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The majority strategy on graphs

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Abstract

In a tree one can find the median set of a profile simply by starting at an arbitrary vertex and then moving to the majority of the profile. This strategy is formulated for arbitrary graphs. The graphs for which this strategy produces always the median set $M(\pi)$, for each profile π, are precisely the median graphs.

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1. Introduction

Most of the centrality notions on graphs were first introduced for trees. The center and centroid of a tree were already introduced and characterized by Jordan [5] in 1869. He proved that the center as well as the centroid of a tree is a single vertex or an edge. The median set of a tree was first characterized by Zelinka [17], and was proven (for trees) to coincide with the centroid. In arbitrary graphs these three notions are quite distinct. A median vertex minimizes the sum of the distances to all other vertices.

A first generalization is to consider a profile, i.e. a sequence of vertices, and to find the median set of the profile (see [6]). In this case a median vertex minimizes the sum of the distances to all the elements of the profile (taking into account multiple ocurrences). Now the median set in a tree is just a path. It is either a single vertex such that in each of its branches there is only a minority of the profile, or it is the maximal path with ends, say, u and v such that exactly half of the profile lies 'left' from u and the other half lies 'right' from v. A simple strategy is available, and belongs probably to the folklore of graph theory. One starts at an arbitrary vertex and moves to the (not necessarily strict) majority of the profile. In the case of a single median vertex x one arrives at x and gets stuck there. In the case of a median path, one gets to this path,

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and then one can still move back and forth along this path, but one cannot leave the path.

In this paper we formulate a Majority Strategy that works for arbitrary graphs and, on trees, reduces to the above strategy. Two questions arise:

(i) does the outcome depend on the vertex where we started our move to majority?

(ii) in which graphs do we find the median set of a profile using this strategy?

In the answers to both questions median graphs play an essential role. Such graphs are characterized by the fact that profiles of length 3 always have a unique median vertex.

2. Preliminaries

All graphs considered in this paper are finite, connected, undirected graphs without loops or multiple edges.

The *distance* $d(u, v)$ between two vertices u and v of a graph $G = (V, E)$ is the usual shortest path distance. A subgraph H of G is *isometric* if the distance between any two vertices in H equals their distance in G, i.e. H inherits its distance function from G. The *interval* $I(u, v)$ between u and v is defined by

$$I(u, v) = \{w \mid d(u, w) + d(w, v) = d(u, v)\},$$

i.e., it consists of all vertices 'between' u and v. We set

$$I(u, v, w) = I(u, v) \cap I(v, w) \cap I(w, u).$$

A *profile* of *length* p on G is a finite sequence $\pi = v_1, v_2, \ldots, v_p$ of vertices of G. We set $p = |\pi|$. A *median* of π is a vertex x minimizing

$$D(x, \pi) = \sum_{1 \leqslant i \leqslant p} d(x, v_i),$$

and the *median set* $M(\pi)$ of π consists of all medians of π. Note that, if $I(u, v, w) \neq \emptyset$, then $M(u, v, w) = I(u, v, w)$.

A *median graph* is a graph G such that $|I(u, v, w)| = 1$, for any three vertices u, v, w of G. Note that this means that each profile of length three has a unique median. Clearly median graphs are connected, and it is easily seen that they are bipartite. Median graphs were independently introduced by Avann [1, 2], Nebeský [16], and Mulder and Schrijver [15]. For a survey of some fifty characterizations of median graphs and several structures related to median graphs (e.g. in terms of ternary algebras, semilattices, set functions, hypergraphs, convexities, geometries, conflict models) the reader is referred to [10]. In [11–14] a structure theory for median graphs is developed. We need some of it in our proofs below. Fast recognition algorithms for median graphs based on this theory were given by Jha and Slutzky [9], and by Hagauer et al. [7], see also [8]. Median graphs are precisely the graphs in which each profile of odd length has a unique median, see [3]. Median graphs are the natural common generalization of trees and hypercubes, see e.g. [14], where the following *Metaconjecture* was formulated: *each property that is 'sensibly' shared by trees and hypercubes is shared by all median graphs.* An interesting characterization of median graphs was given by Chung et al. [4]: median graphs are the optimal graphs in a problem on dynamic search in graphs, where one moves around in the graph in search of vertices in demand, while at any moment the next two demands are known beforehand.

