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RS-Invariant All-Orders Renormalon Resummations for some QCD Observables

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Abstract

We propose a renormalon-inspired resummation of QCD perturbation theory based on approximating the renormalization scheme (RS) invariant effective charge beta-function coefficients by the portion containing the highest power of $b=\frac{1}{6}(11N-2N_f)$, for SU(N) QCD with N_f quark flavours. This can be accomplished using exact large-N_f all-orders results. The resulting resummation is RS-invariant and the exact next-to-leading order (NLO) and next-to-NLO (NNLO) coefficients in any RS are included. This improves on a previously employed naive resummation of the leading-b piece of the perturbative coefficients which is RS-dependent, making its comparison with fixed-order perturbative results ambiguous. The RS-invariant resummation is used to assess the reliability of fixed-order perturbation theory for the e^+e^- R-ratio, the analogous τ-lepton decay ratio R_τ, and Deep Inelastic Scattering (DIS) sum rules, by comparing it with the exact NNLO results in the effective charge RS. For the R-ratio and R_τ, where large-order perturbative behaviour is dominated by a leading ultra-violet renormalon singularity, the comparison indicates fixed-order perturbation theory to be very reliable. For DIS sum rules, which have a leading infra-red renormalon singularity, the performance is rather poor. In this way we estimate that at LEP/SLD energies ideal data on the R-ratio could determine $\alpha_s(M_Z)$ to three-significant figures, and for the R_τ we estimate a theoretical uncertainty $\delta\alpha_s(m_\tau) \simeq 0.008$ corresponding to $\delta\alpha_s(M_Z) \simeq 0.001$. This encouragingly small uncertainty is much less than has recently been deduced from comparison with the ambiguous naive resummation.

1 Introduction

There has been a great deal of interest recently in the possibility of identifying and resumming to all-orders the Feynman diagrams which dominate the large-order asymptotics of perturbation theory [1–6].

More precisely consider some generic dimensionless QCD observable $D(Q)$, dependent on the single dimensionful scale Q, with a perturbation series

$$D(Q) = a + d_1 a^2 + d_2 a^3 + \cdots + d_k a^{k+1} + \cdots \ , \tag{1}$$

where $a \equiv \alpha_s/\pi$ is the renormalization group (RG) improved coupling. The perturbative coefficients d_k can themselves be written as polynomials of degree k in the number of quark flavours, N_f; we shall assume massless quarks.

$$d_k = d_k^{[k]} N_f^k + d_k^{[k-1]} N_f^{k-1} + \cdots + d_k^{[0]} \ . \tag{2}$$

The leading $d_k^{[k]}$ coefficient corresponds to the evaluation in each order of perturbation theory of a gauge-invariant set of Feynman diagrams containing chains of k fermion bubbles. Techniques for evaluating such diagrams exactly in all-orders have been developed, and rather compact results for QCD vacuum polarization [7, 8], Deep Inelastic Scattering (DIS) sum rules [9], and heavy quark decay widths and pole masses [10, 11] obtained.

The resummation of such diagrams provides a gauge-invariant effective charge in QED, but in QCD one would need to include chains of gluon bubbles and ghosts, and the isolation of a gauge-invariant subset of diagrams providing an analogous QCD effective charge is problematic [12]. One knows on the grounds of gauge invariance, however, that part of the result should be proportional to b^k, where b is the first beta-function coefficient, $b{=}\frac{1}{6}(11N{-}2N_f)$, for SU($N$) QCD. Since for large-$N_f$ one must obtain the QED result one can substitute $N_f{=}(\frac{11}{2}N - 3b)$ in the 'N_f-expansion' of equation (2) to obtain a 'b-expansion'

$$d_k = d_k^{(k)} b^k + d_k^{(k-1)} b^{k-1} + \cdots + d_k^{(0)} \ , \tag{3}$$

where $d_k^{[k]}{=}(-1/3)^k d_k^{(k)}$, so that exact knowledge of the leading-N_f $d_k^{[k]}$ to all-orders implies exact knowledge of the leading-b $d_k^{(k)}$ to all-orders as well.

The existence of so-called infra-red (IR) and ultra-violet (UV) renormalon singularities in the Borel plane at positions $z{=}2l/b$ with l a positive or negative integer, respectively, means that in large-orders the perturbative coefficients are expected to grow as $d_k \sim b^k k!$. As shown in reference [13] this singularity structure leads to the expectation that the 'leading-b' term when expanded in powers of N_f should, asymptotically, reproduce the sub-leading coefficients. That is, expressing $d_k^{(k)} b^k$ as

$$d_k^{(k)} b^k = d_k^{[k]} N_f^k + \tilde{d}_k^{[k-1]} N_f^{k-1} + \cdots + \tilde{d}_k^{[k-r]} N_f^{k-r} + \cdots + \tilde{d}_k^{[0]} \ , \tag{4}$$

one can show that,

$$\tilde{d}_k^{[k-r]} \sim d_k^{[k-r]} \left[1 + O(\frac{1}{k}) \right] \ , \tag{5}$$

so that for fixed r and large k the sub-leading 'N_f-expansion' coefficients are reproduced.

As demonstrated in reference [1] for both the e^+e^- Adler D-function and DIS sum rules this asymptotic dominance of the leading-b term is already apparent in comparisons with the exact next-to-leading order (NLO) and next-to-NLO (NNLO) perturbative coefficients, d_1 and d_2. For instance for SU(N) QCD the first two perturbative coefficients for the Adler D-function are given by [14, 15]

$$
\begin{aligned}
d_1 &= -.115N_f + \left(.655N + \frac{.063}{N}\right) , && (6)\\
d_2 &= .086N_f^2 + N_f\left(-1.40N - \frac{.024}{N}\right) + \left(2.10N^2 - .661 - \frac{.180}{N^2}\right) . && (7)
\end{aligned}
$$

These are to be compared with the leading-b terms

$$
\begin{aligned}
d_1^{(1)}b &= .345b = -.115N_f + .634N , && (8)\\
d_2^{(2)}b^2 &= .776b^2 = .086N_f^2 - .948N_fN + 2.61N^2 . && (9)
\end{aligned}
$$

The subleading N and N_fN, N^2 coefficients approximate well in sign and magnitude those in the exact expressions in equations (6) and (7).

Notice that the level of accuracy with which the sub-leading coefficients $d_k^{[k-r]}$ are reproduced is far in excess of that to be anticipated from the asymptotic expectation of equation (5). This is a rather weak statement which implies only that $d_k^{[k-r]}$ should be reproduced to $O(1/k)$ accuracy for fixed r and large k on expanding $d_k^{(k)}b^k$, whereas the $d_k^{[0]}$ (r=k) coefficient, which is leading in the $1/N$ expansion for a large number of colours, is reproduced accurately for k=1 and k=2.

The above observations suggest that there should be some merit in resumming to all-orders the 'leading-b' terms, even though many features of the approximation remain to be clarified. In a number of recent papers [1–6] such a programme has been carried out. We shall refer mainly to the results of reference [1].

Following [1] we define

$$D^{(L)} \equiv \sum_{k=0}^{\infty} d_k^{(L)} a^{k+1} \tag{10}$$

where $d_k^{(L)} \equiv d_k^{(k)}b^k$ ($d_0^{(L)} \equiv 1$). We can also consider the complementary sum over the sub-leading b terms

$$D^{(NL)} \equiv \sum_{k=1}^{\infty} d_k^{(NL)} a^{k+1} \tag{11}$$

where $d_k^{(NL)} \equiv d_k - d_k^{(L)}$. Hence

$$D = D^{(L)} + D^{(NL)} .$$

In reference [1] the summation in equation (10) was defined using Borel summation, with the IR renormalon singularities principal value (P.V.) regulated. Whilst such a summation can be performed the results for $D^{(L)}$ and $D^{(NL)}$ are separately dependent on the chosen renormalization scheme (RS), the sum of the two being formally RS-invariant. Changing the RS changes the definition of the renormalization group (RG)-improved coupling and hence the 'a' appearing in the summation changes. In the full sum this change is precisely compensated by the RG transformations of the coefficient d_k under the change in RS. However, by restricting oneself to the 'leading-b' piece of the coefficients this exact compensation is destroyed and the resulting sum $D^{(L)}$ is fatally RS-dependent [1, 16]. One response to this, which has been adopted in references [2–6], is to artificially restrict the RG-transformation of 'a' to that contributed by the first term in the beta-function only. With this restriction $D^{(L)}$ is then RS-invariant since RS changes are exactly compensated for at the 'leading-b' level. In our view, however, one can and should do much better than this. For several QCD observables one has exact results for the first two perturbative coefficients d_1 and d_2, usually in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme with renormalization scale μ=Q [14, 15, 17, 18]. What is clearly needed is a resummation in which the exact NLO and NNLO contributions are included, and an approximate resummation of the higher orders performed, in such a way that the full sum is explicitly RS-invariant under the full QCD RG transformations. In this way one would have a test bed for assessing the reliability of fixed-order perturbation theory in any RS by seeing how it differed from the RS-invariant resummed result. As one reduced the energy scale Q (e.g. the centre of mass-energy in e^+e^- annihilation) one could also assess how the reliability of fixed-order perturbation theory deteriorates.

In this paper we shall show in section 2 how such an improved resummation can be carried out, and will use it in section 3 to assess the reliability of fixed-order perturbation theory for the e^+e^- R-ratio, the analogous τ-decay ratio R_τ, and DIS sum rules. Section 4 contains discussion of results and section 5 our conclusions.

2 RS Dependence and RS-Invariants

We begin by briefly reviewing the RS dependence of the coupling 'a' and the perturbative coefficients d_k. We refer the reader to reference [19] for more details.

The RG improved coupling 'a' satisfies the beta-function equation

$$\frac{\mathrm{d}a}{\mathrm{d}\ln\tau} = -a^2(1 + ca + c_2a^2 + \cdots + c_ka^k + \cdots) \equiv -\beta(a)\ , \tag{12}$$

where $\tau \equiv b\ln\frac{\mu}{\Lambda}$, with μ the renormalization scale and Λ the dimensional transmutation mass parameter of QCD. Here b and c are universal with

$$\begin{aligned} b &= \frac{1}{6}(11C_A - 2N_f)\ , \\ c &= \left[-\frac{7}{8}\frac{C_A^2}{b} - \frac{11}{8}\frac{C_AC_F}{b} + \frac{5}{4}C_A + \frac{3}{4}C_F\right]\ , \end{aligned} \tag{13}$$

where for SU(N) QCD C_A=N and C_F=$(N^2$–$1)/2N$.

