Biconic semi-copulas with a given section

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Abstract

Inspired by the notion of biconic semi-copulas, we introduce biconic semi-copulas with a given section. Such semi-copulas are constructed by linear interpolation on segments connecting the graph of a continuous and decreasing function to the points (0, 0) and (1, 1). Special classes of biconic semi-copulas with a given section such as biconic (quasi-)copulas with a given section are considered. Some examples are also provided.

Keywords: Biconic copula, Conic copula, Quasicopula, Copula, Linear interpolation

1. Introduction

Semi-copulas have recently gained importance in several areas of research, such as reliability theory, fuzzy set theory and multi-valued logic [6, 10, 12]. Special classes of semi-copulas, such as quasi-copulas and copulas, are widely studied. For instance, quasi-copulas appear in fuzzy set theoretical approaches to preference modeling and similarity measurement [3, 4, 5]. Due to Sklar’s theorem [23], copulas have received ample attention from researchers in probability theory and statistics [14].

Recall that a semi-copula [8, 9] is a function $S : [0, 1]^2 \to [0, 1]$ satisfying the following conditions:

(i) for any $x \in [0, 1]$, it holds that

$$S(x, 0) = S(0, x) = 0, \quad S(x, 1) = S(1, x) = x;$$

(ii) for any $x, x', y, y' \in [0, 1]$ such that $x \leq x'$ and $y \leq y'$, it holds that $S(x, y) \leq S(x', y')$.

In other words, a semi-copula is nothing else but a binary aggregation function with neutral element 1. The functions $T_M$ and $T_D$ given by $T_M(x, y) = \min(x, y)$ and $T_D(x, y) = \min(x, y)$ whenever $\max(x, y) = 1$, and $T_D(x, y) = 0$ elsewhere, are examples of semi-copulas. Moreover, for any semi-copula $S$ the inequality $T_D \leq S \leq T_M$ holds.

A semi-copula $Q$ is a quasi-copula [11, 13, 20] if it is 1-Lipschitz continuous, i.e. for any $x, x', y, y' \in [0, 1]$, it holds that

$$|Q(x', y') - Q(x, y)| \leq |x' - x| + |y' - y|.$$
2. Biconic functions with a given section

In this section we introduce the definition of a biconic function with a given section. We denote the (linear) segment with endpoints \( x, y \in [0, 1]^2 \) as

\[(x, y) = \{\theta x + (1-\theta)y \mid \theta \in [0, 1]\}.

We denote the set of continuous and strictly decreasing functions \( f : [0, 1] \rightarrow [0, 1] \) that satisfy \( f(x) \leq 1 - x \) for any \( x \in [0, 1] \) as \( U \).

Let \( f \in U \), \( f(0) = d' \) and \( d \) be the smallest value in \([0,1]\) such that \( f(d) = 0 \). We introduce the following notations

\[S_f = \{(x, y) \in [0, d] \times [0, 1] \mid y < f(x)\} \]
\[\Delta_{d'} = \Delta_{f([0,d'],(0,1),(1,1))} \]
\[\Delta_d = \Delta_{f((d,0),(1,1),(1,0))} \]
\[F_f = [0, 1]^2 \setminus (S_f \cup \Delta_{d'} \cup \Delta_d).

The sets \( S_f \) and \( F_f \) as well as the triangles \( \Delta_{d'} \) and \( \Delta_d \) are depicted in Figure 1. Let \( C \) be a semi-copula and \( g_C : [0, 1] \rightarrow [0, 1] \) be defined by \( g_C(x) = C(x, f(x)) \). Then the function \( A_{f,g_C} : [0,1]^2 \rightarrow [0,1] \) defined by \( A_{f,g_C}(x,y) = \)

\[
\begin{aligned}
gc(x_0) x, & \quad \text{if } (x, y) \in S_f \setminus \{(0, 0)\}, \\
1 - \frac{1-gc(x_1)}{1-x_1}(1-x), & \quad \text{if } (x, y) \in F_f \setminus \{(1, 1)\}, \\
\min(x, y), & \quad \text{otherwise},
\end{aligned}
\]

where \( (x_0, f(x_0)) \) (resp. \( (x_1, f(x_1)) \)) is the unique point such that \((x, y)\) is located on the segment \( (\{0,0\}, (x_0, f(x_0))) \) (resp. \((x_1, f(x_1)), (1, 1))\), is well defined. The function \( A_{f,g_C} \) is called a biconic function with section \( f,g_C \) since \( A_{f,g_C}(t, f(t)) = gc(t) \) for any \( t \in [0, 1] \), and since it is linear on each segment \( \{(0,0), (t, f(t))\} \) on \( S_f \) as well as on each segment \( \{(t, f(t)), (1, 1)\} \) on \( F_f \).

