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# DIMENSION OF MARKOV TOWERS FOR NON UNIFORMLY EXPANDING ONE-DIMENSIONAL SYSTEMS

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**Abstract.** We prove that a non uniformly expanding one-dimensional system defined by an interval map with an ergodic non atomic Borel probability $\mu$ with positive Lyapunov exponent can be reduced to a Markov tower with good fractal geometrical properties. As a consequence we approximate $\mu$ by ergodic measures supported on hyperbolic Cantor sets of arbitrarily large dimension.

1. **Introduction.** It was argued in [7] that a hyperbolic measure $\mu$ preserved by a $C^2$ smooth surface diffeomorphism $f$ can be approximated by ergodic measures supported on large hyperbolic invariant sets, so describing the dynamics of $(f, \mu)$ by a limiting process. The central idea is that hyperbolic behavior of the linearized system, extended to certain regular neighborhoods, can be combined with non-trivial recurrence and the existence of some positive exponential growth rates, as topological or metrical entropy, to produce a rich and complicated orbit structure characterized by the abundance of horseshoes. In this work we apply these ideas to non uniformly expanding one-dimensional systems to get the following.

**Theorem A** *Let $f$ be a $C^2$ interval transformation with finitely many non degenerate critical points leaving invariant an ergodic non atomic Borel probability $\mu$ with positive Lyapunov exponent. Then there is a sequence of $f$-invariant hyperbolic Cantor sets $\Lambda_n$ and hyperbolic measures $\mu_n$ supported on $\Lambda_n$ such that*

1. *the sequence $\{\mu_n\}$ converges to $\mu$ in the weak topology;*
2. $\dim_{\mathcal{H}}(\Lambda_n) \uparrow \dim_{\mathcal{H}}(\mu)$ *as* $n \to +\infty$
3. *the Hausdorff measure of $\Lambda_n$ is uniformly bounded from below, that is: $\mathcal{H}_{\alpha(n)}(\Lambda_n) \geq a > 0$ for every $n > 0$.*

Compare [7, Theorem S.5.9] and [13]. Throughout this work $\dim_{\mathcal{H}}(X)$ denotes the *Hausdorff dimension of a set* $X$ and $\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(\mu)(X) : \mu(X) = 1\}$ the *Hausdorff dimension of* $\mu$. See [5] for definitions and [14] for a comprehensive introduction to the dimension theory of dynamical systems.

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Approximations of one-dimensional systems by Markov subsets were considered in [6] to compute local dimensions of Borel probabilities left invariant by a piecewise monotonic interval transformation. In contrast with Katok's work [7] and Hofbauer's [6] our approach is semi-local and closer in spirit to induction techniques used in one-dimensional dynamics. Roughly speaking we look for suitable returns to a fixed neighborhood of a point in the support of $\mu$ so inducing a Markov tower structure having the Hausdorff measure as reference measure.

A *Markov tower* is an abstract system $T : (\Delta, m) \circlearrowleft$ defined by a piecewise expanding map of a compact subset $\Delta_0$ onto itself, an integer valued *time of return function* $R$ and a *reference measure* $m$. The subset $\Delta_0 \subset \Delta$ is called the base of the tower. The return time $R$ decomposes $\Delta_0$ into countably many blocks $\Delta_{0,i}$ such that $R_i = R \mid \Delta_{0,i}$ is a positive integer and such that $T^{R_i} : \Delta_{0,i} \longrightarrow \Delta$ is one to one and onto with expansion coefficient bounded from below. Moreover, the map $T^R$ defined as $T^R(x) = T^{R(x)}(x)$ has a Jacobian with respect to $m$ the reference measure having bounded backward and forward non linear distortion, up to certain separation time $s(x, y)$ which is the smallest $n \geq 0$ such that $T^n(x)$ and $T^n(y)$ lies in distinct $\Delta_{0,i}$. The map $T$ is the discrete time semiflow over $(\Delta_0, T^R)$ with respect to the height function $R = R(x)$.

**Theorem B** *Let $(f, \mu)$ be a non uniformly expanding interval transformation satisfying the hypotheses of Theorem A. Then, for every $x_0 \in supp\ \mu$ there is an interval $J$ containing $x_0$ and a compact set $\Delta_0 \subset J$ which is the base of a Markov tower $T : \Delta \circlearrowleft$ with the following properties:*

1. *The Hausdorff measure of $\Delta_0$ is finite and positive. Even more, let $d = \dim_{\mathcal{H}}(\Delta_0)$ be the Hausdorff dimension of the base of the tower, then we can find a constant $C > 1$ such that*
$$C^{-1} \leq \frac{\mathcal{H}_d(\Delta_0 \cap B(x, r))}{r^d} \leq C \tag{1}$$
*for every $x \in \Delta_0$ and $0 < r < 1$;*
2. *The base of the tower has positive $\mu$-measure. Moreover, $\mu$ restricted to $\Delta_0$ is equivalent to the Hausdorff measure and $\mu \mid \Delta_0$ is exactly dimensional with $d_\mu(x) = \dim_{\mathcal{H}}(\Delta_0)$ for $\mu$-a.e. $x \in \Delta_0$ and*
$$\dim_{\mathcal{H}}(\Delta_0) = \frac{h_\mu f}{\int \ln |f'| d\mu}; \tag{2}$$
3. *The time of return function $R = R(x)$ is $\mathcal{H}_d$-integrable, that is:*
$$\int R(x) d\mathcal{H}_d(x) < \infty\ . \tag{3}$$

As a consequence, $\mu$ can be recovered as the unique invariant measure provided by [18, Theorem 1] which is absolutely continuous w.r.t. $\mathcal{H}_d$. The *local dimension of* $\mu$ is defined as the limit
$$d_\mu(x) = \lim_{r \to 0^+} \frac{\ln \mu(B(x, r))}{\ln r}\ , \tag{4}$$
which is proved to exist for almost every point in the support of the measure $\mu$
$$d_\mu(x) = \frac{h_\mu f}{\int \ln |f'| d\mu} \quad \mu - a.e.\ x \in [0, 1]\ . \tag{5}$$

In particular, $\mu \mid \Delta_0$ is exactly dimensional with $d_\mu(x) = \dim_{\mathcal{H}}(\mu)$ almost surely. To my knowledge the formula (5) was first proved in [10] for invariant measures preserved by interval transformations with positive entropy. Compare also [6].

Tower constructions can be followed back to Kakutani and Rokhlin. More recently they have been used by several authors to get a deeper understanding of some statistical properties of systems having large non invariant hyperbolic subsets without being uniformly hyperbolic. The idea is to choose a subset with strong hyperbolic properties observing that returns to this set allows the system to recover the decay of hyperbolicity due to critical points or homoclinic tangencies, for instance. Estimations on rates of convergence to the equilibrium and decay of correlations can be obtained if some information on the tails of the return $R$ time $\{x : R(x) \geq n\}$ is available. See [18].

Theorem B provides a general approach to construct Markov towers for one-dimensional dynamical system with weak expansion properties, looking at the geometrical properties of the base of the tower *regardless of a finer control on the rate of decay of the tail of the return time.* This provides fractal approximations of $\mu$ supported on large horseshoes.