As shown by Stevenson [20] the RS is labelled by the parameters τ, c_2, c_3, $\ldots$; the higher beta-function coefficients are not universal and characterise the RS. Integrating up equation (12) with a suitable choice of boundary condition [20] one finds that $a(\tau, c_2, c_3, \cdots, c_k)$ is the solution of the transcendental equation

$$\tau = \frac{1}{a} + c\ln\frac{ca}{1+ca} + \int_0^a \mathrm{d}x\left[-\frac{1}{x^2B(x)} + \frac{1}{x^2(1+cx)}\right]\ , \tag{14}$$

where $B(x) \equiv (1 + cx + c_2x^2 + c_3x^3 + \cdots + c_kx^k + \cdots)$.

For consistency of perturbation theory one finds that $d_1(\tau)$, $d_2(\tau, c_2)$, $d_3(\tau, c_2, c_3)$, $\cdots$, $d_k(\tau, c_2, c_3, \cdots, c_k)$, $\cdots$. The combination

$$\rho_0(Q) = \tau - d_1(\tau) \equiv b \ln \frac{Q}{\tilde{\Lambda}} \tag{15}$$

is RS-invariant.

The explicit functional dependence of the d_k on RS is conveniently obtained by considering the special RS, the effective charge (EC) scheme [21], in which d_1=d_2=$\cdots$=d_k=$\cdots$=0, so that D=a, and in this scheme the renormalized coupling is the observable itself. From equation (15) this RS will correspond to the choice of parameter τ=ρ_0 (ensuring d_1=0). To determine the remaining parameters, c_2=ρ_2, c_3=ρ_3, $\cdots$, c_k=ρ_k, $\cdots$, characterizing the EC RS, one proceeds as follows. From equation (12) we have that for two RS's, barred and unbarred,

$$\overline{\beta}(\overline{a}) = \frac{\mathrm{d}\overline{a}}{\mathrm{d}a}\beta(a(\overline{a})) \;, \tag{16}$$

where a, $\overline{a}$ denote the couplings in the respective schemes RS and $\overline{\mathrm{RS}}$. If the barred RS is chosen to be the EC scheme then

$$\overline{\beta}(\overline{a}) = \rho(\overline{a}) = \overline{a}^2(1 + c\overline{a} + \rho_2\overline{a}^2 + \cdots + \rho_k\overline{a}^k + \cdots) \;, \tag{17}$$

with $\overline{a}$=D. Then equation (16) gives

$$\rho(D) = \frac{\mathrm{d}D}{\mathrm{d}a}\beta(a(D)) \;, \tag{18}$$

where $a(D)$ is the inverted perturbation series.

By expanding both sides of equation (18) as power series in D and equating coefficients one obtains

$$\begin{aligned} \rho_2 &= c_2 + d_2 - cd_1 - d_1^2 \\ \rho_3 &= c_3 + 2d_3 - 4d_1d_2 - 2d_1\rho_2 - cd_1^2 + 2d_1^3 \\ &\vdots \end{aligned} \tag{19}$$

Since ρ_0=$\tau - d_1$ is RS-invariant we can use d_1 itself, rather than τ, to label the RS. Rearranging equation (19) we can then obtain

$$\begin{aligned} d_2(d_1, c_2) &= d_1^2 + cd_1 + (\rho_2 - c_2) \\ d_3(d_1, c_2, c_3) &= d_1^3 + \frac{5}{2}cd_1^2 + (3\rho_2 - 2c_2)d_1 + \frac{1}{2}(\rho_3 - c_3) \\ &\vdots \end{aligned} \tag{20}$$

The result for $d_n(d_1, c_2, \cdots, c_n)$ is a polynomial of degree n in d_1 with coefficients involving ρ_n,ρ_{n-1},$\cdots$,c and c_2,c_3,$\cdots$,c_n; such that $d_n(0,\rho_2,\rho_3,\cdots,\rho_n)$=0. The ρ_2,ρ_3,$\cdots$,ρ_n,$\cdots$, are RS-invariants which completely characterise the QCD observable D. They are independent of the energy scale Q but do depend on the number of active quark flavours, N_f. Given just these numbers the perturbative coefficients in any RS can be obtained from equations

(20). To construct RS-invariant resummations the strategy will be to approximate the RS-invariants ρ_k, and then use equations (20) to obtain the approximate perturbative coefficients in any arbitrary RS. In this way invariance under the full RG transformations of QCD is guaranteed.

The ρ_k invariants can be organised as an expansion in b, with

$$\rho_k = \rho_k^{(k)} b^k + \rho_k^{(k-1)} b^{k-1} + \cdots + \rho_k^{(0)} + \rho_k^{(-1)} b^{-1} \, . \tag{21}$$

The b^{-1} term arises from the fact that in a 'regular' RS such as minimal subtraction the d_k are polynomials in b of degree k [22], whereas the corresponding beta-function coefficients c_k are polynomials in b of degree k-1 with additional b^{-1} terms (c.f. the expression for c=c_1 in equation (13)). The RS-invariant combinations in equation (19) in principle could contain arbitrary inverse powers of b, but RG considerations guarantee that only b^{-1} terms remain [22]. Thus $b\rho_k$ is a polynomial of degree k+1 in b.

The effective charge beta-function $\rho(D)$ (equation (17)) will contain Borel plane singularities at the same positions as those in $D(a)$ [23] and hence one should expect a weak asymptotic result analogous to equation (5), with the $\rho_k^{(k)} b^{k+1}$ term asymptotically reproducing the N_f-expansion coefficients of $b\rho_k$. For the Adler D-function and DIS sum rules the level at which this works is again far in excess of that to be anticipated from the asymptotic result. The $\rho_k^{(k)}$ term involves only combinations of the $d_k^{(k)}$, with for instance $\rho_2^{(2)}$=$d_2^{(2)} - (d_1^{(1)})^2$, and so the $\rho_k^{(k)}$ can be obtained to all-orders given the exact leading-N_f all-orders results.

For the Adler D-function ($\tilde{D}$ [1]) one has the exact result for SU(N) QCD (where the "light-by-light" contribution $\tilde{\tilde{D}}$ is excluded, see [1])

$$\begin{aligned} b\rho_2(\tilde{D}) &= -0.0243 N_f^3 + (0.553N - 0.00151\frac{1}{N}) N_f^2 \\ &\quad +(-3.32N^2 + 0.344 + 0.0612\frac{1}{N^2}) N_f \\ &\quad +(3.79N^3 - 1.45N - 0.337\frac{1}{N}) \, . \end{aligned} \tag{22}$$

This is to be compared with the 'leading-b' piece

$$\begin{aligned} b^3 \rho_2^{(2)}(\tilde{D}) &= b^3(d_2^{(2)} - (d_1^{(1)})^2) = 0.656 b^3 \\ &= -0.0243 N_f^3 + 0.401 N N_f^2 - 2.21 N^2 N_f + 4.04 N^3 \, . \end{aligned} \tag{23}$$

Notice the good agreement of the sub-leading NN_f^2, $N^2 N_f$, and N^3 coefficients.

For the DIS sum rules (polarized Bjorken or GLS, $\tilde{K}$ [1]) one has the exact result

$$\begin{aligned} b\rho_2(\tilde{K}) &= -0.0221 N_f^3 + (0.513N + 0.00665\frac{1}{N}) N_f^2 \\ &\quad +(-3.29N^2 + 0.505 + 0.0143\frac{1}{N^2}) N_f \\ &\quad +(3.85N^3 - 1.73N - 0.337\frac{1}{N}) \, , \end{aligned} \tag{24}$$

which is to be compared with

$$\begin{aligned} b^3 \rho_2^{(2)}(\tilde{K}) &= 0.597 b^3 \\ &= -0.0221 N_f^3 + 0.365 N N_f^2 - 2.01 N^2 N_f + 3.68 N^3 \, , \end{aligned} \tag{25}$$

again the NN_f^2, N^2N_f, and N^3 coefficients are well reproduced.

In both cases in the large-N limit (N_f=0) the RS-invariant ρ_2 is approximated to better than 10% accuracy. The 20% level agreement of the sub-leading coefficients does not, unfortunately, guarantee that the overall RS-invariant is reproduced to the same accuracy for all N, N_f since there are large numerical cancellations. For instance for N=3 and N_f=5 one has $\rho_2(\tilde{D})$=-2.98 (exact), whereas $b^2\rho_2^{(2)}(\tilde{D})$=9.64.

