Solution of Linear and Nonlinear PDEs by the He's Variational Iteration Method

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Abstract: -Using the He's variational iteration method, it is possible to find the exact solutions or better approximate solutions of the partial differential equations. In this method, a correction functional is constructed by a general Lagrange multiplier, which can be identified via variational theory. In this paper, this method is used for solving a nonlinear partial differential equation, a three-dimensional linear parabolic partial differential equation and the one-dimensional parabolic-like equation with variable coefficients and with given initial conditions. The solutions obtained by VIM show the accuracy and efficiency of the method.

Key-Words: - Variational Iteration Method, Linear and Nonlinear Partial Differential Equations, Lagrange multiplier, Analytical solutions.

1. Introduction

It is well known that there are many nonlinear partial differential equations which are used in the study of several fields for example physics, mechanics, etc. The solutions of these equations can give more understanding of the described process. But because of the complexity of the nonlinear partial differential equations and the limitations of mathematical methods, it is difficult to obtain the exact solutions for these problems. Thus, this complexity hinders further applications of nonlinear partial differential equations [1, 2]. A broad class of analytical and numerical methods were used to handle these problems such as Backlund transformation [3], Hirota's bilinear method [4, 5], Darboux transformation [6], Symmetry method [7], the inverse scattering transformation [8], the tanh method [9, 10], the Adomian decomposition method [11, 12], the improved Adomian decomposition method [13], the exp-function method [14] and other asymptotic methods [15] for strongly nonlinear equations. In 1978, Inokuti et al. [16] proposed a general use of Lagrange multiplier to solve nonlinear problems, which was intended to solve problems in quantum mechanics. Subsequently, in 1999, the variational iteration method (VIM) was first proposed by Ji-Huan He [17, 18]. The idea of the VIM is to construct an iteration method based on a correction functional that includes a generalized Lagrange multiplier. The value of the multiplier is chosen using variational theory so that each iteration improves the accuracy of the solution.

The initial approximation (trial function) usually includes unknown coefficients which can be determined to satisfy any boundary and initial conditions. This method is now widely used by many researchers to study linear and nonlinear partial differential equation [19, 20]. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise a few approximations can be used for numerical purposes.

In this paper, we have applied the variational iteration method (VIM) [21] to solve a nonlinear partial differential equation [22], three-dimensional linear parabolic equation [23] and the one-dimensional parabolic-like equation with variable coefficients [24] and with given initial conditions. The main advantage of the method is the fact that it provides its user with an analytical solution.

2. Analysis of the method

We begin by considering a differential equation in the formal form:

$$Lu + Nu = g(x,t), \qquad (1)$$

where L and N are linear and nonlinear operators, respectively, and $g(x,t)$ is the source inhomogeneous term. He's [17, 18] introduced the variational iteration method where a correction

functional for (1) can be written as
$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)(Lu_n(x,\xi) + N\widetilde{u_n}(x,\xi) - g(x,\xi))\,d\xi,$ (2)

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\widetilde{u_n}$ is a restricted variation which means $\delta\,\widetilde{u_n} = 0$.

It is obvious now that the main steps of He's variational iteration method require first the determination of the Lagrangian multiplier λ that will be identified optimally. Having determined the Lagrangi multiplier, the successive approximations $u_{n+1}, n \geq 0$, of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution is

$$u = \lim_{n\to\infty} u_n. \quad (3)$$

In other words, the correction functional (2) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

3. Applications

In this section, we will solve three problems using the variational iteration method.

3.1. We consider the nonlinear partial differential equation [22]:

$$u_t + uu_x = x + xt^2, \quad (4)$$

with initial condition $u(x,0) = 0$.