Theorem B also allow us to establish a connection between Ledrappier-Young characterization of smooth measures as those satisfying the Rochlin-Pesin entropy formula $h_\mu f = \int \ln |f'| d\mu$ and existence of induced Markov transformations with integrable return time for an endomorphism $f$ preserving an absolutely continuous measure with positive entropy. Cf. [3], [9], [12].

**Corollary C** *Suppose that $(f, \mu)$ satisfies the hypotheses of Theorem A. Then, the following conditions are equivalent:*

1. *The measure $\mu$ is absolutely continuous;*
2. *It holds the Rochlin-Pesin entropy formula, $h_\mu f = \int \ln |f'| d\mu$;*
3. *Given a point $x_0 \in$ supp $\mu$ there is an interval $J$ such that $f$ induces countably many expanding branches $\{f^{n_i} : J_i \longrightarrow J\}$ defining a Markov transformation with $Leb(J - \bigcup_i J_i) = 0$, whose return time $R = R(x)$ is Lebesgue integrable, i.e. $\sum_i |J_i| < \infty$.*

The present approach can be extended to higher dimensions using Pesin's theory for non invertible smooth endomorphisms of compact manifolds. This remark may sustain a growing believe that Markov tower constructions can be used to get a deeper understanding of geometrical and statistical properties of hyperbolic measures in the setting of Pesin's theory. We refer the reader to [1] and [2] for alternative constructions of Markov structures for multidimensional non uniformly expanding endomorphisms.

2. **Outline of the proof: pseudo-Markov property for interval transformations.** The proof of Theorem B will be based on an extension of the pseudo-Markov property introduced in Definition S.4.15 and Theorem S.4.16 in [7] for non uniformly hyperbolic diffeomorphisms of compact surfaces. See Lemma 4.1 in Section 4 below. Compare also Lemma 2.1 [17].

A first problem to overcome is the lack of inverse for $f$. For this we switch to $F : (Z, \overline{\mu}) \circlearrowleft$ the *Rokhlin extension* of $(f, \mu)$. The set $Z$ is the inverse limit of $f$ endowed with a measure $\overline{\mu}$ which is the unique lift of $\mu$ to $Z$ making $(Z, \overline{\mu})$ isomorphic to $([0, 1], \mu)$ as measure spaces. See next Section for details. Non uniform expansion of the system $(f, \mu)$ decompose $Z$ into $F$-invariant unstable manifolds so defining a measurable lamination. Moreover, $Z$ may be endowed with a measurable partition $\xi$ subordinated to the unstable lamination. This partition is increasing, that is $F^{-1}\xi \geq \xi$. Thus, in some sense, we can think of the Rokhlin extension, endowed with this partition made of uncountably many elements, as a non uniformly expanding Markov transformation. By Luzin's theorem, we can find large non invariant compact sets $Z_\delta \subset Z$, such that the atoms of $\xi$ vary continuously on $Z_\delta$ and such that $F^n : [F^{-n}\xi](z) \longrightarrow \xi(F^n(z))$ is uniformly expanding, whenever $z \in Z_\delta$ and $F^n(z) \in Z_\delta$.

We use this setup to construct a class closed subsets $\Sigma \subset Z$, which shall be called *rectangles*. Rectangles $\Sigma$ are homeomorphic to the product $K \times J$ of an interval $J \subset [0, 1]$ and a suitable Cantor set $K$. This homeomorphism allows to decompose $\Sigma$ into *level sets* $J_z$, homeomorphic to $J$ and "fibers" $K_z$ homeomorphic to $K$. Indeed, let $\pi : Z \longrightarrow [0, 1]$ be the projection onto the first factor of the inverse limit and let $K_z = \pi^{-1}\pi(z) \cap \Sigma$ be the fiber of $\Sigma$ over $\pi(z)$. The level sets $J_z$ are given by a continuous family of intervals $J_z \subset \xi(z)$, contained in atoms of $\xi$, whose projections onto $[0, 1]$ cover a fixed closed interval $J \subset [0, 1]$, i.e. $\pi(J_z) \supset J$ for every $z \in \Sigma$. This is a consequence of the continuity properties of the Pesin set $Z_\delta$. The decomposition of a rectangle into level sets and Cantor fibers defines a *hyperbolic product structure* (see [18, Definition 1]) with the following *pseudo-Markov property*: given a point $x \in \Sigma$ which returns to $\Sigma$ in $n$ iterates, there is an stable "vertical" rectangle $S \subset \Sigma$ containing $x$ and an unstable "horizontal" rectangle $U \subset \Sigma$ containing $F^n(x)$ such that $F^n$ maps $S$ onto $U$ contracting uniformly the fibers $\mathcal{F}_z$ and expanding $S$ uniformly along unstable directions. Even more, $F^n$ have non linear distortion along unstable manifolds bounded by constants only depending on $Z_\delta$. Here "vertical" means that for every $z \in S$ it holds $K_z \cap S = K_z$, that is $S$ extends fully in the "stable" direction which coincides with the fiber $\mathcal{F}_z = \pi^{-1}\pi(z)$. Similarly so, "horizontal" means that $U$ extends fully in the unstable direction. In particular, $F^n : S \longrightarrow U$ preserves the hyperbolic product structure of $\Sigma$. See Lemma 4.1, Section 4.

We use pseudo-Markov property to show that first return maps to closed rectangles projects onto piecewise expanding Markov transformations, thus defining a tower structure. Indeed, the first return map to $\Sigma$ defines a piecewise hyperbolic mapping $\bigcup_i F^{n_i} : \bigcup_i S_i \longrightarrow \bigcup_i U_i$ with infinitely many branches projecting onto a Markov transformation defined by countably many expanding branches $f^{n_i} : J_i \subset J \longrightarrow J$.

A main point here is that the time of return of $z \in S_i$ does not depend on the level set $J_z \subset S$, so that hyperbolic branches at the Rohklin extension level project onto uniquely defined expanding branches $f^n : J' \subset J \longrightarrow J$. See Lemma 4.2 in Section 4. Compare also [3, Lemma 2].

We choose $\Delta_0$, the base of the tower, as the limit set of the iterated function system defined by the contractions $\tau_i = (f^{n_i})^{-1}$. Bounded non linear distortion property implies that $0 < \mathcal{H}_d(\Delta_0) < +\infty$. Even more, the Hausdorff measure is uniformly distributed, bounded by constants depending only in the distortion and the size of the interval $J$. This is a well known result for Cantor sets defined by finitely many expanding branches. See [14, Chapter 7, Section 20]. In the present case we have infinitely many branches, so we use a limiting argument, approximating $\Delta_0$ by a sequence of hyperbolic Cantor sets defined by finitely many expanding branches. We prove indeed that $\Delta_0$ can be approximated by dynamically defined Cantor sets $\Delta_n \subset \Delta$ having a measure of maximal dimension $\mu_n$ converging to $\mu \mid \Delta_0$ such that$\mu_n(B(x,r)) \asymp r^{\dim_{\mathcal{H}}(\Delta_n)}$ uniformly, meaning that the ratios $\mu_n(B(x,r))/r^{\dim_{\mathcal{H}}(\Delta_n)}$ are bounded uniformly independent of $x$ and $n$. We conclude that $\mu(B(x,r)) \asymp r^{\dim_{\mathcal{H}}(\Delta_0)}$ for every $x \in \Delta_0$ and $0 < r < 1$. This shows that $\mu \mid \Delta_0$ is equivalent to the Hausdorff measure and exactly dimensional.