3 RS-Invariant Leading-b Resummations

Approximating the RS-invariants ρ_k by $\rho_k^{(L)} \equiv b^k\rho_k^{(k)}$ one can now define the RS-invariant resummation

$$D^{(L*)} \equiv \sum_{k=0}^{\infty} d_k^{(L*)} a^{k+1} \;, \tag{26}$$

where in a general RS $d_k^{(L*)}(d_1, c_2, c_3, \cdots, c_k)$ is obtained by replacing ρ_k in equations (20) by $\rho_k^{(L)}$, so that

$$\begin{aligned} d_0^{(L*)} &= 1 \\ d_1^{(L*)} &= d_1 \\ d_2^{(L*)}(d_1, c_2) &= d_1^2 + cd_1 + (\rho_2^{(L)} - c_2) \\ d_3^{(L*)}(d_1, c_2, c_3) &= d_1^3 + \frac{5}{2}cd_1^2 + (3\rho_2^{(L)} - 2c_2)d_1 + \frac{1}{2}(\rho_3^{(L)} - c_3) \\ &\vdots \;\; \vdots \end{aligned} \tag{27}$$

Notice that, unlike the strict 'leading-b' approximation of equation (10), the NLO coefficient d_1 is now included exactly. If an exact NNLO calculation exists then the exact ρ_2 can be used and ρ_3,ρ_4,$\cdots$ approximated by $\rho_3^{(L)}$,$\rho_4^{(L)}$,$\cdots$, so that d_2 (in any RS) is included exactly. In any case the all-orders sum in equation (26) is formally RS-invariant, and can be compared with the NLO, NNLO, N^{3}LO,$\cdots$,N^{n}LO,$\cdots$ fixed-order perturbative approximations to assess the accuracy of the fixed-order results,

$$D^{(L*)(n)} \equiv \sum_{k=0}^{n} d_k^{(L*)} a^{k+1} \;. \tag{28}$$

The next task is to define the all-orders resummation in equation (26). If we consider the EC RS then a=D and

$$\tau = \rho_0 = b \ln \frac{Q}{\Lambda_{\overline{\rm MS}}} - d_1^{\overline{\rm MS}}(\mu = Q) \;,$$

where for later convenience we have expressed the RS invariant ρ_0 in terms of the $\overline{\rm MS}$ scheme NLO coefficient with μ=Q, $d_1^{\overline{\rm MS}}(\mu = Q)$, which is customarily what is computed, and where $\Lambda_{\overline{\rm MS}}$ is the universal dimensional transmutation parameter of QCD. Equation (14) in the EC scheme with $x^2B(x)$=$\rho(x)$, the EC beta-function of equation (17), then yields

$$\frac{1}{D} + c \ln \frac{cD}{1+cD} = b \ln \frac{Q}{\Lambda_{\overline{\rm MS}}} - d_1^{\overline{\rm MS}}(\mu = Q) - \int_0^D {\rm d}x \left[-\frac{1}{\rho(x)} + \frac{1}{x^2(1+cx)} \right] \;. \tag{29}$$

As discussed in reference [19] the EC beta-function $\rho(x)$ is of fundamental significance since

$$\frac{\mathrm{d}D(Q)}{\mathrm{d}\ln Q} = -b\rho(D(Q)) \;, \tag{30}$$

and so it can be partially reconstructed from measurements of the energy evolution of the observable. Given $\rho(x)$, $D(Q)$ is specified by the solution of the transcendental equation (29). The resummation $D^{(L*)}$ of equation (26) will correspond to the solution of equation (29) with $\rho(x)$ replaced by $\rho^{(L*)}(x)$, where

$$\rho^{(L*)}(x) \equiv x^2(1 + cx + \rho_2 x^2 + \sum_{k=3}^{\infty} \rho_k^{(L)} x^k) \;. \tag{31}$$

For the observables to which we shall apply the resummation exact NNLO results exist and so we have included the exact ρ_2, rather than $\rho_2^{(L)}$.

We can define $\rho^{(L*)}$ using the principal value (P.V.) regulated Borel sum results for $D^{(L)}(a)$ of equation (10) obtained in reference [1].

$$D^{(L)}(a) = P.V. \int_0^{\infty} \mathrm{d}z\, \mathrm{e}^{-z/a} B[D^{(L)}](z) \;, \tag{32}$$

where $B[D^{(L)}](z)$ denotes the Borel transform which potentially contains poles at $z{=}z_l{=}\frac{2l}{b}$ ($l{=}1, 2, 3, \cdots$) corresponding to infra-red renormalons (IR_l), and at $z{=}{-}z_l$ corresponding to ultra-violet renormalons (UV_l). The IR_l singularities are intimately connected with the operator product expansion (OPE) for the observable in question, and the chosen regulation of the IR singularities determines the definition of non-perturbative condensates [24].

In reference [1] results have been derived for the e^+e^- Adler D-function ($\tilde{D}$) and the polarized Bjorken (or GLS) DIS sum rules ($\tilde{K}$). For these Euclidean quantities one can obtain the regulated Borel sum of equation (32) as sums of exponential integral functions $Ei(Fz_l)$ and $Ei(-Fz_l)$, where $F \equiv \frac{1}{a}$. The resulting expressions for $\tilde{D}^{(L)}(F)$ and $\tilde{K}^{(L)}(F)$ split into UV and IR contributions are given in equations (48,49) and equations (52,53) respectively in reference [1]. Results are also obtained for two Minkowski quantities, the e^+e^- R-ratio ($\tilde{R}$) and the analogous τ-decay ratio ($\tilde{R}_\tau$). Expressions for $\tilde{R}^{(L)}(F)$ and $\tilde{R}_\tau^{(L)}(F)$ are given in equations (60,64) and equations (69,70) respectively of reference [1], in terms of generalized exponential integral functions.

Equation (18) at the leading-b level ($\beta(a){=}a^2$) yields

$$\rho^{(L)}(x) = (a^{(L)}(x))^2 \frac{\mathrm{d}D^{(L)}(a)}{\mathrm{d}a}\bigg|_{a=a^{(L)}(x)} \;, \tag{33}$$

where $a^{(L)}(x)$ is the inverse function to $D^{(L)}(a)$, i.e. $D^{(L)}(a^{(L)}(x)){=}x$, and explicitly

$$\rho^{(L)}(x) \equiv x^2(1 + \sum_{k=2}^{\infty} \rho_k^{(L)} x^k) \;. \tag{34}$$

$\rho^{(L)}(x)$ can then be straightforwardly obtained from the corresponding $D^{(L)}(F)$ expressions of [1] for the various observables. For a given x one numerically solves

$$D^{(L)}(F(x)) = x \;, \tag{35}$$

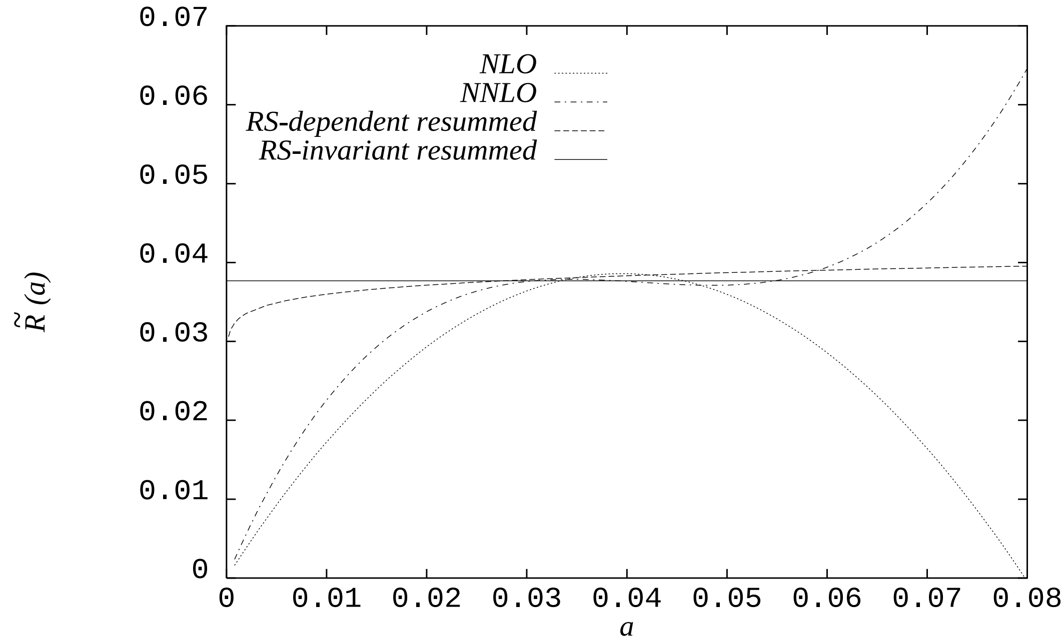

Figure 1(a): NLO and NNLO fixed-order perturbation theory, the naive RS-dependent leading-b resummation, and RS-invariant leading-b resummation, for $\tilde{R}$ at $Q = 91$ GeV plotted against 'a'; $c_2 = 0$ has been assumed.

to obtain $F(x)$, and then from equation (33)

$$\rho^{(L)}(x) = -\frac{\mathrm{d}}{\mathrm{d}F} D^{(L)}(F)\bigg|_{F=F(x)} . \tag{36}$$

Finally comparing equations (31) and (34) one has

$$\rho^{(L*)}(x) = \rho^{(L)}(x) + cx^3 + \rho_2^{(NL)} x^4 , \tag{37}$$

where $\rho_2^{(NL)} \equiv \rho_2 - \rho_2^{(L)}$. This $\rho^{(L*)}(x)$ can be inserted in equation (29) and the integral performed numerically. Given a value of $\Lambda_{\overline{\rm MS}}$ and including the known exact NLO result for $d_1^{\overline{\rm MS}}(\mu = Q)$ one can then solve the transcendental equation (29) for $D{=}D^{(L*)}$. Conversely given $D{=}D^{(L*)}{=}D_{data}$, from the experimental measurement of the observable, one can solve equation (29) for $\Lambda_{\overline{\rm MS}}$. By varying Q, with $\Lambda_{\overline{\rm MS}}^{(N_f)}$ and $d_1^{\overline{\rm MS}}(\mu = Q)$ evaluated with the number of active quark flavours, N_f, changing across quark thresholds, one can study the Q-dependence of $D^{(L*)}(Q)$. The resummed result $D^{(L*)}$ can also be compared with N^{n}LO fixed-order perturbative results. Since d_1 and d_2 are exactly included in any RS one has $D^{(L*)(1)}{=}D^{(1)}$, $D^{(L*)(2)}{=}D^{(2)}$; where $D^{(L*)(n)}$ denotes the truncations of equation (28), and $D^{(1)}$, $D^{(2)}$ denote the exact NLO and NNLO results.