Note that for \( gc(x) = 0 \), the class of binary conic functions is retrieved [18]. Note also that for \( f(x) = 1 - x \), the class of biconic functions with a given opposite diagonal section is retrieved [16].

Let us introduce, for a biconic function \( A_{f,g_C} \), the functions \( \varphi_f, \hat{\varphi}_f, \psi_{g_C}, \psi_{g_C} : ]0, d[ \rightarrow \mathbb{R} \) defined by

\[\varphi_f(x) = \frac{x}{f(x)}, \quad \hat{\varphi}_f(x) = \frac{1-x}{1-f(x)}, \]
\[\psi_{g_C}(x) = \frac{gc(x)}{x}, \quad \psi_{g_C}(x) = \frac{1-gc(x)}{1-x}.
\]

These functions will be used along the paper.

3. Biconic semi-(resp. quasi-)copulas with a given section

Here, we characterize the class of biconic semi-(resp. quasi-)copulas with a given section.

4. Biconic copulas with a given section

Next, we characterize for specific cases the class of biconic copulas with a given section. A function

Figure 1: Illustration of the sets \( F_f \) and \( S_f \) as well as the triangles \( \Delta_{d'} \) and \( \Delta_d \).

**Proposition 1** Let \( f \in U \) and \( C \) be a semi-copula. The function \( A_{f,g_C} \) defined in (1) is a semi-copula if and only if

(i) the functions \( \varphi_f \psi_{g_C} \) and \( \psi_{g_C} \) are increasing and decreasing, respectively;

(ii) the functions \( \hat{\varphi}_f \hat{\psi}_{g_C} \) and \( \hat{\psi}_{g_C} \) are decreasing and increasing, respectively.

**Proposition 2** Let \( f \in U \) and \( C \) be a quasi-copula. The function \( A_{f,g_C} \) defined in (1) is a quasi-copula if and only if

(i) the conditions of Proposition 1 are satisfied;

(ii) the functions \( \frac{1}{\varphi_f} - \psi_{g_C} \) and \( \varphi_f(1 - \psi_{g_C}) \) are decreasing and increasing, respectively;

(iii) the functions \( \frac{1}{\hat{\varphi}_f} - \hat{\psi}_{g_C} \) and \( \hat{\varphi}_f(1 - \hat{\psi}_{g_C}) \) are increasing and decreasing, respectively.

**Example 1** Let \( f \in U \) and \( C = TL \). One easily verifies that the conditions of Proposition 1 are satisfied and the corresponding biconic function \( A_{f,g_C} \) is a semi-copula. On the other hand, \( A_{f,g_C} \) is a quasi-copula if and only if the functions \( \frac{f(x)}{1-x} \) and \( \frac{1-f(x)}{x} \) are decreasing and increasing on the interval \([0, d[\), respectively.

**Example 2** Let \( f \in U \) and \( C = \Pi \). One easily verifies that the conditions of Proposition 2 are satisfied and the corresponding biconic function \( A_{f,g_C} \) is a quasi-copula, and hence a semi-copula. Consequently, when the considered semi-copula is \( \Pi \), the class of biconic semi-copulas with a given section and the class of biconic quasi-copulas with a given section coincide.
Proposition 3 Let \( f \in \mathcal{U} \) such that \( f \) is piecewise linear and \( C = \Pi \). The function \( A_{f,gc} \) defined in (1) is a copula if and only if the functions \( \varphi_f \) and \( \hat{\varphi}_f \) are convex.

Example 3 Let \( f : [0,1] \to [0,1] \) be defined by \( f(x) = 1 - x \) for any \( x \in [0,1] \), and let \( C = \Pi \). One easily verifies that the functions \( \varphi_f \) and \( \hat{\varphi}_f \) are convex and the corresponding biconic function \( A_{f,gc} \) is a copula, and is given by

\[
A_{f,gc}(x,y) = \begin{cases} \frac{xy}{x+y}, & \text{if } y \leq 1-x, \\ xy - (x+y-1)^2, & \text{otherwise}. \end{cases}
\]

Using the same technique as in [18], Proposition 3 can be generalized for any element from \( \mathcal{U} \).