The equation (4) can be written in the formal form (1), where the notations $Lu = \frac{\partial u}{\partial t}$, $Nu = u\frac{\partial u}{\partial x}$ and $g(x,t) = x + xt^2$ symbolize the linear, nonlinear and inhomogeneous terms, respectively. The correction functional for the equation (4) reads:

$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)(\frac{\partial}{\partial \xi}(u_n(x,\xi)) + \widetilde{u_n}(x,\xi)\frac{\partial}{\partial x}(u_n(x,\xi)) - g(x,\xi))\,d\xi\,,\; n \geq 0.$ (5)

Taking variation with respect to the independent variable u_n, noticing that $\delta N(\widetilde{u_n}) = 0$,

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\xi)(\frac{\partial}{\partial \xi}(u_n(x,\xi)) + \widetilde{u_n}(x,\xi)\frac{\partial}{\partial x}(u_n(x,\xi)) - g(x,\xi))\,d\xi$$
$$= \delta u_n(x,t) + [\lambda(\xi)\delta u_n(x,\xi)]_{\xi=t} - \int_0^t \lambda'(\xi)\delta u_n(x,\xi)\,d\xi.$$

This yields the stationary conditions

$$1 + \lambda(\xi)\,|_{\xi=t} = 0, \quad (6)$$

$$[\lambda'(\xi)]_{\xi=t} = 0. \quad (7)$$

Equation (7) is called the Lagrange–Euler equation, and equation (6) is the natural boundary condition. The Lagrange multiplier, therefore, can be identified as $\lambda = -1$. Now by putting the value of $\lambda = -1$ in equation (5), the following variational iteration formula can be obtained as

$u_{n+1}(x,t) = u_n(x,t) - \int_0^t (\frac{\partial}{\partial \xi}(u_n(x,\xi)) + u_n(x,\xi)\frac{\partial}{\partial x}(u_n(x,\xi)) - g(x,\xi))\,d\xi,\; n \geq 0.$ (8)

The zeroth approximation $u_0(x,t) = 0$ is selected by using the given initial value.

Therefore, we obtain the following successive approximations:

$$u_1(x,t) = xt + \frac{1}{3}xt^3,$$
$$u_2(x,t) = xt - \frac{2}{15}xt^5 - \frac{1}{63}xt^7,$$
$$u_3(x,t) = xt - \{Small\ terms\}, \qquad |t| \leq 1,$$
$$u_4(x,t) = xt - \{Small\ terms\}, \qquad |t| \leq 1,$$
$$u_5(x,t) = xt - \{Small\ terms\}, \qquad |t| \leq 1,$$
$$\vdots$$
$$u_n(x,t) = xt - \{Small\ terms\}, \qquad |t| \leq 1,$$

Now using $u(x,t) = \lim_{n\to\infty} u_n(x,t)$, we obtain

$$u(x,t) = xt, \qquad |t| \leq 1.$$

This gives the exact solutions of equation (4).

3.2. The three-dimensional linear parabolic equation

The three-dimensional linear parabolic equation [23] is

$$u_t - u_{xx} + u_{yy} + u = (1+t)\,sinh(x+y), \quad (9)$$

with the initial condition

$$u(x,y,0) = sinh(x+y). \quad (10)$$

The equation (9) can be written in the formal form (1), where the notations $Lu = \frac{\partial u}{\partial t}$, $g(x,y,t) = (1+t)\,sinh(x+y)$ symbolize the linear and inhomogeneous terms, respectively. The correction functional for the equation (9) reads:

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda(\xi)\{\frac{\partial}{\partial \xi}\big(u_n(x,y,\xi)\big) - \frac{\partial^2}{\partial x^2}\big(u_n(x,y,\xi)\big) + \frac{\partial^2}{\partial y^2}\big(u_n(x,y,\xi)\big) + \widetilde{u_n}(x,y,\xi) - g(x,y,\xi)\}\,d\xi\,,\; n \geq 0, \quad (11)$$

where λ is the Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the n^{th}-order approximation. $\widetilde{u_n}$ is consider as the restricted variation, i.e. $\delta\,\widetilde{u_n}=0$.