In Figure 1(a) we have plotted as the dashed curve the leading-b resummation $\tilde{R}^{(L)}(a)$ versus the coupling 'a' for the e^+e^- R-ratio with Q=91 GeV, the t'Hooft scheme corresponding to $B(x){=}1 + cx$, and minimal subtraction have been assumed with $\Lambda_{\overline{\rm MS}}^{(5)}{=}200$

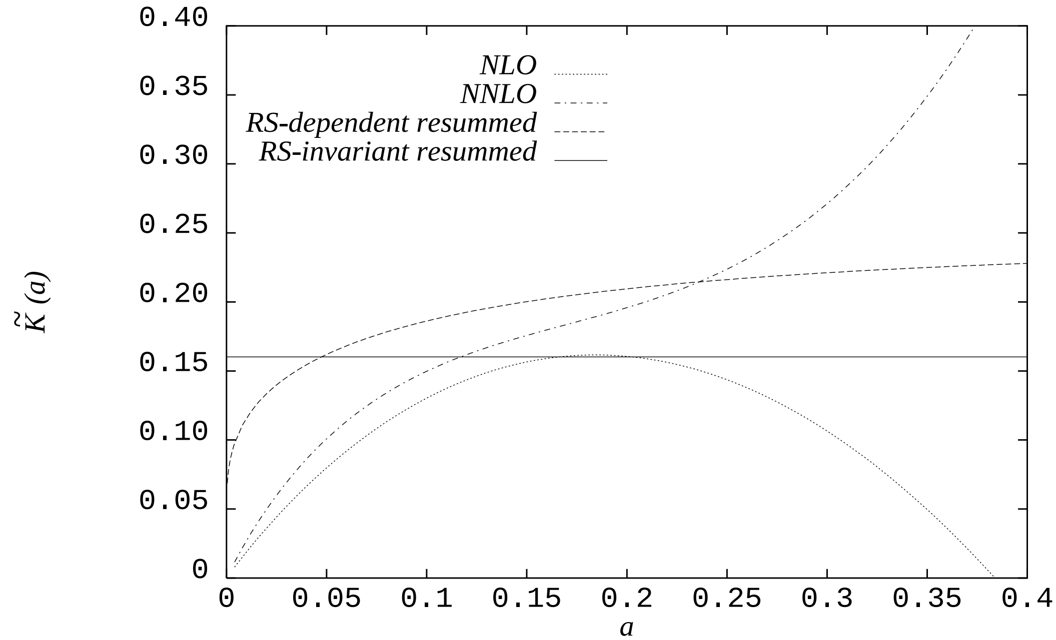

Figure 1(b): As for Figure 1(a) but for $\tilde{K}$ at $Q = 1.5$ GeV.

MeV. There is a monotonic RS-dependence as discussed in reference [1]. Noting that τ is related to 'a' using equation (14) one can use 'a' to label the exact NLO and NNLO approximants, $\tilde{R}^{(1)}(a)$, $\tilde{R}^{(2)}(a, c_2)$. The dotted line shows $\tilde{R}^{(1)}(a)$, and the dashed-dotted line gives $\tilde{R}^{(2)}(a, 0)$. We have chosen c_2=0 to avoid adding an extra axis to the plot. The solid line gives the RS-invariant resummation $\tilde{R}^{(L*)}$. We note that the fixed-order results agree best with the $\tilde{R}^{(L*)}$ resummation in the vicinity of the stationary points with respect to variation of the RS. This is to be anticipated since the Principal of Minimum Sensitivity (PMS) [20] choice of RS avoids the inclusion of potentially large UV logarithms connected with the choice of renormalization scale [19]. A similar statement holds for the NLO and NNLO results in the EC scheme [19], $\tilde{R}^{(1)}(EC)$ and $\tilde{R}^{(2)}(EC)$, corresponding to solutions of equation (29) with $\rho^{(L*)}$ in equation (31) truncated. These are numerically very close to the PMS approximants. The 'optimized' PMS/EC fixed-order NNLO approximant is thus seen to be very close to to the RS-invariant resummed result for the R-ratio at LEP energy, indicating that the approximated effect of including N^{3}LO and higher corrections is small, and thus suggesting that one can in principle accurately determine $\Lambda^{(5)}_{\overline{\rm MS}}$ given ideal data.

In Figures 1(b) and 1(c) the analogous plots for the DIS sum rule $\tilde{K}$ at Q=1.5 GeV, and for the τ-decay ratio $\tilde{R}_\tau$ (Q=m_τ=1.78 GeV) have been given. $\Lambda^{(3)}_{\overline{\rm MS}}$=320 MeV has been assumed.

In contrast to Figure 1(a) the differences between the fixed-order results and the RS-invariant resummations are clearly much larger. Thus at these lower values of Q the significance of N^{3}LO and higher effects is apparently much greater, and the reliability

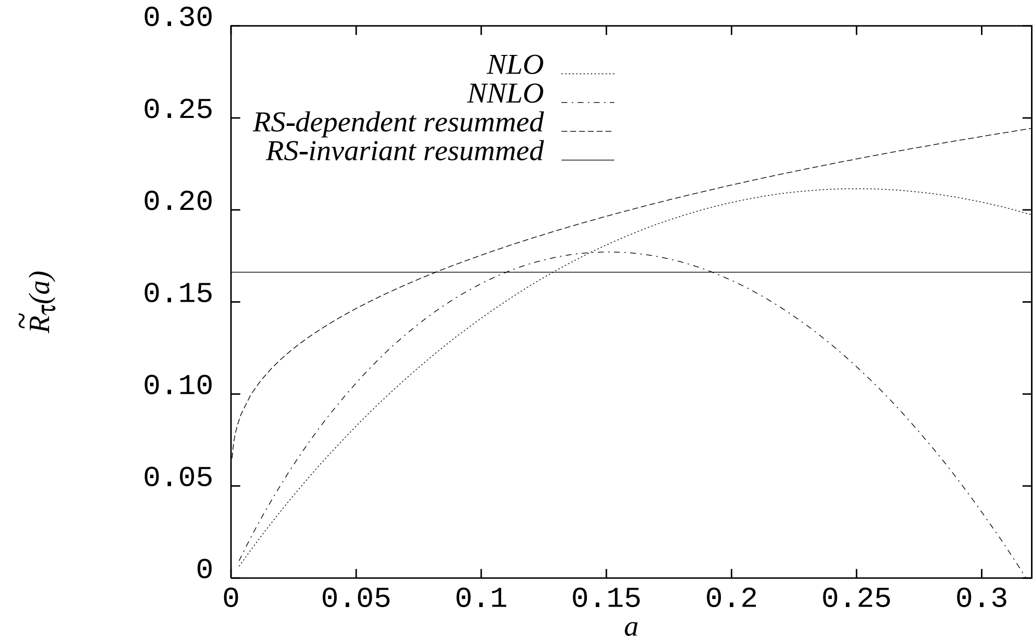

Figure 1(c): As for Figure 1(a) but for $\tilde{R}_\tau$ at $Q = 1.78$ GeV.

with which $\Lambda^{(3)}_{\overline{\rm MS}}$ can be determined correspondingly less. We shall quantify this more carefully in just a moment.

In Figure 2(a) we plot, for the e^+e^- R-ratio at Q=91 GeV, the fixed-order perturbative results $\tilde{R}^{(L*)(n)}(EC)$ (equation (28)) for n=2 (NNLO) and higher orders (crosses) compared with the RS-invariant resummed result $\tilde{R}^{(L*)}$ (dashed line). $\Lambda^{(5)}_{\overline{\rm MS}}$=200 MeV has again been assumed. We could of course have chosen to plot the fixed-order approximants in any RS, for instance $\overline{\rm MS}$ with μ=Q, but as discussed in connection with Figure 1(a), we expect the 'optimized', EC or PMS, choice of RS to approach the resummed result more rapidly. We stress that the fixed-order $\tilde{R}^{(L*)(n)}(EC)$ approximants correspond to the solutions of equation (29) with $\rho(x)$ replaced by the truncation of $\rho^{(L*)}(x)$ in equation (31). Since ρ_2 is included exactly $\tilde{R}^{(L*)(2)}(EC)$=$\tilde{R}^{(2)}(EC)$.

As can be seen from Figure 2(a) the N^{3}LO and higher fixed-order results are indistinguishable from the resummed result with the chosen vertical scale, and there is only a small shift between the NNLO and resummed results. Evidently fixed-order perturbation theory in the EC scheme seems to be working very well for the R-ratio at LEP/SLD energies.

In Figure 2(b) we show a similar plot for $\tilde{R}$ at Q=5 GeV. $\Lambda^{(4)}_{\overline{\rm MS}}$=279 MeV has been assumed. Clearly the approach to the resummed result is somewhat less rapid. The slight oscillation of successive fixed-order approximants above and below the resummed result is explained by the dominance of the UV$_1$ singularity at z=$-\frac{2}{b}$ in the Borel plane, which is the closest to the origin for the R-ratio. This singularity is responsible for alternating sign factorial growth of the perturbative coefficients. Beyond order n=12 the amplitude of the

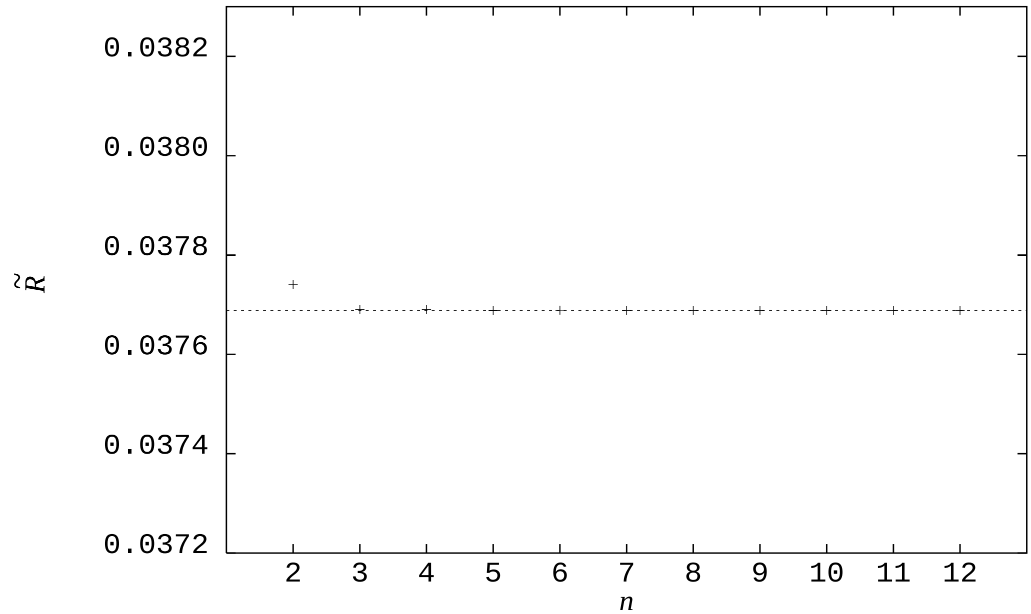

Figure 2(a): Comparison of fixed-order EC perturbation theory (crosses) with the RS-invariant resummation (dashed line) for $\tilde{R}$ at $Q = 91$ GeV.