3. The majority strategy

Let $T = (V, E)$ be a tree, and let π be a profile on T. We can find the median set $M(\pi)$ of π as follows. Assume we are in a vertex u, and let v be a neighbor of u. If at least half of the elements of π is nearer to v than to u, then we have $D(v, \pi) \leqslant D(u, \pi)$. So, in moving from u to v, we improve our position. We proceed in this way, and we will arrive at a vertex x of $M(\pi)$. If $|M(\pi)| = 1$, then, for each neighbor y of x, there is a strict minority at the side of y, that is, there are strictly less elements of the profile nearer to y than to x, and we will not move to y. If π is even, then it is possible that we have an edge xy such that at both sides of this edge there lies exactly half of the profile. In this case both x and y must be in $M(\pi)$, and we can move back and forth along the edge xy. Now $M(\pi)$ is a path containing xy, and for each edge on this path exactly half of the profile lies on one side of the edge and exactly half lies on the other side. So we can move freely along this path without increasing the distance to the profile. But for each neighbor z of this path, there is only a strict minority at the side of z and there is a strict majority at the side of the path. So we will not move away from the path. Thus, we can formulate the following stopping rule: either we are stuck at a single vertex, or we visit vertices at least twice, and for each neighbor z of a vertex x that is visited at least twice, either z is also visited at least twice, or at the side of z there is a strict minority.

Loosely speaking, our majority strategy on trees reads as follows: move to majority; at a vertex, where we get stuck or where we get twice, we *park* and we erect a *traffic sign* that reads *median*. When we stop according to our stopping rule, we will have parked and erected traffic signs at all median vertices.

For a profile π and an edge wv in a graph G, we denote by π_{wv} the subprofile of π consisting of the elements of π nearer to w than to v.

The majority strategy on graphs.

Let G be a connected graph, and let π be a profile on G.

- Start at an *initial vertex* v.
- If we are in v and w is a neighbor of v with $|\pi_{wv}| \geqslant \frac{1}{2}|\pi|$, then we move to w.
- We move only to a vertex visited at least twice if there is no alternative.
- We stop when

(i) we are stuck at a vertex v (i.e. $|\pi_{wv}| < \frac{1}{2}|\pi|$, for any neighbor w of v) or

(ii) we have visited vertices at least twice, and, for each vertex v visited at least twice and each neighbor w of v, either w is also visited at least twice or $|\pi_{wv}| < \frac{1}{2}|\pi|$.

- We park at the vertex where we get stuck or at each vertex visited twice and erect a traffic sign reading *median.*

We say that the Majority Strategy *produces*, for π, from initial vertex v, the set of vertices where we parked and erected traffic signs. If the Majority Strategy produces the same set S for π from any initial position, then we just say that it *produces* S for π.

We will now answer the questions for which graphs the Majority Strategy produces the median set for each profile, and for which graphs it produces a set independent of the initial position, for each profile.

Lemma 1. *Let $G = (V, E)$ be a connected graph.* If, *for each profile π on G, the Majority Strategy produces $M(\pi)$, then G is bipartite.*

Proof. First we prove that G is triangle-free. Assume the contrary, and let u, v, w induce a triangle in G. Consider the profile $\pi = u, v, w$. Then $D(x, \pi) = 2$ for x in π, and $D(x, \pi) \geqslant 3$ for any x outside π, so that $M(\pi) = \{u, v, w\}$. We apply the Majority Strategy from initial position v with respect to π. Only u is nearer to u than to v, so we do not move from v to u. Similarly, we do not move to w. Let x be any neighbor of v different from u of w. Then none of the profile is nearer to x than to v, so we do not move to x either. Hence we are stuck at v, that is, from initial position v we do not get all of $M(\pi)$.