The rest of the paper is organized as follows. In Section 3 we recall some key facts and terminology of Pesin's theory in the setting of one-dimensional systems using the Rokhlin extension of $(f,\mu)$. In Section 4 we prove the pseudo-Markov property for the class of closed 'rectangles' introduced above. Main theorems will be proved in Section 5. Section 6 shows how we get approximating ergodic measures supported on hyperbolic Cantor sets using ideas of [7]. Finally, Section 7 provides a proof of a technical lemma concerning certain geometrical properties of $\overline{\mu}$ which should be well known. We provide a proof for the sake of completeness.

3. **Pesin's theory and the Rokhlin canonical extension of a non invertible endomorphism.** Throughout this work $f : [0,1] \circlearrowleft$ denotes a $C^2$ smooth interval transformation with finitely many critical points $0 \leq a_0 < \cdots < a_n \leq 1$. We shall suppose that these singularities are *non degenerate* i.e. there are real numbers $k_i^{\pm} > 0$ such that

$$\left| \ln \frac{|f'(x)|}{|x-a_i|^{k_i^{\pm}}} \right| \qquad \text{is bounded in a left (right) neighborhood of } a_i \ ,$$

for every $i = 1, \cdots, n$. For the sake of brevity and following the terminology of [9] we call these interval transformations *C-maps.* Hereafter, by a non uniformly expanding interval transformation $(f,\mu)$ we shall mean a C-map $f$ endowed with an invariant *non atomic* Borel probability $\mu$ with positive Lyapunov exponent.

Now, we recall the notion the Rohklin extension of $(f,\mu)$ in order to introduce some notation and terminology. Let $F(z) = (f(x_0), x_0, x_1, \cdots)$ be defined on $Z = \{ \{x_n\}_{n\geq 0} : f(x_{n+1}) = x_n \}$, the set of *preorbits of* $f$. The set $Z$, the limit inverse of $f$, is a compact metric space endowed with the metric $d(z,z') = \sum_{n=0}^{+\infty} 2^{-n}|z_n - z'_n|$. The map $F$ acts as an homeomorphism on $Z$ with inverse $F^{-1}(z) = (x_1, x_2, x_3, \cdots)$. Now, let $\pi(z) = x_0$ be the projection onto the first factor. Clearly, $\pi \circ F = f \circ \pi$, that is, $F : Z \circlearrowleft$ is an extension of $f$. Moreover, there is a one-to-one correspondence between Borel probabilities $\mu$ on $[0,1]$ and Borel probabilities $\overline{\mu}$ of $Z$ where $\overline{\mu}$ is the unique Borel probability on $Z$ such that $\pi^*\mu = \overline{\mu}$ and it makes $(Z,\overline{\mu})$ isomomorphic to $([0,1],\mu)$ as measure spaces. Furthermore, $\overline{\mu}$ is invariant (resp. ergodic, mixing) if and only if $\mu$ is invariant (resp. ergodic, mixing). We shall call $F : (Z, \overline{\mathcal{B}}, \overline{\mu}) \circlearrowleft$ the Rokhlin extension of $(f,\mu)$. All this is standard and well known. See [15] for instance. Compare [3, Theorem 5] or [8] where sufficient conditions to lift an f-invariant measure of the interval to an invariant measure on of the Hofbauer extension are given.

Let $(f,\mu)$ be a non uniformly expanding C-map and $(F,\overline{\mu})$ its Rokhlin extension. Let us suppose in addition that $\ln|f'|$ is $\mu$-integrable and $\int \ln|f'|d\mu \geq \chi$. Under these hypotheses it is proved in [9, Proposition 7] that $\mu$ is *non-degenerate*, that is, the preorbits of $f$ approaches the singularities of the map at a subexponential speed. Namely, let $\delta(z) = \inf_i |\pi(z) - a_i|$, then it holds

$$\lim_{n\to+\infty} \frac{1}{n} \ln \delta(F^{-n}(z)) = 0 \qquad \overline{\mu} - a.e. \tag{6}$$

Indeed, [9, Proposition 7] says that $\liminf_{n\to+\infty} 1/n \ln \delta(F^{-n}(z)) = 0$. But this implies (6) as you can easily check using that $0 < \delta(F^{-n}(z)) < 1$ almost surely. Moreover, by the same arguments used in [9, Lemma 5.3] we can prove that positive orbits of $f$ approach the critical points at subexponential speed.

Related to subexponential growth properties as those mentioned before we shall use the following terminology and facts. We say that a measurable function $\phi_\epsilon$ is *tempered* if it satisfies

$$e^{-\epsilon} \le \frac{\phi_\epsilon(F(z))}{\phi_\epsilon(z)} \le e^{\epsilon} \ . \tag{7}$$

The *tempering kernel lemma* states that, given a positive real number $\epsilon > 0$ and a measurable function $\phi > 0$ with subexponential growth, we can find an $\epsilon$-tempered function $\phi_\epsilon > 0$ such that $\phi_\epsilon \ge \phi$. See [7, Lemma S.2.12].

The preorbit space $Z$ comes endowed with a decomposition $\Xi$ defined naturally by $\mathcal{S} = \{I_i\}$, the decomposition of $[0,1]$ into the intervals $I_i = [a_i, a_{i+1})$, where $a_i$ are the singularities of the map. Namely, two preorbits $z$ and $z'$ belongs to the same $\Xi$-atom if and only if they have the same $\mathcal{S}$-itinerary.

*Remark* 3.1. Let $\Xi_n = \bigvee_{k=0}^{n-1} F^k \pi^{-1}\mathcal{S}$. $\{\Xi_n\}$ is an increasing sequence of measurable partitions and $\Xi = \bigvee_{n\ge 0} \Xi_n$. We state for further use the following easy to check properties:

$$\Xi_n(z) = F^{-1}\Xi_{n+1}(F(z)) \quad \text{and} \quad F(\Xi_n(z)) \subset \Xi_{n+1}(F(z)) \ . \tag{8}$$

Also notice that $F^{-1} : \mathcal{F}_{F(z)} \longrightarrow \mathcal{F}_z$ is a fiber contraction, where $\mathcal{F}_z = \pi^{-1}\pi(z)$. Namely,

$$d(F^{-1}(w), F^{-1}(w')) \le \frac{1}{2} d(w,w') \quad \forall\, w,w' \in \mathcal{F}_z \quad \text{and every } z \in Z \ . \tag{9}$$

The next proposition states that non uniformly expanding C-maps have an unstable lamination subordinated to $\Xi$. See [9, Theorem 8] for proof.