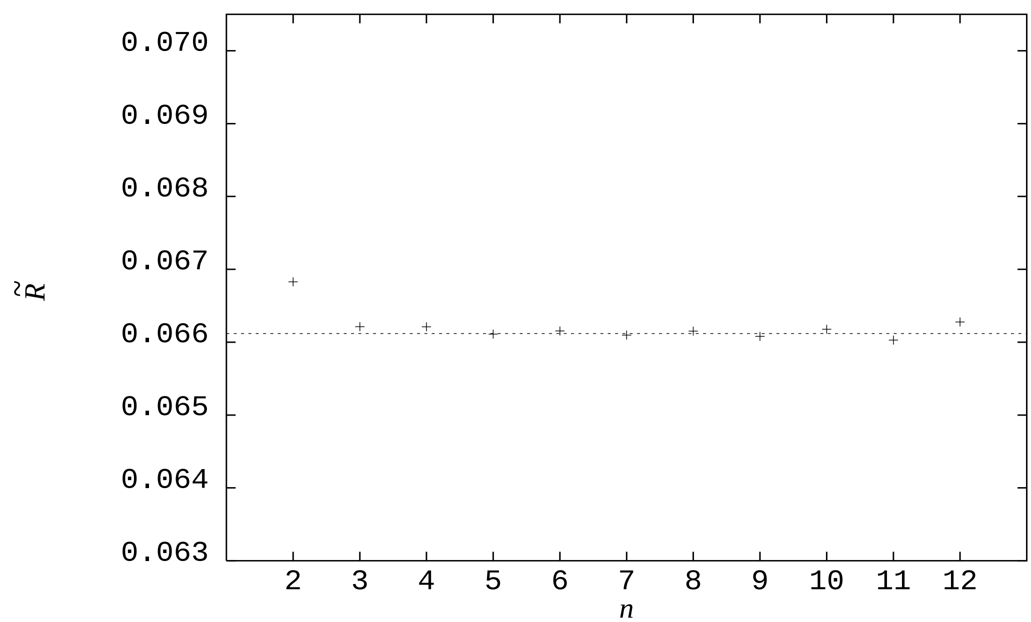

Figure 2(b): As for Figure 2(a) except at $Q = 5$ GeV.

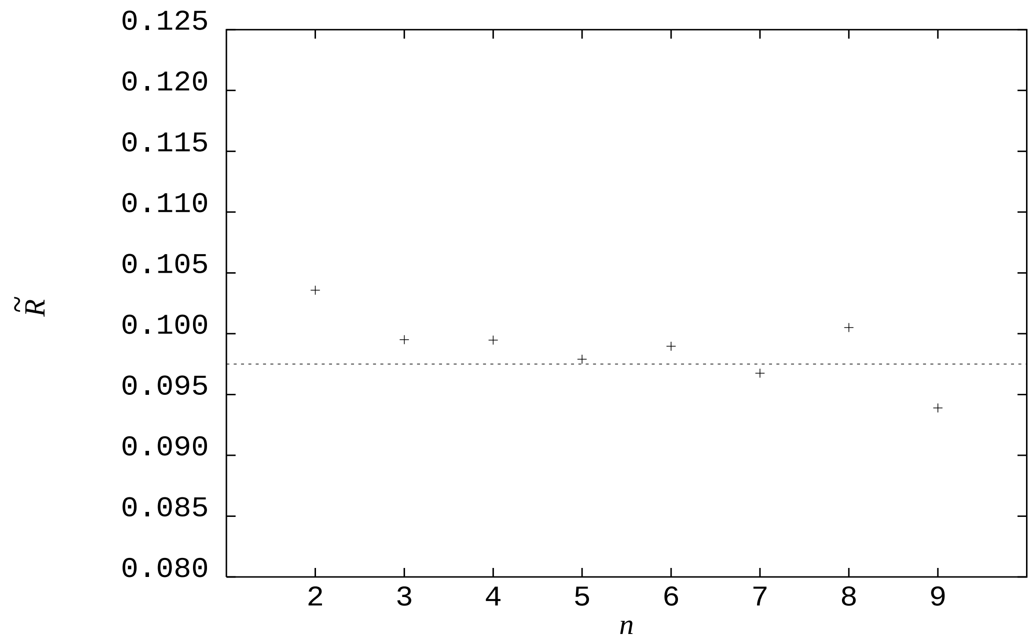

Figure 2(c): As for Figure 2(a) except at $Q = 1.5$ GeV.

oscillations increases dramatically, and the fixed-order approximants diverge increasingly from the resummation. This is precisely what one would expect to see on comparing the Borel sum of an alternating sign asymptotic series with its truncations. We note that a similar oscillating behaviour with eventual wild oscillations setting in would also have been apparent in Figure 2(a) had we used a finer vertical scale. ¿From the large-order behaviour one would not expect the wild oscillations to set in until $n > 50$.

Figure 2(c) finally shows the corrsponding plot for $\tilde{R}$ at Q=1.5 GeV, with $\Lambda^{(3)}_{\overline{\rm MS}}$ as above. The approach to the resummed result is still slower, and the oscillations have only just become established when they increase wildly beyond n=9. Nonetheless even at this low energy fixed-order perturbation theory is approximating the resummed results, albeit much less well.

This reasonable performance of fixed-order perturbation theory for $\tilde{R}$ is to be contrasted with the situation for the DIS sum rules $\tilde{K}$. Figures 3(a), 3(b), and 3(c) are plotted at the same values of Q as the corresponding Figures 2 for $\tilde{R}$, and with the same vertical scales to enable direct comparisons. In Figure 3(a) at Q=91 GeV we see a much slower approach to the resummed result. The fixed-order EC approximations then track the resummed result between sixth and tenth order and then for n=12 there is a dramatic breakdown. The Borel plane singularities nearest the origin for $\tilde{K}$ are now [1] IR_1 at $z=\frac{2}{b}$, and UV_1 at $z=-\frac{2}{b}$. It is the presence of the IR_1 singularity which leads to fixed-sign factorial growth of the perturbative coefficients and a consequent deterioration in the performance of fixed-order perturbation theory. The relative deterioration compared to the R-ratio can be seen even more clearly in Figure 3(b) at Q=5 GeV. There is a

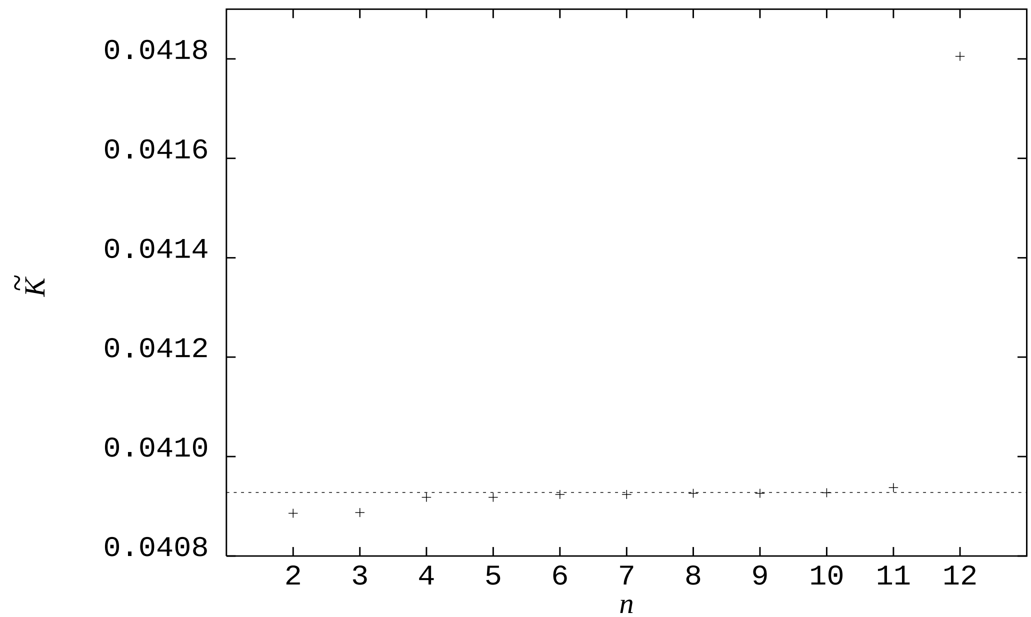

Figure 3(a): Comparison of fixed-order EC perturbation theory (crosses) with the RS-invariant resummation (dashed line) for $\tilde{K}$ at $Q = 91$ GeV.

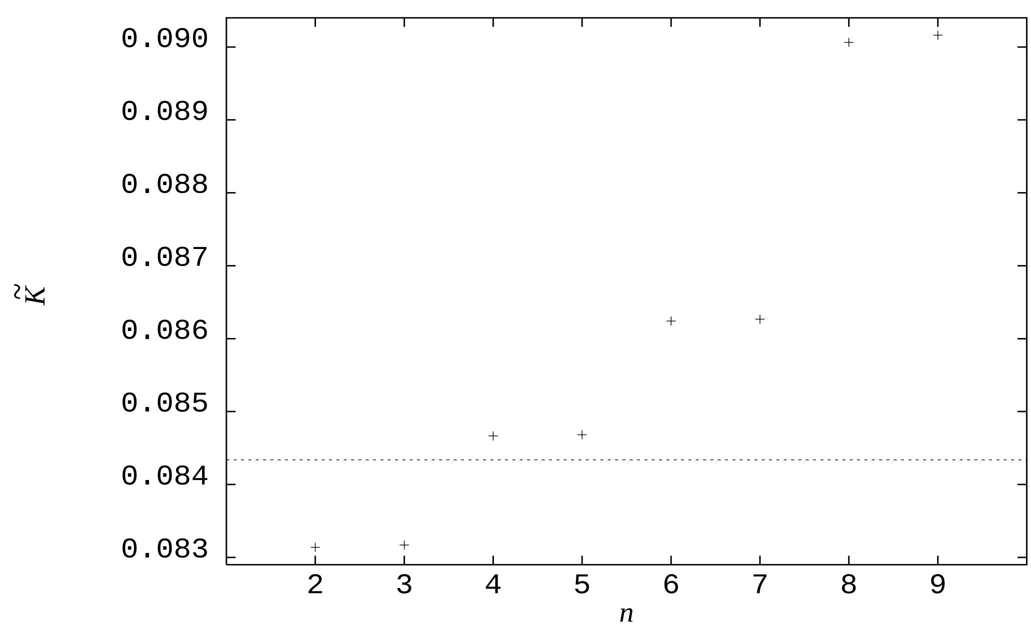

Figure 3(b): As for Figure 3(a) except at $Q = 5$ GeV.