Assume that G is not bipartite, and let C be a smallest odd cycle in G of length $2k + 1 > 3$. Then C is an isometric cycle in G. Take any vertex u of C, and let v and w be the vertices on C at distance k from u. Again we take $\pi = u, v, w$. Now we have $D(v, \pi) = D(w, \pi) = k + 1$. Take any vertex x distinct from v and w. Since G is triangle-free, x cannot be adjacent to both v and w, say $d(x, w) \geqslant 2$. Because of the triangle inequality, we have $d(x, u) + d(x, v) \geqslant k$, whence $D(x, \pi) \geqslant k + 2$. So we have $M(\pi) = \{v, w\}$. We apply the Majority Strategy from initial position v with respect to π. Let x be any neighbor of v. If $x = w$, then only x is nearer to x than to v. If $x \neq w$, then only u could be nearer to x than to v. Hence we do not move to x, so that we are stuck at v. Again we do not get all of $M(\pi)$. □

Lemma 2. *Let $G = (V, E)$ be a connected graph. If, for each profile π on G, the Majority Strategy produces $M(\pi)$, then $|I(u, v, w)| = 1$, for any three vertices u, v, w of G with $d(v, w) = 2$.*

Proof. First we prove that G does not contain $K_{2,3}$ as induced subgraph. Assume the contrary, and let $\{x, y\}$ and $\{u, v, w\}$ be the two independent sets of an induced $K_{2,3}$. Note that this $K_{2,3}$ is isometric. Consider the profile $\pi = u, v, w$. Then $D(z, \pi) = 3$, for any common neighbor z of u, v, and w, whereas, G being triangle-free, we have $D(z, \pi) \geqslant 4$, for any z outside of $I(u, v, w)$. So x and y lie in $M(\pi) = I(u, v, w)$. Now we apply the Majority Strategy from initial position x with respect to π. Clearly, only u is nearer to u than to x, so we do not move to u. And, similarly, we do not move to v or w. Since G is triangle-free, any other neighbor of x is not adjacent to either u or v or w. Hence we will also not move to another neighbor of x. Again we are stuck at our initial position and we do not get all of $M(\pi)$.

Take three vertices u, v, w of G with $d(v, w) = 2$, and $k = d(u, v) \leqslant d(u, w)$. If $d(u, v) < d(u, w)$, then, by Lemma 1, we have $d(u, v) = d(u, w) - 2$, and $I(u, v, w) = \{v\}$. So let $d(u, v) = d(u, w) = k$.

If $k = 1$, then u is between v and w, whence $I(u, v, w) = \{u\}$.

Let $k = 2$. Assume that $I(u, v, w) = \emptyset$. Let x be a common neighbor of u and v, let y be a common neighbor of v and w, and let z be a common neighbor of w and u. Consider the profile $\pi = u, v, w$. Since G is bipartite, it follows that the cycle $u \to x \to v \to y \to w \to z \to u$ is isometric. Hence we have $D(u, \pi) = 4 = D(v, \pi) = D(w, \pi)$. Let s be any vertex not in π. Then s cannot be simultaneously adjacent to all three verices of π, say $d(s, w) \geqslant 2$. Because of the triangle inequality, we have $d(s, u) + d(s, v) \geqslant 2$. If we have $d(s, u) + d(s, v) = 2$, then, G being bipartite, we have $d(s, w) = 3$. So $D(s, \pi) = 5$. Otherwise, we have $d(s, u) + d(s, v) \geqslant 3$, and it follows that $D(s, \pi) \geqslant 5$. So $M(\pi) = \{u, v, w\}$. We apply the Majority Strategy from initial position v with respect to π. Let t be any neighbor of v. If u is nearer to t than to v, then we have $d(t, w) = 3$, so that we will not move to t. Otherwise we will for sure not move to t. So we are stuck at v, and we do not get all of $M(\pi)$. Hence $I(u, v, w) \neq \emptyset$. Since $K_{2,3}$ does not occur in G, it follows that $|I(u, v, w)| = 1$.