$$\delta u_{n+1}(x,y,t) = \delta u_n(x,y,t) + \delta\int_0^t \lambda(\xi)\{\frac{\partial}{\partial\xi}\big(u_n(x,y,\xi)\big) - \frac{\partial^2}{\partial x^2}\big(u_n(x,y,\xi)\big) + \frac{\partial^2}{\partial y^2}\big(u_n(x,y,\xi)\big) + \widetilde{u_n}(x,y,\xi) - g(x,y,\xi)\}\,d\xi\,,\ \ n\ge 0.$$

$$\delta u_{n+1}(x,y,t) = \delta u_n(x,y,t)\ + [\lambda(\xi)\delta u_n(x,y,\xi)]_{\xi=t} - \int_0^t \lambda'(\xi)\delta u_n(x,y,\xi)\,d\xi.$$

This yields the stationary conditions

$$1+\lambda(\xi)\mid_{\xi=t} = 0, \quad (12)$$

$$[\lambda'(\xi)]_{\xi=t} = 0. \quad (13)$$

Therefore, solving (12) and (13), the Lagrange multiplier is obtained as $\lambda=-1$. Now by putting the value of $\lambda=-1$ in equation (11), the following variational iteration formula can be obtained as

$$u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t\{\frac{\partial}{\partial\xi}\big(u_n(x,y,\xi)\big) - \frac{\partial^2}{\partial x^2}\big(u_n(x,y,\xi)\big) + \frac{\partial^2}{\partial y^2}\big(u_n(x,y,\xi)\big) + u_n(x,y,\xi) - g(x,y,\xi)\}\,d\xi\,,\ \ n\ge 0. \quad (14)$$

The zeroth approximation $u_0(x,y,t) = sinh(x+y)$ is selected by using the given initial value. We obtain the following successive approximations:

$$u_1(x,y,t) = sinh(x+y) + \frac{t^2}{2\,!}Sinh(x+y)$$

$$u_2(x,y,t) = sinh(x+y) + \frac{t^2}{2!}sinh(x+y) - \frac{t^3}{3\,!}sinh(x+y),$$

$$u_3(x,y,t) = sinh(x+y) + \frac{t^2}{2!}sinh(x+y) - \frac{t^3}{3\,!}sinh(x+y) + \frac{t^4}{4\,!}sinh(x+y),$$

and similarly we have obtained the other approximations:

$$u_4(x,y,t) = sinh(x+y) + \frac{t^2}{2!}sinh(x+y) - \frac{t^3}{3\,!}sinh(x+y) + \frac{t^4}{4\,!}sinh(x+y) - \frac{t^5}{5\,!}sinh(x+y),$$

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$$u_n(x,y,t) = sinh(x+y) + \frac{t^2}{2!}sinh(x+y) - \frac{t^3}{3\,!}sinh(x+y) + \frac{t^4}{4\,!}sinh(x+y) - \frac{t^5}{5\,!}sinh(x+y) + \cdots(-1)^{n+1}\frac{t^{n+1}}{(n+1)\,!}sinh(x+y).$$

Now using $u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t)$, we obtain

$$u(x,y,t) = sinh(x+y) + \frac{t^2}{2!}sinh(x+y) - \frac{t^3}{3\,!}sinh(x+y) + \frac{t^4}{4\,!}sinh(x+y) - \frac{t^5}{5\,!}sinh(x+y) + \cdots$$

$$= sinh(x+y)\Big((1-\frac{t^1}{1!}+\frac{t^2}{2!}-\frac{t^3}{3\,!}+\frac{t^4}{4\,!}-\frac{t^5}{5\,!}+\cdots)+t\Big).$$

Therefore $u(x,y,t) = sinh(x+y)(e^{-t}+t)$ is the exact solution of the equation (9).