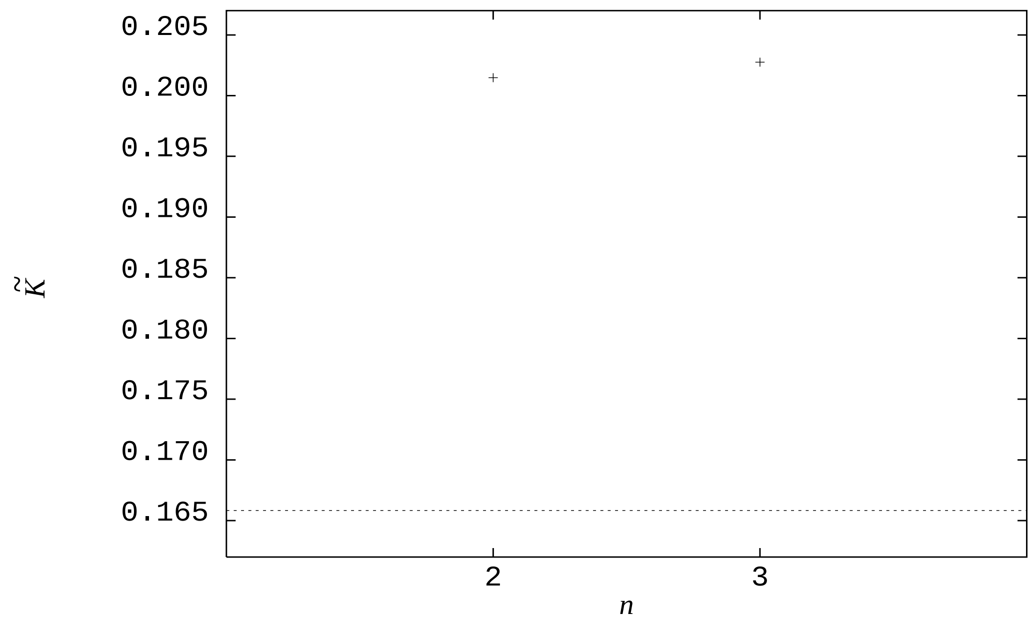

Figure 3(c): As for Figure 3(a) except at $Q = 1.5$ GeV.

monotonic increase in successive orders with no tendency to track the resummed result. The stair-like pattern, with neighbouring odd and even orders roughly similar in low orders, follows from a partial cancellation between the fixed-sign (IR_1) and alternating-sign (UV_1) behaviours.

In Figure 3(c) we see that at Q=1.5 GeV fixed-order EC perturbation theory is a poor approximation to the resummed result for the DIS sum rules. Only n=2 and n=3 are shown since for $n \geq 4$ fixed-order perturbation theory is not defined in the EC scheme. If $\rho(x)$ has a zero at x=x^* (where $x^* > 0$ is the closest zero to the origin) then equation (29) has a solution $D = D^*$, with $D^* < x^*$. This will be the case if the expansion coefficients of $\rho(x)$ have alternating factorial behaviour, at least in either odd or even orders. If, however, the coefficients have fixed-sign factorial growth, as is the case for the DIS sum rules, then $\rho(x)$ will have no positive zeros. In this case equation (29) may fail to have a solution, the condition for this being that in the limit as $D \to +\infty$ the right-hand side of equation (29) is negative.

In Figure 4 we show the analogous plot for the hadronic τ-decay ratio, R_τ. Here evidently $Q = m_\tau = 1.78$ GeV, and the same $\Lambda^{(3)}_{\overline{\mathrm{MS}}}$ as above has been assumed. Fixed-order EC perturbation theory is seen to be working reasonably well with oscillating behaviour around the resummed result which becomes wild for $n \geq 5$. Notice, however, that the performance is much worse than that of $\tilde{R}$ at the comparable $Q = 1.5$ GeV in Figure 2(c).

We can summarize the behaviours exhibited in the foregoing figures by plotting the energy dependence of $\tilde{R}$ and $\tilde{K}$. Figure 5(a) shows $\tilde{R}^{(L*)}$ (solid line), $\tilde{R}^{(1)}(EC)$ (dotted line), and $\tilde{R}^{(2)}(EC)$ (dashed-dotted line), plotted versus $\ln Q/GeV$ over a range equivalent

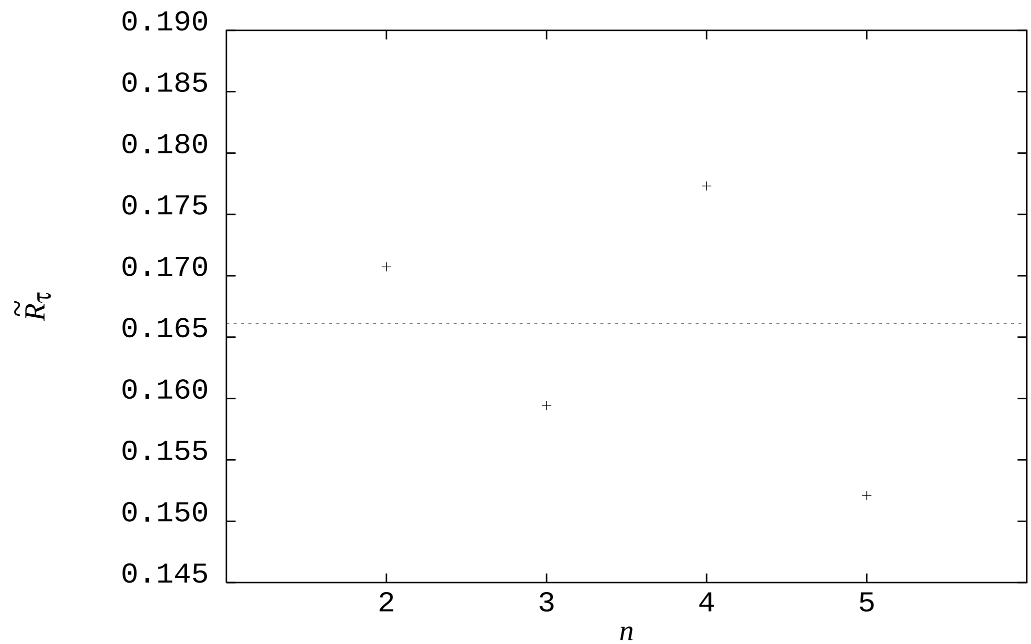

Figure 4: Comparison of fixed-order EC perturbation theory (crosses) with the RS-invariant resummation (dashed line) for $\tilde{R}_\tau$.

to $Q = 1 - 91$ GeV. Flavour thresholds in Q at $m_b = 4.5$ GeV, $m_c = 1.25$ GeV, have been assumed and values of $\Lambda_{\overline{\rm MS}}^{(N_f)}$ chosen as above. Figure 5(b) gives a similar plot for $\tilde{K}$. We note the reasonably satisfactory behaviour for $\tilde{R}$, in particular at all energies the NNLO EC approximation is closer to the resummed result than the NLO, as one would hope. In contrast for $\tilde{K}$ below $Q \sim 3$ GeV the NLO becomes closer than NNLO to the resummed result, making the use of fixed-order perturbation theory dubious. Notice in addition that the vertical scale in Figure 5(b) is much coarser than that in Figure 5(a).

We would finally like to use the RS-invariant all-orders resummation to assess the likely accuracy to which $\Lambda_{\overline{\rm MS}}^{(N_f)}$ can be determined for various observables at various energies.

In Table 1 we have given the $\Lambda_{\overline{\rm MS}}^{(N_f)}$ values obtained by fitting the NLO, NNLO EC fixed-order perturbative and the RS-invariant resummed results to the central values of the data for $\tilde{R}(Q{=}91$ GeV) [25], $\tilde{R}(Q{=}9$ GeV) [26], $\tilde{R}_\tau$, and $\tilde{K}(Q^2{=}5$ GeV^2). The value for $\tilde{R}_\tau$ is that obtained in [6] by averaging those obtained from the leptonic branching ratio [27] and τ-lifetime measurements [28]; we have corrected for the small estimated power corrections [5].

For $\tilde{K}(Q^2 = 5\text{GeV}^2)$ we have taken the GLS sum rule result of the CCFR collaboration [29] corrected by subtracting off the Q^{-2} higher-twist corrections suggested by reference [30], so that

$$\tilde{K}(Q) = \tilde{K}_{CCFR}(Q) + \frac{(0.27 \pm 0.14)}{Q^2} \; . \qquad (38)$$

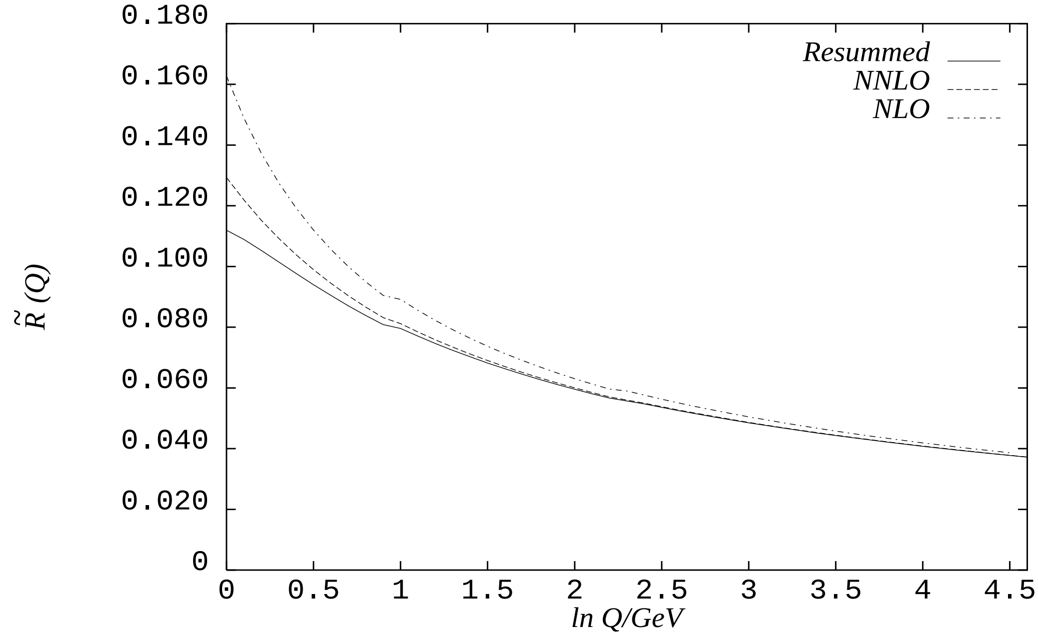

Figure 5(a): NLO, NNLO fixed-order results in the EC scheme, and RS-invariant resummation, for $\tilde{R}$ plotted versus $\ln Q/\mathrm{GeV}$ over the range $1 \leq Q \leq 91$ GeV.