Proposition 4 Let \( f \in \mathcal{U} \) and \( C = \Pi \). The function \( A_{f,gc} \) defined in (1) is a copula if and only if the functions \( \varphi_f \) and \( \hat{\varphi}_f \) are convex.

Example 4 Let \( f : [0,1] \to [0,1] \) be defined by

\[
f(x) = \begin{cases} (1-2x)^2, & \text{if } x \leq 1/2, \\ 0, & \text{otherwise}, \end{cases}
\]

and let \( C = \Pi \). One easily verifies that the functions \( \varphi_f \) and \( \hat{\varphi}_f \) are convex and the corresponding biconic function \( A_{f,gc} \) is a copula.

Rather than using the product copula, we consider now any copula \( C \) but we will suppose that \( f \) is piecewise linear.

Proposition 5 Let \( f \in \mathcal{U} \) such that \( f \) is piecewise linear, and let \( C \) be a copula. Let \( a \in [0,d] \) be the unique value such that \( f(a) = a \). The function \( A_{f,gc} \) defined in (1) is a copula if and only if

(i) for any \( x_1, x_2, x_3 \in [0,d] \) such that \( x_1 < x_2 < x_3 \), it holds that

\[
\begin{vmatrix} 1 - x_1 & 1 - y_1 & 1 - z_1 \\ 1 - x_2 & 1 - y_2 & 1 - z_2 \\ 1 - x_3 & 1 - y_3 & 1 - z_3 \end{vmatrix} \geq 0 \quad (2)
\]

and

\[
\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \geq 0, \quad (3)
\]

where \( f(x_i) = y_i \) and \( g_C(x_i) = z_i \) for any \( i \in \{1,2,3\} \).

(ii) the function \( \xi : [0,a] \cup [a,d] \to \mathbb{R} \) defined by

\[
\xi(x) = \frac{x - g_C(x)}{x - f(x)}
\]

is decreasing on the interval \([0,a]\) as well as on the interval \([a,d]\).

In order to retrieve the class of conic copulas and the class of biconic copulas with a given opposite diagonal section, we need the following two lemmas.

Lemma 1 Let \( f \) be a real-valued function defined on the interval \([a,b]\). Then it holds that \( f \) is convex if and only if

\[
\begin{vmatrix} 1 - x_1 & 1 - f(x_1) & 1 \\ 1 - x_2 & 1 - f(x_2) & 1 \\ 1 - x_3 & 1 - f(x_3) & 1 \end{vmatrix} \geq 0 \quad (4)
\]

holds for any \( x_1, x_2, x_3 \in [a,b] \) such that \( x_1 < x_2 < x_3 \).

Lemma 2 Let \( g \) be a real-valued function defined on the interval \([a,b]\). Then it holds that \( g \) is concave if and only if

\[
\begin{vmatrix} 1 - x_1 & 1 - g(x_1) & 1 \\ 1 - x_2 & 1 - g(x_2) & 1 \\ 1 - x_3 & 1 - g(x_3) & 1 \end{vmatrix} \geq 0. \quad (5)
\]

holds for any \( x_1, x_2, x_3 \in [a,b] \) such that \( x_1 < x_2 < x_3 \).

Remark 1

(i) For \( g_C(x) = 0 \), inequality (2) is equivalent to the convexity of \( f \) (see Lemma 1) and hence, the class of conic copulas (when the upper boundary curve of the zero-set is piecewise linear) is retrieved [18].

(ii) For \( f(x) = 1 - x \), inequality (2) is equivalent to the concavity of \( g_C \) (see Lemma 2) and hence, the class of biconic copulas with a given opposite diagonal section (when the opposite diagonal section is piecewise linear) is retrieved [16].

5. Conclusions

We have introduced biconic semi-copulas with a given section. We have also characterized the class of biconic quasi-copulas with a given section. Under some assumptions, we have characterized biconic copulas with a given section. Some known classes of semi-copulas, such as binary conic semi-copulas and biconic semi-copulas with a given opposite diagonal section, turn out to be special cases of biconic semi-copulas with a given section.

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References