Let $k \geqslant 3$. Assume that $I(u, v, w) = \emptyset$, and that k is as small as possible under this condition. Because of the minimality of k, we have $I(u, v) \cap I(u, w) = \{u\}$. Let z be a common neighbor of v and w. Then, G being bipartite, we have $d(u, z) = k + 1$. Consider the profile $\pi = u, v, w$. Then we have $D(v, \pi) = D(w, \pi) = k + 2$. For any common neighbor z of v and w, we have $D(z, \pi) = k + 3$. Let x be a vertex not adjacent to both v and w, say $d(x, w) \geqslant 2$. By the triangle inequality, we have $d(x, u) + d(x, v) \geqslant k$. If $d(x, u) + d(x, v) = k$, then x lies in $I(u, v)$. Now, because G is bipartite and $I(u, v) \cap I(u, w) = \{u\}$, it follows that $d(x, w) \geqslant 3$. Otherwise we have $d(x, u) + d(x, v) \geqslant k + 1$. In each case, it follows that $D(x, \pi) \geqslant k + 3$. So $M(\pi) = \{v, w\}$. We apply the Majority Strategy from initial position v with respect to π. Let x be any neighbor of v. If x is adjacent to w, then only w is nearer to x than to v. Otherwise, only u could be nearer to x than to v. So we are stuck at v, and we do not get all of $M(\pi)$. Hence we infer that $I(u, v, w) \neq \emptyset$. If there would exist two distinct vertices x and y in $I(u, v, w)$, then, by minimality of k, there would exist a vertex s in $I(u, x, y)$, which must be a common neigbor of x and y. But then s, v, w, x, y would induce a $K_{2,3}$ in G. This impossibility concludes the proof. □

Note that, in the above proofs, every time that we constructed a contradiction, the following situation arose: we considered the profile $\pi = u, v, w$, we applied the Majority Strategy at initial position v, respectively x, and we got stuck at our initial position. If we would have chosen w, resp. y, as our initial position, then we also would have gotten stuck at the initial position. Thus we have the following result.

Lemma 3. *Let $G = (V, E)$ be a connected graph. If, for any profile π on G, the Majority Strategy produces the same set from any initial position, then G is bipartite and $|I(u, v, w)| = 1$, for any three vertices u, v, w of G with $d(v, w) = 2$.*

Clearly, the above results also hold if we only require that the Majority Strategy produces the same set (resp. the median set) from any initial position, for each profile $\pi = u, v, w$ with $d(v, w) \leqslant 2$.

The next result was first proved in the author's thesis, which was published as [13]. Because this book may not be available everywhere, we include a full proof of the theorem here.

Proposition 4 (Mulder [13, Theorem 3.1.8]). *Let $G = (V, E)$ be a connected triangle-free graph. If $|I(u, v, w)| = 1$, for any three vertices u, v, w of G with $d(v, w) = 2$, then G is a median graph.*

Proof. Note that $K_{2,3}$ is not a subgraph of G.

First we prove that G is bipartite. Assume the contrary, and let C be a smallest odd cycle of length $2k + 1$. Take any vertex u on C. Let v be a vertex at distance k from u on C, and let w be the vertex on C at distance 2 from v and at distance $k - 1$ from u. Clearly, for these vertices u, v and w, we have $I(u, v, w) = \emptyset$, which is impossible. So G is bipartite.

Take any three vertices u, v and w of G, with $d(v, w) \leqslant d(u, v) \leqslant d(u, w)$. If $d(v, w) \leqslant 1$, then, G being bipartite, we infer that v is between u and w, so that $I(u, v, w) = \{v\}$. If $d(v, w) = 2$, then we are done by the condition of the theorem.