3.3. The one-dimensional parabolic-like equation with variable coefficients

We consider the one-dimensional parabolic-like equation with variable coefficients [24]

$$u_t(x,t) - \tfrac{1}{2}x^2u_{xx}(x,t) = 0, \quad (15)$$

with initial condition

$$u(x,0) = x^2. \quad (16)$$

The equation (15) can be written in the formal form (1), where the notations $Lu=\frac{\partial u}{\partial t}$, $Nu = -\frac{1}{2}x^2u_{xx}(x,t)$ and $g(x,t)=0$. The correction functional for the equation (15) reads:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)(\frac{\partial}{\partial\xi}(u_n(x,\xi)) - \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}(\widetilde{u_n}(x,\xi)))\,d\xi,\quad n\ge 0. \quad (17)$$

where λ is the Lagrange multiplier, the subscript n denotes the n^{th} –order approximation and $\widetilde{u_n}$ is considered as the restricted variation, i.e. $\delta\,\widetilde{u_n}=0$.

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Taking variation with respect to the independent variable u_n , we obtain

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\xi)(\frac{\partial}{\partial \xi}(u_n(x,\xi)) - \frac{1}{2}x^2 \frac{\partial^2}{\partial x^2}(\widetilde{u_n}(x,\xi)))\, d\xi, \quad n \geq 0.$$

$$= \delta u_n(x,t) + [\lambda(\xi)\delta u_n(x,\xi)]_{\xi=t} - \int_0^t \lambda'(\xi)\delta u_n(x,\xi)\, d\xi.$$

This yields the stationary conditions

$$1 + \lambda(\xi)|_{\xi=t} = 0, \qquad (18)$$

$$[\lambda'(\xi)]_{\xi=t} = 0. \qquad (19)$$

Therefore, solving (18) and (19), the Lagrange multiplier is obtained as $\lambda = -1$. Now by putting the value of $\lambda = -1$ in equation (17), the following variational iteration formula can be obtained as

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t (\frac{\partial}{\partial \xi}(u_n(x,\xi)) - \frac{1}{2}x^2 \frac{\partial^2}{\partial x^2}(u_n(x,\xi)))\, d\xi, \quad n \geq 0. \qquad (20)$$

The zeroth approximation $u_0(x,t) = x^2$ is selected by using the given initial value. Therefore, we obtain the following successive approximations:

$$u_1(x,t) = x^2 + x^2 t,$$

$$u_2(x,t) = x^2 + x^2 t + x^2 \frac{1}{2!} t^2,$$

$$u_3(x,t) = x^2 + x^2 t + x^2 \frac{1}{2!} t^2 + x^2 \frac{1}{3!} t^3,$$

$$u_4(x,t) = x^2 + x^2 t + x^2 \frac{1}{2!} t^2 + x^2 \frac{1}{3!} t^3 + x^2 \frac{1}{4!} t^4,$$

$$\vdots$$

$$u_n(x,t) = x^2 + x^2 t + x^2 \frac{1}{2!} t^2 + x^2 \frac{1}{3!} t^3 + x^2 \frac{1}{4!} t^4 + \cdots + x^2 \frac{1}{n!} t^n.$$

Now using $u(x,t) = \lim_{n\to\infty} u_n(x,t)$, we obtain

$$u(x,t) = x^2 + x^2 t + x^2 \frac{1}{2!} t^2 + x^2 \frac{1}{3!} t^3 + x^2 \frac{1}{4!} t^4 + \cdots,$$

$$u(x,t) = x^2 \left\{1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \frac{1}{4!} t^4 + \cdots\right\},$$

$$u(x,t) = x^2 e^t.$$

This gives the exact solution of equation (15).

4. Conclusion

We have employed the variational iteration method (VIM) to solve a nonlinear partial differential equation, three-dimensional linear parabolic equation and the one-dimensional parabolic-like equation with variable coefficients and with given initial conditions. The VIM reduced the size of calculations without any need to transform the nonlinear terms. It is obvious that the method gives rapidly convergent successive approximations through determining the Lagrange multiplier. He's variational iteration method gives several successive approximations by using the iteration of the correction functional. The VIM uses the initial values for selecting the zeroth approximation. The results obtained confirm the accuracy and efficiency of the method.

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