**Lemma 3.1.** *There are measurable real functions $\alpha > 0$ and $1 < \beta < +\infty$ with subexponential growth along the orbits of $F$ such that:*

1. $W^u_\alpha(z) = \{w \in \Xi(z) : |\pi(z) - \pi(w)| < \alpha(z)\}$ *is a neighborhood of $z$ and*
$$|\pi(F^{-n}(z)) - \pi(F^{-n}(w))| \le \beta(z) e^{-n\chi} |\pi(z) - \pi(w)| \qquad \forall\, n \ge 0 \tag{10}$$
*for every $w \in W^u_\alpha(z)$;*
2. $|(F^n)'(z)| \ge (\beta(F^n z))^{-1} e^{n\chi}$ *for every $n \ge 0$ and almost every $z \in Z$, where $F'(w) = f'(\pi(w))$ for every $w \in W^u_\alpha(z)$.*

Backward contraction along unstable manifolds provides a measurable function $\gamma = \gamma(z)$ with subexponential satisfying:

$$|\pi(w) - \pi(w')| \le d(w,w') \le \gamma(z)|\pi(w) - \pi(w')|, \tag{11}$$

for every $w, w' \in W^u_\alpha(z)$. Namely, by (10)

$$\begin{aligned}\sum_{n=0}^{+\infty} \frac{|w_n - w_n'|}{2^n} &\le \sum_{n=0}^{+\infty} \beta(z)|\pi(w) - \pi(w')| \frac{e^{-n\chi}}{2^n} \\ &= \frac{2e^{\chi}\beta(z)|\pi(w) - \pi(w')|}{2e^{\chi} - 1}.\end{aligned}$$

Thus, local unstable leaves $W^u_\alpha(z) \subset \Xi(z)$, endowed with the metric $d = d(z,z')$ of $Z$, can be seen as isometric copies of $[0,1]$, up to measurable corrections depending on $z$.

**Lemma 3.2.** *There exists a measurable partition $\xi$ of $Z$ with the following properties:*

1. $\xi(z) \subseteq W^u_\alpha(z)$ *for every* $z \in Z$ *and it contains a neighborhood of* $z$*, i.e. it is* subordinated to the unstable lamination. *Namely, there is a measurable function* $\rho > 0$ *such that, almost surely*
$$\{z' \in W^u_\alpha(z) : |\pi(z) - \pi(z')| < \rho(z)\} \subseteq \xi(z);$$
*is a neighborhood in* $W^u_\alpha(z)$*;*
2. $\xi$ *is increasing, that is,* $F^{-1}\xi \geq \xi$*. Moreover, it generates the* $\sigma$*-algebra of Borel subset of* $Z$ *and* $h_\mu f = H(\xi, F^{-1}\xi)$*.*

Cf. [9, Proposition 3.2]. From now on $\xi(x)$ will denote the atom of a Pesin partition $\xi$ containing $x$. The following non linear distortion estimates follows from non uniformly expansion along unstable leaves. See [9, Theorem 8].

**Lemma 3.3.** *There are measurable functions* $\gamma_1, \gamma_2 : Z \longrightarrow (0, +\infty)$*, finite and positive* $\overline{\mu}$*-a.e., with subexponential growth, such that*
$$\prod_{i=0}^{+\infty} \left| \frac{F'(F^{-i}w)}{F'(F^{-i}w')} \right| \leq \exp\{\gamma_1(z) d(w, w')\} \ , \tag{12}$$
*for every* $w, w' \in \xi(z)$ *and such that*
$$\left| \frac{(F^n)'(w)}{(F^n)'(w')} \right| \leq \exp\{\gamma_2(F^n(z)) d(F^n(w), F^n(w'))\} \ , \tag{13}$$
*for every* $n > 0$ *and every* $w, w' \in [F^{-n}\xi](z)$*.*

*Proof.* Non linear backward distortion bound (12) follows from contraction properties of backward iterates of $F$ along the local unstable manifolds and is a standard fact. See for instance [9, Theorem 8].

Nevertheless, due to the singularities of the map $f$, some care must be taken to get (13). Indeed, we claim that for every $0 < \epsilon \ll 1$ there is a constant $C > 0$ such that, for every $n \geq 0$ there holds:
$$|\ln |F'(F^{-n}z)| - \ln |F'(F^{-n}w)|| \leq C e^{n\epsilon} |\pi(F^{-n}z) - \pi(F^{-n}w)| \tag{14}$$
for $\overline{\mu}$-a.e. $z \in Z$ and every $w \in W^u_\alpha(z)$. Let us suppose (14) proven, then:
$$\begin{aligned} \ln \left| \frac{(F^n)'(w)}{(F^n)'(w')} \right| &= \ln \prod_{i=1}^{n} \left| \frac{F'(F^{-i}(F^n(w)))}{F'(F^{-i}(F^n(w')))} \right| \\ &\leq 2C\beta(F^n(z)) \sum_{i=0}^{n-1} e^{-i\chi/2}\, d(F^n(w), F^n(w')) \ , \end{aligned}$$
for every $n > 0$ and every $w, w' \in [F^{-n}\xi](z)$, where we use (14) and (10), choosing $0 < \epsilon < \chi/2$. Therefore,
$$\gamma_2(z) = \frac{2C\beta(z)e^{-\chi/2}}{1 - e^{-\chi/2}} \tag{15}$$
does work.

Now, we prove (14). First we choose, for every $z \in Z$, a closed interval $U(z) \subseteq [0,1] - \{a_0, \cdots, a_n\}$ such that
$$\lim_{n \to +\infty} \frac{1}{n} \ln m(U(\pi(F^{-n}z)) = 0 \ . \tag{16}$$

Even more, given $z \in Z$ and $w \in W^u_\alpha(z)$ we may suppose that $\pi(F^{-n}w) \in U(\pi(F^{-n}z)$ for $n > 0$ large enough. For this we define $U(z) = \{ x \in [0,1] : |x - \pi(z)| \le \delta(z)/4 \}$. As $f$ is a C-map, for every in $x \in [0,1]$ and $i = 1, \cdots, n$ it holds that $\ln|f'(x)| \asymp \ln|x - a_i|$ bounded by some constant $C > 1$ not depending on $x$ neither $a_i$. Therefore, $\ln|F'(z)| \asymp \ln\delta(z)$ for $\overline{\mu}$-a.e. $z \in Z$. Using these remarks we get

$$\begin{aligned} |\ln|F'(w)| - \ln|F'(w')|| &\asymp |\ln\delta(w) - \ln\delta(w')| \\ &\le \frac{C_0}{\delta(z)}|\pi(w) - \pi(w')| \ , \end{aligned}$$

for every $w, w' \in W^u_\alpha(z)$ with $\pi(w), \pi(w') \in U(z)$ and a suitable constant $C_0 > 0$. $\delta(z)$ has subexponential growth, since $\mu$ is non degenerate, so we get a measurable function $L = L(z)$ such that $|\ln|F'(w)| - \ln|F'(w')|| \le L(z)|\pi(w) - \pi(w')|$ and

$$\lim_{n\to+\infty} \frac{1}{n} \ln\, L(F^{-n}z)) = 0 \ , \tag{17}$$

for every pair of points $w, w' \in W^u_\alpha(z)$ projecting onto the closed interval $U(z)$. Therefore, by the tempered kernel lemma, given $0 < \epsilon \ll 1$ we can find $C > 1$ such that

$$|U(\pi(F^{-n}z))| \ge C^{-1}e^{-n\epsilon} \qquad \text{and} \qquad L(\pi(F^{-n}z)) \le Ce^{n\epsilon},$$

which proves (14). □

Proof of the following technical lemma will be left to the end.

