A \approx \alpha$. Then $\int A \approx \int \alpha$.

Also, Stokes' theorem is easily generalized to the new setting by componentwise application of the classical theorem.

Theorem 10 *Let X be a manifold with boundary and $A \in \mathcal{G}^{\wedge^{n-1}}(X)$ with compact support. Then*

$$\int\limits_X dA = \int\limits_{\partial X} A$$

$\square$

Let us now turn to the task of generalizing symplectic geometry. Let (X, ω) be a symplectic manifold, i.e., suppose that X is furnished with smooth nodegenerate and closed 2-form ω. Generalizing ω to be distributional or even an element of $\mathcal{G}^{\wedge^2}$ does not seem feasible since in that setting a distributional analogue of Darboux' theorem is not attainable (cf. [26], §7). However, by Theorem 7, $\omega \in \Omega^2(X) \subseteq \mathcal{G}^{\wedge^2}$ induces a $\mathcal{G}(X)$-bilinear alternating map $\mathcal{G}^1_0(X) \times \mathcal{G}^1_0(X) \to \mathcal{G}(X)$. This in turn allows us to define the following extension of the classical isomorphism between vector fields and one-forms induced by ω:

$$\begin{array}{rcl} \omega_\flat : \mathcal{G}^1_0(X) & \to & \mathcal{G}^0_1(X) \\ \omega_\flat(\Xi)(H) & := & \omega(\Xi, H). \end{array}$$

This map is even a $\mathcal{G}(X)$-linear isomorphism. We denote its inverse by $\omega_\sharp$ and set $\Xi^\flat = 2\omega_\flat(X)$ and $A^\sharp = \frac{1}{2}\omega_\sharp(A)$. Then we have $\Xi^\flat = i_\Xi \omega \in \mathcal{G}^0_1(X)$, $\Xi^\flat(Z) = -\Xi(Z^\flat) \in \mathcal{G}(X)$ and $A^\sharp(B) = -A(B^\sharp) \in \mathcal{G}(X)$ for $A, B \in \mathcal{G}^0_1(X)$ and $\Xi, Z \in \mathcal{G}^1_0(X)$. Moreover, if $\Xi \approx \xi \in \mathcal{D}^1_0(X)$ resp. $A \approx \alpha \in \mathcal{D}^0_1(X)$ then $\Xi^\flat \approx \xi^\flat$ resp. $A^\sharp \approx \alpha^\sharp$.

For any $H \in \mathcal{G}(X)$ we call the generalized vector field defined by

$$\Xi_H := (dH)^\sharp$$

the generalized Hamiltonian vector field with energy function H. If $H \approx h \in \mathcal{D}'(X)$ then we have $\Xi_H \approx X_h$, where X_h is defined according to [26], Prop. 7.3.

Let $F = \mathrm{cl}[(f_\varepsilon)_\varepsilon]$, $G = \mathrm{cl}[(g_\varepsilon)_\varepsilon] \in \mathcal{G}(X)$. We define the Poisson bracket of F and G by

$$\{F, G\} := \mathrm{cl}[(\{f_\varepsilon, g_\varepsilon\})_\varepsilon].$$

Literally all classical properties carry over. In particular, $\{\,,\,\}$ is antisymmetric, the Jacobi identity holds and we have $\{F, G\} = L_{\Xi_G} F = -L_{\Xi_F} G = -i_{\Xi_F} i_{\Xi_G} \omega$ and $\Xi_{\{F,G\}} = -[\Xi_F, \Xi_G]$. We note that in contrast to the distributitional setting ([26], Prop. 7.4), where ill-defined products of distributions have to be avoided carefully, in our present framework *both* factors F and G may be generalized functions. There is of course a result analogous to Proposition 3 concerning consistency with respect to the smooth resp. distributional setting in the sense of association.

Example 2 We close this section by discussing a simple example from nonsmooth mechanics to indicate the usefulness of the present setting. Let $X = \mathbb{R}^2$ and consider the generalized Hamiltonian function $H(p, q) = \frac{p^2}{2} + D(q)$, where D denotes a generalized delta function in the sense of [16], i.e., we suppose that D possesses a representative δ_ε with $\mathrm{supp}(\delta_\varepsilon) \to \{0\}$, $\int \delta_\varepsilon \to 1$ and $\int |\delta_\varepsilon| \leq C$ for ε small. Clearly, every generalized delta function is associated to δ. Nets δ_ε possessing the above mentioned properties provide a general and flexible means of modeling delta-type singularities (so-called *strict delta nets*, cf. [27], chap. II, §7). The Hamiltonian equations for this setup take the form

$$\dot{p} = -\frac{\partial H}{\partial q} = -D'(q) \qquad \dot{q} = \frac{\partial H}{\partial p} = p\,,$$

leading to

$$\begin{aligned} \ddot{q} + D'[q] &= 0 \\ q(0) = q_0,\ \dot{q}(0) &= \dot{q}_0. \end{aligned} \tag{9}$$

This initial value problem has been studied in detail in [16, 17]. It was shown that provided D satisfies certain growth restrictions, a solution in the Colombeau algebra exists and is unique for arbitrary initial conditions $q_0,\ \dot{q}_0 \in \mathcal{R}$. The limiting behavior of this unique solution will in general depend on the chosen regularization for δ. For example, if we choose $\delta_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ with $\rho \in \mathcal{D}(\mathbb{R})$ we get the picture of pure reflection at the origin, i.e., the unique solution to (9) is associated to the function $t \to \mathrm{sign}(q_0)|q_0 + \dot{q}_0 t|$. (The proof consists in a rather technical analysis of the limiting behavior of the trajectories, establishing that they are neither delayed nor trapped at the origin as $\varepsilon \to 0$.) For generalized delta functions of different type, a more complicated limiting behavior can be observed: For any given finite subset S of $(0, \infty)$ there exists a generalized delta function such that the solutions to (9) with $x_0 \neq 0$ and $\dot{x}_0 = -\mathrm{sign}(x_0)\sqrt{2s}$ with $s \in S$ are trapped at the origin after time $t = -\frac{x_0}{\dot{x}_0}$.

Furthermore, (9) possesses a unique flow which itself is a Colombeau generalized function. Although problematic in the distributional picture ([26], §8), energy conservation in our present setting is immediate from $\{H, H\} = 0$.

The main applications of Colombeau's special algebra on manifolds so far have occurred in general relativity with the purpose of studying singular spacetimes (see [31] for a survey). Based on the framework developed in the present article, a satisfying theory for analyzing the geometry of these spacetimes can be given. A thorough investigation of such generalized semi-Riemannian geometries is deferred to a separate paper ([25]).

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