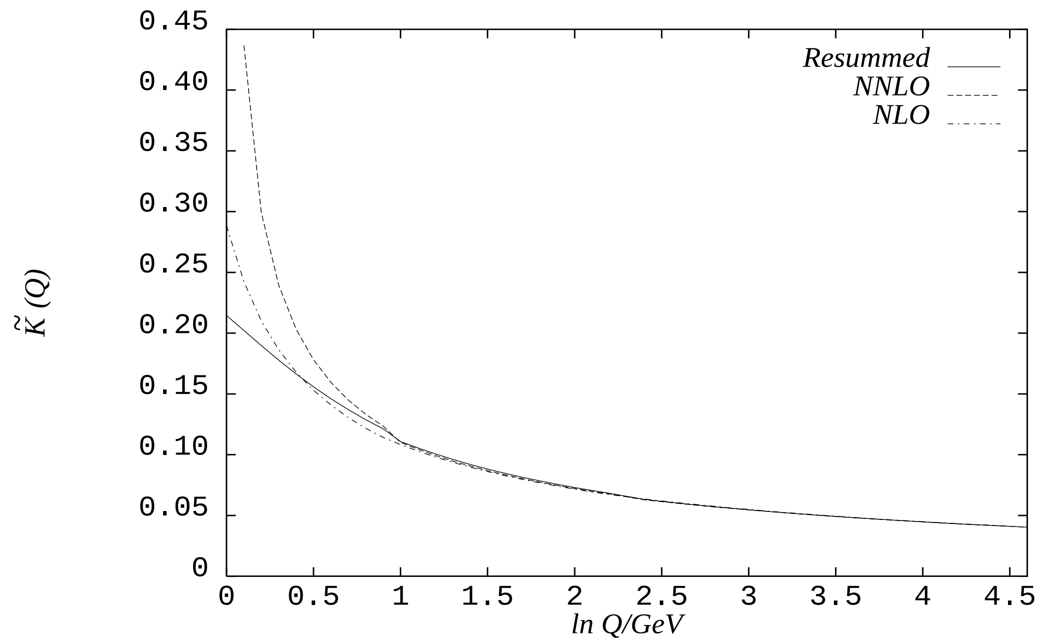

Figure 5(b): As for Figure 5(a) except for $\tilde{K}$.

Observable	Energy Q/GeV	N_f	Data	$\Lambda_{\overline{\rm MS}}^{(N_f)}$/MeV fitted to experiment		
				NLO	NNLO	Resummed
$\tilde{R}$	91	5	0.040 ± 0.004	252^{+190}_{-126}	293^{+228}_{-149}	296^{+232}_{-150}
	9	5	0.073 ± 0.024	399^{+478}_{-322}	516^{+735}_{-423}	537^{+823}_{-443}
$\tilde{R}_\tau$	1.78	3	0.205 ± 0.006	319^{+7}_{-7}	387^{+11}_{-11}	404^{+12}_{-12}
$\tilde{K}$	2.24	3	0.154 ± 0.075	437^{+187}_{-299}	379^{+138}_{-252}	426^{+386}_{-303}

Table 1: Values of $\Lambda_{\overline{\rm MS}}^{(N_f)}$ adjusted to fit the predictions of NLO, NNLO fixed-order results in the EC scheme, and the RS-invariant resummation to the experimental data for $\tilde{R}$, $\tilde{R}_\tau$ and $\tilde{K}$.

The errors have been combined in quadrature.

The results in Table 1 are encouraging in that they indicate rather small differences between the NNLO EC and resummed fits. From these differences one would estimate that one could determine $\Lambda_{\overline{\rm MS}}^{(5)}$ to an accuracy of $\sim \pm 3$ MeV at LEP/SLD energies given ideal data for $\tilde{R}$, corresponding to determining $\alpha_s(M_Z)$ to three significant figures. Needless to say even with ideal data undetermined finite quark mass effects would in fact introduce far larger uncertainties.

At $Q = 9$ GeV $\Lambda_{\overline{\rm MS}}^{(5)}$ would apparently be determined to an accuracy of $\sim \pm 20$ MeV. The data for $\tilde{R}(Q = 91$ GeV) imply NNLO $\alpha_s(M_Z)$ ($\overline{\rm MS}$) values of $\alpha_s(M_Z)$=0.122$\pm$0.012; NNLO EC and resummed are the same to the quoted number of significant figures.

For $\tilde{R}_\tau$ a comparison of the NNLO EC and resummed fits would suggest that $\Lambda_{\overline{\rm MS}}^{(3)}$ could be determined with a precision of $\sim \pm 15$ MeV. One finds $\alpha_s(m_\tau)$=0.320 $\pm$ 0.005 (NNLO EC) and $\alpha_s(m_\tau)$=0.328$\pm$0.005 (resummed). Evolving up from $N_f = 3$ to $N_f = 5$ assuming the flavour thresholds noted above yields $\Lambda_{\overline{\rm MS}}^{(5)}$=$253^{+9}_{-9}$ MeV (NNLO EC) and $\Lambda_{\overline{\rm MS}}^{(5)}$=$267^{+10}_{-10}$ MeV (resummed), corresponding to $\alpha_s(M_Z)$=0.119$\pm$0.001 (NNLO EC) and $\alpha_s(M_Z)$=0.120$\pm$0.001 (resummed). A conservative estimate of the theoretical uncertainty is then $\delta\alpha_s(M_Z)$=0.001.

If taken seriously the above estimate of the accuracy with which $\Lambda_{\overline{\rm MS}}^{(5)}$ ($\alpha_s(M_Z)$) can be determined from R_τ measurements is very reassuring, and clearly indicates that this is indeed potentially the most reliable determination. The uncertainty is somewhat smaller than has been assumed based on more naive estimates of the size of the $O(a^4)$ perturbative coefficient [31]. It is much smaller than $\delta\alpha_s(m_\tau) = 0.05$ inferred by Neubert in reference

[5] based on comparison of the exact $O(\alpha_s^3)$ NNLO perturbative result in the $\overline{\text{MS}}$ scheme with $\mu{=}m_\tau$, and a straightforward resummation of the leading-b terms, which is essentially our $\tilde{R}_\tau^{(L)}$ (c.f. equation (10)), with $a{=}\frac{\alpha_s(m_\tau)}{\pi}$. As can be seen from Figure 1(c) the dashed curve $\tilde{R}_\tau^{(L)}(a)$ lies above the RS-invariant resummation (solid line) for $a{=}\frac{\alpha_s(m_\tau)}{\pi} \simeq 0.12$, and there is a strong 'a' dependence in this region. The NNLO EC result, and indeed the $\overline{\text{MS}}$ $\mu = m_\tau$ NNLO result, are both much closer to the RS-invariant resummation. The implication then is that the rather large difference between the exact fixed-order and naive resummed leading-b results found in [5] is just a reflection of the inadequacy of the naive resummation, which was our original motivation for improving it.

We finally turn to the GLS sum rule results in Table 1. Whilst the $\Lambda_{\overline{\text{MS}}}^{(3)}$ values for NLO, NNLO, and resummed are in reasonable agreement with that obtained for $\tilde{R}_\tau$, we note once again the worrying feature that the NLO result is closer to the resummed than the NNLO. We have also had to assume and correct for sizeable power corrections, based on the modelled suggestion of reference [30]. Also note the very large errors which reflect the difficulty in reconstructing the sum rule by combining data from various DIS experiments [29]. Clearly $\tilde{K}$ will not be competitive with $\tilde{R}_\tau$ as a way of determining $\Lambda_{\overline{\text{MS}}}$.

4 Discussion

Before giving a summary of our main conclusions we would like to discuss several ways in which we could improve or extend the RS-invariant resummations, and mention some technical issues related to them.