**Lemma 3.4.** *Let $\{\overline{\mu}_{\xi(z)}\}$ be the family of conditional measures obtained from the disintegration of $\overline{\mu}$ with respect to the measurable partition $\xi$. Similarly so $\{\overline{Leb}_{\xi(z)}\}$, where $\overline{\mu}$ (resp. $\overline{Leb}$) is the lifting of an $f$-invariant measure $\mu$ (resp. the Lebesgue measure in $[0,1]$) to $Z$. Then, there are measurable functions $M, L : Z \longrightarrow (1, +\infty)$ such that:*

$$\frac{1}{M(z)} \le \frac{\overline{\mu}_{\xi(z)}(B \cap \xi(z))}{\mu(\pi_z(B \cap \xi(z))} \le M(z) \qquad \overline{m} - a.e.\ z \in Z \tag{18}$$

*where $\pi_z = \pi \mid \xi(z)$ and*

$$\frac{1}{L(z)} \le \frac{\overline{Leb}_{\xi(z)}(B \cap \xi(z))}{Leb(\pi_z(B \cap \xi(z))} \le L(z) \qquad \text{almost surely,} \tag{19}$$

*for every Borelian subset $B \subseteq Z$.*

Therefore, $\xi(z)$ endowed with the conditional measure $\overline{\mu}_{\xi(z)}$ can be seen as an isomorphic copy of the abstract measure space $([0,1], \mu)$, bounded by non uniform corrections.

We prove later Lemma 3.4 introducing a family of conditional measures $\mu_{\xi(z)}$ which produce a new $\sigma$-finite Borel measure $\mathbf{M}$ equivalent to $\overline{\mu}$. Namely, $\mu_{\xi(z)}$ is the pullback $\pi_z^*\mu$ normalized to a probability on $\xi(z)$, and

$$\mathbf{M}(B) = \int_{Z_\xi} \mu_\xi(B) d\overline{\mu}_\xi \ .$$

where $\pi_z = \pi \mid \xi(z)$ and $(Z_\xi, \overline{\mu}_\xi)$ is the quotient space $(Z, \overline{\mu})/\xi$. See Section 5 for details.

Now, we can precise the notion of a Pesin set which shall be used later.

**Lemma 3.5.** *Given $0 < \delta < 1$ there exists a compact subset $Z_\delta \subset Z$ with $\overline{\mu}(Z_\delta) \ge 1 - \delta$ and a constant $C = C(\delta) > 0$ such that:*

1. *The family of unstable leaves* $z \longmapsto W^u_\alpha(z)$ *is continuous on* $Z_\delta$*;*
2. *The functions* $\alpha$*,* $\beta$*,* $\gamma_1$*,* $\gamma_2$ *and* $\rho$ *introduced at Lemmas 3.1 and Lemma 3.2, the functions* $L, M$ *introduced in Lemma 3.4, the function* $\gamma$ *in (11) and* diam$(\xi(z))$ *are continuous on* $Z_\delta$*;*
3. *Let* $z \in Z_\delta$ *and suppose that* $F^n(z) \in Z_\delta$*, then*
$$|(F^n)'(w)| \geq Ce^{n\chi} \qquad \forall\ w \in [F^{-n}\xi](z)\ ; \tag{20}$$
*even more, for every* $w, w' \in [F^{-n}\xi](z)$ *and* $0 \leq k < n$ *it holds:*
$$d(F^k(w), F^k(w')) \leq C^{-1}e^{(-n+k)\chi}d(F^n(w), F^n(w'))\ ; \tag{21}$$
4. *Given* $z \in Z_\delta$ *and* $n > 0$ *such that* $F^n(z) \in Z_\delta$ *then*
$$\left|\frac{(F^n)'(w)}{(F^n)'(w')}\right| \leq \exp(Cd(F^n(w), F^n(w')) \qquad \forall\ w, w' \in [F^{-n}\xi](z)\ . \tag{22}$$

**Outline of the proof**
Linear and non linear expansion estimates (20) and (21) follow from Lemma 3.1 and Lemma 3.3 using the definition of $Z_\delta$ as a quotient space and standard arguments.

The existence of $Z_\delta$ and the continuity of the Borel functions mentioned in Lemma 3.5 follows from Luzin's theorem. For the continuity of the local unstable manifolds we argue as follows. First recall that local unstable manifolds were defined by local contracting branches of $f^{-1}$. Indeed, it is proved in [9, Theorem 8] that for $\overline{\mu}$-a.e. $z \in Z$ there is a sequence of $C^2$ functions $\{g^n_z\}_{n\geq 0}$ satisfying the following:

1. $g^n_z : U(z) \longrightarrow [0,1]$ is a $C^2$ smooth embedding defined in an open interval $U(z) = (\pi(z) - \alpha(z), \pi(z) + \alpha(z))$ contained in $[0,1]$;
2. for every $x \in U(z)$ and $n \geq 0$ it holds $f^n(g^n_z(x)) = x$;
3. $|g^n_z(x) - g^n_z(y))| \leq \beta(z)e^{-n\chi}|x - y|$ for every $n \geq 0$ and $x\,,\, y \in \pi(W^u_\alpha(z))$.

Therefore, $\phi_z(x) = \{g^n_z(x)\}_{n\geq 0}$ is a continuous parametrization of $W^u_\alpha(z)$. Moreover, the map $\pi_z = \pi \mid W^u_\alpha(z)$ is an embedding from $W^u_\alpha(z)$ into $[0,1]$ with inverse $\phi_z$. Each $g^n_z$ is a $n$-fold composition of local branches of $f^{-1}$ defined over suitable open intervals. See [9, Theorem 8] and [16, Theorem 5.1] for details. Now, let $E = C^2[0,1]^{\mathbb{Z}^+}$ be the space of sequences $\{g_n\}_{n\geq 0}$ of $C^2$ smooth mappings $g_n : [0,1] \longrightarrow [0,1]$. $E$ carries a natural structure of separable Banach space using the $C^2$ norm at the coordinate spaces. We identify the family of unstable leaves $z \longmapsto W^u_\alpha(z)$ as a Borel measurable map $z \longmapsto \{g^n_z\}_{n\geq 0}$ taking values in $E$. Then, we use Egorov-Luzin theorem to get $Z_\delta$ with the claimed properties.

## 4. **Rectangles and pseudo-Markov property: proofs of main technical lemmas.**

**Definition 4.1.** Let $J \subset [0,1]$ be an interval and $X \subset Z$ a subset. A family of subsets of $Z$, $\{J_z \subset Z\}_{z\in X}$, is a *continuous family of intervals* of base $J$ parametrized by $X$ if, for every $z \in X$ the projection $\pi_z := \pi \mid J_z$ is a homeomorphism onto $J$ and the function $\Phi(z, z') = (z, \pi_z(z'))$ defined on $X \times \bigcup_{z\in X} J_z \subset X \times Z$ is continuous.

**Definition 4.2.** Let $J \subset [0,1]$ be a closed interval, $K \subset \mathcal{F}_z$ a closed subset of the Cantor fiber over $\pi(z)$ and $\{J_z\}_{z\in K}$ a continuous family of intervals of base $J$ parametrized by $z \in K$. The subset $\Sigma(J, K) = \bigcup_{z\in K} J_z$ will be called a *closed rectangle* of base $J$ and fiber $K$.