So let $d(v, w) \geqslant 3$. The proof that $|I(u, v, w)| = 1$ consists of two steps.

Step 1: $I(u, v, w) \neq \emptyset$.

Assume the contrary, and let u, v, w be such a triple with $d(v, w) + d(u, v) + d(u, w)$ as small as possible. Furthermore, amongst these triples choose a triple with $d(v, w)$ as small as possible. It follows from the minimality of $d(v, w) + d(u, v) + d(u, w)$ that

$$I(v, w) \cap I(v, u) = \{v\},$$

and

$$I(v, w) \cap I(w, u) = \{w\}.$$

Let x be a neighbor of v in $I(v, w)$. Since G is bipartite and x is not in $I(v, u)$, we have

$$d(x, u) = 1 + d(v, u).$$

Then u, x, w are vertices with

$$d(x, w) + d(u, x) + d(u, w) = d(v, w) + d(u, v) + d(u, w).$$

Since $d(x, w) = d(v, w) - 1$, it follows from the minimality of $d(v, w)$ that $I(u, x, w) \neq \emptyset$. Now w lies in $I(x, w) \cap I(w, u) \subseteq I(v, w) \cap I(w, u) = \{w\}$, whence w lies in

$I(x, u)$. Therefore we have

$$1 + d(v, u) = d(x, u) = d(x, w) + d(w, u)$$
$$= d(v, w) - 1 + d(w, u) \geqslant 2 + d(w, u),$$

which contradicts the fact that $d(w, u) \geqslant d(v, u)$. This settles Step 1.

Step 2: $|I(u, v, w)| \leqslant 1$.

Again assume the contrary, and let u, v, w be such a triple with $d(v, w) + d(u, v) + d(u, w)$ as small as possible. Let x and y be two distinct vertices in $I(u, v, w)$. Then it follows from the minimality of $d(v, w) + d(u, v) + d(u, w)$ that

$$I(u, x) \cap I(u, y) = \{u\}.$$

Choose a neighbor u_x of u in $I(u, x)$ and a neighbor u_y of u in $I(u, y)$. Then, since G is triangle-free, we have $d(u_x, u_y) = 2$. Furthermore, we have

$$d(u_x, v) = d(u_y, v) = d(u, v) - 1,$$

and

$$d(u_x, w) = d(u_y, w) = d(u, w) - 1.$$

Let m be the unique vertex in $I(u_x, u_y, v)$, and let m' be the unique vertex in $I(u_x, u_y, w)$. If $m \neq m'$, then the vertices u, m, m', u_x and u_y would induce a $K_{2,3}$ in G, which is forbidden. So $m = m'$.

By Step 1, we can find a vertex z in $I(m, v, w)$. Then z cannot equal both x and y, say $z \neq x$. It follows that z and x lie in $I(u_x, v, w)$. Furthermore, we have

$$d(v, w) + d(u_x, v) + d(u_x, w) = d(v, w) + d(u, v) + d(u, w) - 2,$$

which contradicts the choice of u, v and w. This concludes the proof. □

Thus we have shown the following: if, in a connected graph G, the Majority Strategy either produces the median set $M(\pi)$, or produces the same set from every initial position, for each profile $\pi = u, v, w$ with $d(v, w) \leqslant 2$, then G is a median graph.

The converse, that in a median graph the Majority Strategy always produces the median set $M(\pi)$, is an immediate consequence from the structure theory developed for median graphs (see [11–14]). Here we give a sketch of some of the relevant features of this theory. We omit proofs.

Let $G = (V, E)$ be a median graph, and let uw be an arbitrary edge of G. Let U be the set of vertices nearer to u than to w, and let W be the set of vertices nearer to w than to u. Since G is bipartite, the sets U and W partition V. Let us call the sets U and W the *sides* of the edge uw, where U is the *side* of u, and W is the *side* of w, see Fig. 1. Let xy be any edge between U and W, say with x in U and y in W. Then it turns out that U is also the set of vertices nearer to x than to y, and W is the set of vertices nearer to y than to x. So U and W are also the sides of the edge xy with U the side of x and W the side of y. Moreover, it is proved that the sets U and W are convex. Since G is bipartite, it follows that, for every vertex z in W, there is a geodesic from x to z passing through y, and, for every vertex t in U, there is a geodesic from y to t passing through x.