The first concerns the analytical continuation between the Euclidean Adler D-function and the Minkowski quantities $\tilde{R}$ and $\tilde{R}_\tau$. This will imply definite relations between the corresponding RS-invariants ρ_k^D, ρ_k^R, $\rho_k^{R_\tau}$. For instance for $\tilde{R}$ one has [32]

$$\begin{aligned} \rho_2^R &= \rho_2^D - \frac{1}{12}b^2\pi^2 \\ \rho_3^R &= \rho_3^D - \frac{5}{12}cb^2\pi^2 \\ \rho_4^R &= \rho_4^D - \frac{1}{12}(8\rho_2^D + 7c^2)b^2\pi^2 + \frac{1}{360}b^4\pi^4 \\ &\vdots \quad \vdots \end{aligned} \tag{39}$$

The procedure we have used to construct $\tilde{R}^{(L*)}$ involves resumming the effective charge beta-function using the exact ρ_2^R and the leading-b approximations to ρ_k^R, $k > 2$. This means the $-\frac{5}{12}cb^2\pi^2$ analytical continuation term in ρ_3^R, or the $-\frac{7}{12}c^2b^2\pi^2$ in ρ_4^R, have been omitted since they are sub-leading in b. Since ρ_2^D is known exactly we could also improve the resummation by using the exact ρ_2^D in evaluating the $-\frac{2}{3}\rho_2^D b^2\pi^2$ term in ρ_4^R. One could envisage an improved resummation $\rho_R^{(L**)}(x)$ incorporating these extra terms.

$$\rho_R^{(L**)}(x) = \rho_R^{(L*)}(x) + \tilde{\rho}_R(x)\ , \tag{40}$$

where the extra terms to $O(x^7)$ are explicitly

$$\tilde{\rho}_R(x) = -\frac{5}{12}cb^2\pi^2x^5 - (\frac{2}{3}\rho_2^{D(NL)} + \frac{7}{12}c^2)b^2\pi^2x^6 + O(x^7) + \cdots\ . \tag{41}$$

This resummation to all-orders can be accomplished in principle by representing $\tilde{R}$ as a contour integral involving $\tilde{D}$ [32]. Using $\tilde{D}^{(L*)}$ in the integrand would formally produce $\tilde{R}^{(L**)}$ corresponding to the above effective charge beta-function $\rho_R^{(L**)}$, but one would have to evaluate $\tilde{D}^{(L*)}$ at complex values of Q. Similar remarks apply to $\tilde{R}_\tau$. One might note that in the NNLO case, where we can compare with the exact result, $\rho_2^{D(L)}$ is only a good approximation to the exact ρ_2 for $N_f \approx 0$ or for large N_f. Hence one could conclude that the uncertainties in the basic approximation are such that the attempted improvement is not warranted. Nonetheless it would be very worrying if any of our conclusions for R, R_τ changed on including these extra terms. We hope to check this in a future work.

Another aspect of the resummations which requires elucidation is the way the resummed $\rho^{(L*)}$ effective charge beta-function is obtained by numerically inverting the P.V. regulated Borel integral representation of $D^{(L)}(a)$, as detailed in equations (35)-(37). From equation (30) we see that $\rho(D(Q))$ is directly related to the Q-evolution of the observable $D(Q)$, and is therefore of central physical importance in studying power corrections. One might then imagine defining

$$\rho(D) = \text{“}\mathit{Reg.}\text{''} \int_0^\infty \mathrm{d}z \, \mathrm{e}^{-z/D} B[\rho](z) + \rho_{NP}^{Reg}(D) \; , \tag{42}$$

where $B[\rho]$ denotes the perturbatively defined Borel transform of ρ. This will contain singularities at the same positions in the Borel plane as $D(a)$ [23], and to control the IR_l infra-red renormalon singularities on the positive-z axis the integral will have to be regulated, denoted “*Reg.*”. There will be an additional $\rho_{NP}^{Reg}(D)$ incorporating the power corrections ($e^{-1/D}$ terms) whose precise definition will depend on the chosen method of regulation [24].

If $B[\rho]$ is defined in the leading-b approximation we can then ask if the first term in equation (42) with P.V. regulation exactly reproduces the $\rho^{(L*)}$ defined by numerically inverting the P.V. regulated $D^{(L)}(a)$.

This can be reduced to a simpler problem. Consider

$$\begin{aligned} D(x) &\equiv \mathit{P.V.} \int_0^\infty \mathrm{d}z \, \mathrm{e}^{-z/x} B[D](z) \\ a(x) &\equiv \mathit{P.V.} \int_0^\infty \mathrm{d}z \, \mathrm{e}^{-z/x} B[a](z) \end{aligned} \tag{43}$$

where $B[D]$ denotes the perturbatively defined Borel transform of $D(a)$, and $B[a]$ denotes the perturbatively defined Borel transform of the inverse function of $D(a)$, which can unambiguously be defined by formally transforming the coefficients of the power series $D(a)$. With these definitions one can then ask whether $D(a(x)) = x$ exactly or whether there are additional $e^{-1/x}$ terms. Existing results on such problems are in short supply [34], but unfinished work in progress [35] strongly suggests that the relation $D(a(x)) = x$ does hold exactly. Unfortunately the result probably only holds for P.V. regulation. The pragmatic reason for studying this question is that $B[\tilde{D}^{(L)}](z)$ is given by rather simple expressions as a sum of poles [1], whereas $B[\rho](z)$ will have an extremely complicated form. Hence it is impractical to construct $B[\rho](z)$ directly, and the numerical inversion route is the only possibility.

The properties of the function $\rho(x)$ fix the infra-red properties of the observable. For instance if $\rho(D^*) = 0$ then $D \to D^*$ as $Q \to 0$. It has been suggested [36, 37] that such infra-red freezing is supported by a wide body of indirect phenomenological evidence. In reference [38] the assumption of universal infra-red behaviour of an effective coupling $\alpha_{eff}(k)$ has been used to interrelate power corrections for different observables. It is interesting that for all the observables we have studied in this paper $D^{(L)}(a)$ has a maximum value, D_{max}, say. This means that $\rho^{(L)}(x)$ is undefined for $x > D_{max}$. If ρ is to be defined in the infra-red this is presumably a signal that power corrections have to be included beyond a certain point. An interesting consistency check on this interpretation is that if only ultra-violet renormalon singularities are present then $D^{(L)}(a)$ does not have a maximum. In particular if, as is the case for $\tilde{R}$ and $\tilde{R}_\tau$, the UV singularities are single poles, then $D^{(L)}(a)$ increases monotonically and $\rho^{(L)}(x)$ will be defined for all x. The absence of IR renormalons is consistent with there being no power corrections, or at least they are not constrained by the large-order perturbative behaviour. We hope to take up this question of IR behaviour and constraining the form of power corrections in a later work.

A final underlying issue which needs further clarification is the explanation for the excellent performance of the leading-b approximation itself. For all the cases where exact NNLO QCD calculations exist the leading-b approximation not only gives exact results for perturbative coefficients and ρ_k RS-invariants in the large-N_f limit, but remarkably is also an excellent ($\sim 5\%$ level) approximation in the large-N limit of a large number of colours. As pointed out unfortunately it may be a rather poor approximation in-between these extremes, for N_f=5, N=3, for instance. Although the sub-leading N_f-expansion coefficients are reproduced remarkably well ($\sim 20\%$ level).

A possible Feynman-diagrammatic explanation runs as follows. In the large-N limit of QCD only planar diagrams contribute. 't Hooft has shown that if one restricts oneself to UV-finite planar diagrams perturbation theory converges [39]. As far as perturbative estimates are concerned these diagrams can be discarded, therefore, since they do not contribute to $n!$ growth of the coefficients. The remaining UV-divergent planar diagrams will contain among them diagrams containing chains of gluon self-energy insertions and other structures which must be combined with renormalon diagrams with chains of internal fermion bubbles and other structures to produce a gauge-invariant contribution proportional to a power of b, using the pinch technique or background field method [12]. The planar diagrams of interest are those *not* involved in the construction of a gauge-invariant effective charge, therefore. The hope would be to understand why their contribution is 'small'. This would not, unfortunately, explain why the RS-invariant effective charge beta-function coefficients are reproduced so well in the large-N limit, since this involves a combination of perturbative coefficients and beta-function coefficients. The piece that is so well approximated in the large-N limit is that which dominates in a Banks-Zaks expansion in $\varepsilon \equiv (\frac{11N}{2} - N_f)$ around $\varepsilon = 0$ [40] where asymptotic freedom is lost. This has been studied in the context of IR freezing of observables since for suitably small ε there is expected to be an IR fixed point [37].

5 Conclusions

In this paper we have proposed an improvement of the renormalon-inspired 'leading-b' resummations of QCD perturbation theory which have been previously used by various authors [1–6] to assess the reliability of fixed-order perturbative predictions. Such resummations are RS-dependent under the full QCD RG transformations. To avoid this difficulty the strategy is to approximate the RS-invariant effective charge beta-function coefficients by retaining their 'leading-b' part, which is completely determined by exact all-orders large-N_f results. Fixed-order perturbative approximations in any RS can then be obtained from the approximated RS-invariants by using the exact QCD RG. If the exact NNLO invariant is known it can be included. In this way the resummation includes the exact NLO and NNLO perturbative coefficients in any RS.

The RS-invariant resummation was performed for the e^+e^- R-ratio, R_τ the analogous decay ratio for the tau-lepton, and DIS sum rules. Comparison with fixed-order perturbation theory in the effective charge RS revealed impressive convergence to the resummed result for the e^+e^- R-ratio at LEP/SLD energies, $Q = 91$ GeV. As the value of Q was reduced oscillatory behaviour of the fixed-order results above and below the resummed value was increasingly evident, reflecting the alternating-sign factorial growth of the perturbative coefficients resulting from the dominant UV_1 renormalon singularity. Even at $Q = 1.5$ GeV the resummed value was reasonably approximated until ninth order perturbation theory.

For $R_\tau(Q = m_\tau)$, which is also UV_1 dominated, there was also a satisfactory approximation to the resummed value, although with much larger oscillations than for $\tilde{R}$ at a comparable value of Q, and with an earlier breakdown of perturbation theory beyond fifth-order.

In contrast DIS sum rules which have an IR_1 infra-red singularity exhibited much less satisfactory behaviour with successive orders moving steadily away from the resummed result, reflecting the fixed-sign factorial growth of the coefficients.

Using the difference between the exact NNLO EC approximation and the resummation to estimate the uncertainty with which $\Lambda_{\overline{\mathrm{MS}}}$ could be determined indicates that for $\tilde{R}$ at $Q = 91\mathrm{GeV}$ $\alpha_s(M_Z)$ could be determined to three significant figures with ideal data.

For $\tilde{R}_\tau$ one concludes that $\delta\alpha_s(M_Z) = 0.001$ from the NNLO-resummed difference. This is a much smaller uncertainty than deduced by Neubert [5] from a comparison with the naive RS-dependent leading-b resummation. The RS-dependence means that the naive resummation is sensitive to the $\overline{\mathrm{MS}}$ scheme $\alpha_s(m_\tau)$ being assumed for the coupling. Other *a priori* reasonable choices would dramatically change the resummed result, and hence we would argue that this estimate of the uncertainty is too pessimistic.

We regard the impressive performance of fixed-order QCD perturbation theory for the UV-renormalon dominated quantities as the key result of this analysis.

Various technical issues related to the resummation and possibilities for future developments were also discussed.

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