Let $\Sigma$ be a closed rectangle. Given a point $w \in \Sigma$ we define $K_w = \pi^{-1}\pi(w) \cap \Sigma$. This is *the fiber of* $\Sigma$ *over* $\pi(w)$. For any two points $z, z' \in \Sigma$ there is a holonomy map $\phi_{z,z'} : K_z \longrightarrow K_{z'}$ defined as $\phi_{z,z'}(w) = J_w \cap K_{z'}$ which is an homeomorphism between the fibers. In particular we can define an homeomorphism $\Phi : \Sigma \longrightarrow J \times K$, fixing a fiber $K = K_{z_0}$ setting $\Phi(w) = (\pi_z(w), \phi_{z,K}(w))$ for every $w \in J_z$ is an homeomorphism, where $\phi_{z,K} : K_z \longrightarrow K$ is the correspondent holonomy transformation.

A closed rectangle $S \subset \Sigma(J, K)$ is called an *s-rectangle* if $S \cap K_z = K_z$ and $S \cap J_z \subset J_z$ is a closed subinterval, for every $z \in S$. Likewise, a closed rectangle $U \subset \Sigma$ will be called an *u-rectangle* if $U \cap J_z = J_z$ for every $z \in U$ and $U \cap K \subset K$ is a closed subset of $K$.

**Lemma 4.1.** *Let $Z_\delta$ be a hyperbolic Pesin set with $\overline{\mu}(Z_\delta) > 1 - \delta$ given by Lemma 3.5. Then for every $z_0 \in Z_\delta \cap$ supp $\overline{\mu}$ there is a closed rectangle $\Sigma$ containing $z_0$ with $\overline{\mu}(\Sigma) > 0$ and a constant $C = C_\delta > 0$ only depending on $Z_\delta$ satisfying the following* pseudo-Markov property*.*

*Let $z \in \Sigma$ and $n > 0$ such that $F^n(z) \in \Sigma$, then, there exists an s-rectangle $S = S_z$ containing $z$ and an u-rectangle $U = U_{F^n(z)}$ containing $F^n(z)$ such that and the following holds:*

1. *$F^n$ maps $S$ onto $U$ preserving the hyperbolic product structure of $\Sigma$: for every $w \in S$, $F^n : S \cap J_w \longrightarrow J_{F^n(w)}$ is one-to-one and onto and $F^n(K_w) = K_{F^n(w)} \cap U$ for every $w \in S$;*
2. *$F^n : S \longrightarrow U$ is a hyperbolic mapping, that is, it expands uniformly along unstable leaves and it contracts the fibers $K_w$, namely:*
   (a) *$|(F^n \mid J_w)'| \geq Ce^{n\chi}$ for every $w \in S$ for every $w \in S$ and*
   (b) *$d(F^n(w_0), F^n(w_1)) \leq (1/2)^n d(w_0, w_1)$ for every $w_0, w_1 \in K_w$;*
3. *$F^n \mid S$ has bounded non linear distortion along unstable leaves: for any $w \in K$ and for every pair of points $v, v' \in S$ lying on the same level set $J_w \subset \Sigma$ it holds*
$$\left|\frac{(F^n)'(v)}{(F^n)'(v')}\right| \leq \exp(Cd(F^n(v), F^n(v')) \ . \tag{23}$$

*Moreover, there is an s-rectangle $\hat{S}$ containing $S$ and an u-rectangle $\hat{U}$ containing $U$ such that $F^n$ extends to $F^n : \hat{S} \longrightarrow \hat{U}$ and it is hyperbolic and satisfies bounded distortion estimate (23).*

*Proof.* Let $z_0 \in Z_\delta \cap$ *supp* $\overline{\mu}$. Then $\overline{\mu}(B(z_0, r) - \{z_0\}) > 0$ for every $r > 0$. Such a point exists because $\mu$ is non-atomic.

**Claim** There exists a closed interval $J \subset [0, 1]$ and $r_0 > 0$ and

$$J \subset \pi(\xi(z)) \quad \text{for every} \quad z \in B(z_0, r) \cap Z_\delta \ .$$

Let $\tilde{\rho}(z)$ be the radius of the maximal open interval contained in $\pi(\xi(z))$ and let $\zeta = \zeta(z)$ be its "center". That is, $(\pi(\zeta) - \tilde{\rho}(z), \pi(\zeta) - \tilde{\rho}(z)) \subset \pi(\xi(z))$ and it is maximal with respect to that property. Clearly, $z \longmapsto \tilde{\rho}(z)$ is measurable so we may suppose that it is continuous on a hyperbolic Pesin set $Z_\delta \subset Z$, by Luzin-Egorov's theorem. Now, we take $0 < r_0 < \tilde{\rho}(z_0)/10$ such that $|\tilde{\rho}(z_0) - \tilde{\rho}(z)| < \tilde{\rho}(z_0)/10$ for every $z \in B(z_0, r_0) \cap Z_\delta$ and we let $c_0 = \pi(\zeta_0)$ be the projection onto $[0, 1]$ of the "center" $\zeta_0$ of $\xi(z_0)$. Then,

$$J = [c_0 - 2/5\tilde{\rho}(z_0), c_0 + 2/5\tilde{\rho}(z_0)] \subset \pi(\xi(z)),$$

for every $z \in B(z_0, r_0) \cap Z_\delta$. This proves the claim. Now we define

$$\Sigma_0 = \left( \bigcup_{z \in B(z_0,r_0) \cap Z_\delta} \xi(z) \right) \cap \Xi_N(z_0) \quad \text{and} \quad \Sigma = \bigcup_{z \in B(z_0,r_0) \cap Z_\delta} J_z ,$$

where $J_z = \pi_z^{-1}(J) \cap \xi(z)$ and $\Xi_n$ are the measurable partitions introduced at Remark 3.1. $\{J_z\}$ is a continuous family of intervals over $J$ since $\{W^u_\alpha(z)\}$ with $z$ varying in the set $B(z_0, r_0) \cap Z_\delta$ is continuous. Now, let $K = \mathcal{F}_{z_0} \cap \Sigma_0$ be the intersection with the fiber of $z_0$ with $\Sigma$ and $K_z = \mathcal{F}_z \cap \Sigma$. $\Sigma = \bigcup_{z \in K} J_z$ is a rectangle of base $J$ and fiber $K$. Moreover, for every $z \in \Sigma$ the holonomy map $\phi = \phi_{z,z_0} : K_z \longrightarrow K$ defined by the lamination $\{J_z\}$ is a well defined homeomorphism. Clearly $\overline{\mu}(\Sigma_0) > 0$. Moreover, $\mu(J) > 0$ since $\pi(z_0) \in J$ and $\pi(z_0) \in supp\ \mu$. Therefore, $\overline{\mu}(\Sigma) > 0$ since $\mu_{\xi(z)}(J_z) > 0$ almost surely, by Lemma 3.4. Now let $z \in \Sigma$ such that $F^n(z) \in \Sigma$. We define

$$U = \bigcup_{w \in F^n(K_z)} J_{\phi(w)} \quad \text{and} \quad \hat{U} = \bigcup_{w \in F^n(K_z)} \xi(\phi(w)) ,$$

where $\phi : K_{F^n(z)} \longrightarrow K$ is the holonomy map. $U \subset \Sigma$ is a closed u-rectangle contained in $\hat{U} \subset \Sigma_0$. By the Markov property of the Pesin partition $F^n$ maps $[F^{-n}\xi](w)$ one-to-one and onto $\xi(F^n(w))$, for every $w \in K_z$. Therefore, we can find an interval $I_{F^{-n}w} \subset [F^{-n}\xi](F^{-n}w)$ such that

$$F^n \mid I_{F^{-n}w} : I_{F^{-n}w} \longrightarrow J_w \quad \text{is one-to-one and onto} .$$