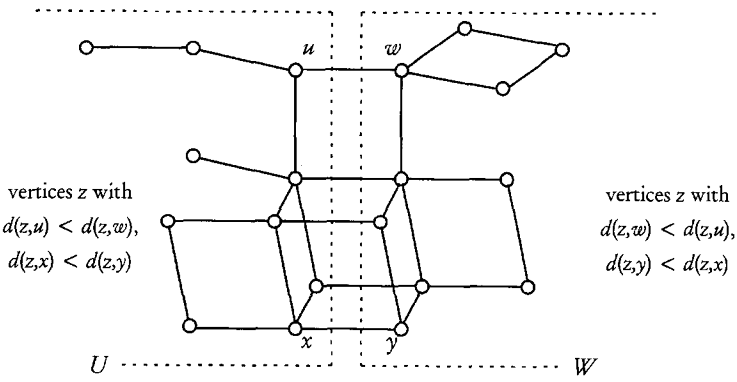

Fig. 1. The sides of edge uw as well as edge xy in a median graph.

Let π be a profile on G. For a set S, let π_S be the subprofile of π consisting of all elements of π in S. If we are in y and we move to x, then we move nearer to all elements of π_U and we move away from all elements in π_W. Assume that there is a majority of π in U. Then, according to the Majority Strategy, we would move from y to x. In this case, it also turns out that $M(\pi)$ is in U, and that, by moving from y to x, we get nearer to $M(\pi)$. So, applying the Majority Strategy, we will always move to the side of an edge where the majority of the profile is, and thus we will get closer to $M(\pi)$, until we are inside $M(\pi)$.

Another feature is that, for a profile π, the median set $M(\pi)$ is precisely the intersection of all sides containing a majority of π. Hence, sides being convex, $M(\pi)$ is convex and thus connected. Moreover, if $M(\pi)$ consists of more than one vertex, then, for any edge xy in $M(\pi)$, both sides of xy contain exactly half of π. And if xy is an edge with x in $M(\pi)$ and y outside $M(\pi)$, then the side of x contains a strict majority of π. So, when applying the Majority Strategy, we will move to $M(\pi)$, and, as soon as we are in $M(\pi)$, we will move around within $M(\pi)$, visit all vertices of $M(\pi)$ at least twice, and never leave $M(\pi)$. That is, we park and erect traffic signs that read "median" at each vertex of $M(\pi)$. Thus we have proved the following theorem.

Theorem 5. *Let $G = (V, E)$ be a connected graph. Then the following conditions are equivalent.*

(i) *G is a median graph,*

(ii) *the Majority Strategy produces the median set $M(\pi)$, for each profile π on G,*

(iii) *the Majority Strategy produces the median set $M(\pi)$, for each profile $\pi = u, v, w$ on G with $d(v, w) \leqslant 2$,*

(iv) *the Majority Strategy produces the same set from any initial position, for each profile on G,*

(v) *the Majority Strategy produces the same set from any initial position, for each profile $\pi = u, v, w$ on G with $d(v, w) \leqslant 2$.*

We note here that, for actually finding $M(\pi)$ in a median graph, the Majority Strategy is not the most efficient procedure. More efficient ones can be constructed using the above mentioned structure theory for median graphs.

Acknowledgements

The idea for this paper arose in Louisville, Kentucky, while the author was visiting F.R. McMorris. We were driving to the University of Louisville along Eastern Parkway. At some stretch there is a beautiful median on Eastern Parkway, with green grass and large trees. Along this median there were traffic signs that read: no parking on the median.

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