Now, let us define

$$S = \bigcup_{w \in F^n(K_z)} I_{F^{-n}w} \quad \text{and} \quad \hat{S} = \bigcup_{w \in F^n(K_z)} [F^{-n}\xi](F^{-n}w).$$

$S$ is an s-rectangle contained in $\hat{S}$ and $F^n$ maps $S$ onto $U$. By the properties of the Pesin set $|(F^n)'(w)| > 1$ for every $w \in S$. Furthermore, $F^n(K_z) \subset K_{F^n(z)}$ using Remark 3.1 and it is contracted uniformly by (9). So $F^n : S \longrightarrow U$ is uniformly hyperbolic. Forward non linear distortion bound (23) follows from (22) Lemma 3.5. □

**Lemma 4.2.** *Every hyperbolic branch $F^n : S \longrightarrow U$ projects onto a uniquely defined expanding branch $f^n : J' \longrightarrow J$.*

Compare with [3, Lemma 2].

*Proof.* Let $z \in \Sigma$ and $n > 0$ such that $F^n(z) \in \Sigma$ and $F^n : S \longrightarrow U$ the correspondent hyperbolic branch. By definition, $S = \bigcup_{w \in F^n(K_z)} I_{F^{-n}w}$. Fix some $w_0 \in F^n(K_z)$ and let us define $J' = \pi(I_{F^{-n}w_0}) \subset J$. As for every $n > 0$ we get an expanding branch $f^n \mid J' : J' \longrightarrow J$. We claim that this expanding branch does depend on the level set $I_{F^{-n}w}$, that is: $J' = \pi(I_{F^{-n}w})$ for every $w \in K_z$.

First recall that $I_{F^{-n}w} \subset [F^{-n}\xi](F^{-n}w)$ were chosen such that the restriction $F^n \mid I_{F^{-n}w}$ is a diffeomorphism of $I_{F^{-n}w}$ onto $J_w$. As $\pi \mid J_w \longrightarrow J$ is a diffeomorphism for every $w$ and $f^n \circ \pi = \pi \circ F^n$ we conclude that $f^n : J' \longrightarrow J$ is one-to-one and onto.

Now, let us suppose by contradiction that there is $w' \in K_z$ such that $J'' = \pi(I_{F^{-n}w'}) \neq J'$. That is, $J' \cap J''$ and the set symmetric difference $J' \Delta J''$ are both non empty. If this happens then, as $f^n : J' \longrightarrow J$ and $f^n : J'' \longrightarrow J$ are diffeomorphisms, then there would be an interior point in one of those intervals which is mapped to the border of $J$, which is absurd. $\square$

As $\overline{\mu}(\Sigma) > 0$ almost every point in $\Sigma$ returns infinitely often to $\Sigma$. So let $\tau$ be the first return map to $\Sigma$: $\tau(z) = F^{R(z)}(z)$, where $F^R(z) \in \Sigma$ and $F^k(z) \notin \Sigma$ for $0 < k < R$. $R = R(z)$ is the first return time to $\Sigma$.

**Lemma 4.3.** *Let $\tau$ be the first return map of $F$ to $\Sigma$. Then, there is countable collection of closed s-rectangles $\{S_i\}$ and a collection of closed u-rectangles $\{U_i\}$ and integers $n_i > 0$, such that:*

1. $\overline{\mu}(\Sigma - \bigcup_i S_i) = 0$ *and* $int\, S_i \cap int\, S_j = \emptyset$ *whenever* $i \neq j$*;*
2. $\tau$ *is well defined on* $\bigcup_i S_i$*, up to a* $\mu$*-zero set. Namely,* $\tau \mid S_i = F^{n_i}$ *for every* $i > 0$*, where* $n_i = R(z)$ *is the return time of* $S_i$*;*
3. *for every* $n > 0$ *the number of s-rectangles* $S_i$ *such that* $n_i = n$ *is finite;*
4. $F^{n_i} : S_i \longrightarrow U_i$ *is a hyperbolic branch with expansion along unstable leaves uniformly bounded from below and bounded non linear distortion.*

*Proof.* First notice that the set of points which first returns at time $t = n$ is covered by finitely many disjoint s-rectangles $\{S_i^n\}$. Actually, if $F^n(z) \in \Sigma$ is the first return of $z \in \Sigma$ and $S = S_z$ the s-rectangle given by Lemma 4.1, then $F^k(S) \cap \Sigma = \emptyset$ for $0 < k < n$. This is by the Markov property of Pesin partition. Indeed, if $F^k(S \cap \xi(z)) \cap \Sigma \neq \emptyset$ then $F^k(S \cap \xi(z)) \subset \xi(F^k(z))$, so every point in $S \cap \xi(z)$ would return to $\Sigma$ at time $t = k$. So, if $R(z) = n$ then $R(z') = n$ for every $z' \in S \cap \xi(z)$. On other hand, $F^k(K_z) \cap K_{F^k(z)} \neq \emptyset$ implies $F^k(K_z) \subset K_{F^k(z)}$ by Remark 3.1. Therefore, first return time is constant at $S = S(z)$, the s-rectangle defined by the first return of $z$ to $\Sigma$. By construction $S \cap \xi(w) \subset [F^{-n}\xi](w)$ for every $w \in S$. Thus, first returns at time $t = n$ are given by at most countably many s-rectangles $\{S_i^n\}$. They are disjoint because $S_i^n \subset \hat{S}_i^n$ and $\hat{S}_i^n \cap \hat{S}_j^n = \emptyset$ for every $i, j$, where we recall that $\hat{S} = \bigcup_{w \in F^n(K_z)} [F^{-n}\xi](F^{-n}w)$ is a disjoint union of $\xi$-atoms. Furthermore, as $S_i^n \cap J_z$ have a diameter bounded from below for every $z \in K$ the subsets, since $\pi_w F^n(S_i^n \cap J_z) = J$ for every $z, w \in K$ we conclude that returns to $\Sigma$ at time $t = n$ are covered by at most finitely many s-rectangles. Now we decompose $\Sigma$, up to measure zero, into countably many subsets $\Sigma_n = \{z \in \Sigma : R(z) = n\}$ defined by their first return time to $\Sigma$ which, by the above remarks, can be covered by at most finitely many s-rectangles $S_j^n$. We relabel this collection to get a family of s-rectangles $S_i$, u-rectangles $U_i$ and return times $n_i$ such that $F^{n_i} : S_i \longrightarrow U_i$ satisfies the claimed properties. $\square$

## 5. **Proof of Theorem A and Theorem B. Proof of Theorem B**

Let us begin constructing $\Delta_0$, the base of the tower. Using Lemma 4.3 and Lemma 4.2 we get countably many expanding branches $f^{n_i} : J_i \longrightarrow J$ defined over closed intervals $J_i \subset J$, where $n_i$ is the time of return of the s-rectangle $S_i$. Let $\Delta_0$ be the maximal invariant subset of the piecewise expanding transformation generated by the branches $f^{n_i} : int\, J_i \longrightarrow int\, J$. Then, $\mu(\Delta_0) > 0$.

Indeed, let $\Delta_0^* = \bigcap_{n \in \mathbb{Z}} \tau^{-n}(\bigcup_i int\, S_i)$ be the maximal invariant subset of $\tau$, the first return map to $\Sigma$, where $int\, S_i = \bigcup_{z \in K} int\, I_z$. Notice that $\overline{\mu}(I_z - int\, I_z) = 0$ so $\bigcup_i S_i = \bigcup_i int\, S_i$, modulo zero. Therefore, $\overline{\mu}(\Delta_0^*) > 0$ since $\overline{\mu}$ is $\tau$-invariant and $\overline{\mu}(\bigcup_i int\, S_i) > 0$. In particular, $\overline{\mu}_{\xi(z)}(\Delta_0^* \cap \xi(z)) > 0$ for a subset of positive measure $z \in \Sigma$. Hence, by Lemma 3.4, $\overline{\mu}_{\xi(z)}(\Delta_0^*) > 0$ for every $z \in \Sigma$. Thus $\mu(\Delta_0) > 0$, since $\Delta_0 = \pi_z(\Delta_0^* \cap \xi(z))$.

Let $\Delta_{i,0} = \Delta_0 \cap J_i$ and $R(x) = n_i$ for every $x \in \Delta_{i,0}$. Then, $\Delta_0 = \bigcup_i \Delta_{0,i}$ (modulo $\mu$-zero sets). The return time function $R : \Delta_0 \longrightarrow \mathbb{Z}^+$ is constant at each $\Delta_{0,i}$ and return maps $f^{n_i} : \Delta_{0,i} \longrightarrow \Delta_0$ are one-to-one and onto.

Let $\Delta = \{ (x, n) \in \Delta_0 \times \mathbb{Z}^+ : n < R(x) \}$ and define $\Delta_l = \{(x, n) \in \Delta : n = l\}$ as the $l$-th level of the tower. $\Delta_{R_i-1,i}$ is at the top of the tower over the block $\Delta_{0,i}$. Now, we define $T : \Delta \circlearrowleft$ as being $T(x, l) = (x, l + 1)$ if $l + 1 < R(x)$ and $T \mid \Delta_{R_i-1,i} = f^{n_i}$, which maps the top of the tower onto $\Delta_0$. The map $T : \Delta \circlearrowleft$ from the abstract measure space $(\Delta, \mathcal{B}, \mathcal{H}_d)$ onto itself is the Markov tower structure we were looking for. With this notation we define $T^R : \Delta_0 \longrightarrow \Delta_0$ as $T^R(x) = T^R(x)(x)$.

**Lemma 5.1.** *Let $\Delta_0$ be the base of the tower and $\mu^*$ the restriction of $\mu$ to $\Delta_0$ normalized to a probability. There exists a sequence of hyperbolic Cantor sets $\Delta_0^n$ contained in $\Delta_0$ and measures of maximal dimension $\mu_n^*$ such that $\Delta_0 = \overline{\bigcup_{n>0} \Delta_0^n}$ and $\mu_n^* \to \mu^*$. Moreover, there is a constant $C > 1$ independent of $n$ such that*

$$C^{-1} \leq \frac{\mu_n^*(B(x, r))}{r^{\alpha(n)}} \leq C \quad \textit{for every } x \in \Delta_0^n \textit{ and } 0 < r < 1 \ . \tag{24}$$

*for every $n > 0$, where $\alpha(n) = \dim_{\mathcal{H}}(\Delta_0^n)$ is the Hausdorff dimension of the set.*

We continue the proof of Theorem B and let the proof of this lemma to the end of this Section.

By Lemma 5.1 the set $\Delta_0$ has positive and finite Hausdorff measure and its uniformly distributed. Indeed, passing to the limit at the inequality (24) $\mu(B(x, r)) \asymp r^{\dim_{\mathcal{H}}(\Delta_0)}$ for $x \in \Delta_0$, $0 < r < 1$. This proves that the Hausdorff measure of $\Delta_0$ is non trivial, i.e. $0 < \mathcal{H}_d(\Delta_0) < +\infty$, by Frostman's Lemma, proving that Hausdorff measure on $\Delta_0$ does work as a reference measure for the tower.

Moreover, as $\mu(B(x, r)) \asymp r^{\dim_{\mathcal{H}}(\Delta_0)}$ we conclude $\mu \mid \Delta_0$ is equivalent to the Hausdorff measure. In particular, $R(x)$ is integrable w.r.t. $\mathcal{H}_d$, since first return time function $R = R(z)$ is $\overline{\mu}$ integrable, by Kac's formula $\int R(z) d\overline{\mu}(z) = (\overline{\mu}(\Sigma))^{-1}$. In particular,

$$\int R(x) d\mathcal{H}_d(x) = \sum_{i+0}^{+\infty} n_i \mathcal{H}_d(\Delta_{0,i}) < +\infty \ , \tag{25}$$

where $d = \dim_{\mathcal{H}}(\Delta_0)$. Moreover,

$$\dim_{\mathcal{H}}(\Delta_0) = \lim_{r \to 0^+} \frac{\ln \mu(B(x, r))}{\ln r}$$

almost everywhere, so proving that the local dimension of the measure $\mu$ at $x$, exists for almost surely for $x \in \Delta_0$ and is constant: $d_\mu(x) = \dim_{\mathcal{H}}(\Delta_0)$. Therefore, $\mu \mid \Delta_0$ is exactly dimensional, so $d_\mu(x) = \dim_{\mathcal{H}}(\mu \mid \Delta_0)$ for $\mu$-a.e. As $\mu$ is ergodic then $\bigcup_j f^j(\Delta_0)$ covers $[0, 1]$, up to a measure zero subset. This proves that $\mu$ is exactly dimensional. Now, by (5) we conclude

$$\dim_{\mathcal{H}}(\Delta_0) = \frac{h_\mu f}{\int \ln |f'| d\mu}$$

so proving (2).

To conclude the proof of Theorem B, we have to show that $T^R$ has a Jacobian w.r.t. the reference measure and that it has bounded distortion. Now, $f^{n_i} : \Delta_{0,i} \longrightarrow \Delta_0$ has a Jacobian $J_{\mathcal{H}} f^{n_i}$ w.r.t. to the Hausdorff measure, indeed $J_{\mathcal{H}} f = |f'|^d$. Therefore $T^R$ has a Jacobian $J_{\mathcal{H}} F^R$ which satisfies a bounded distortion property up to certain *separation time* $s(x, y)$ which is the smallest $n \geq 0$

such that $(T^R)^n(x)$ and $(T^R)^n(y)$ lies in distinct $\Delta_{0,i}$. Namely, let $\wp = \{\Delta_{0,i}\}$ be the partition of $\Delta_0$ defined by the return time and $\wp_n = (T^R)^{-1}\wp$, then:

$$s(x,y) = \sup\{\, n > 0 \,:\, \wp_n(x) = \wp_n(y)\,\}$$

Then, we can find constants $C > 0$ and $0 < \alpha < 1$ such that

$$\left| \frac{J_{\mathcal{H}}(T^R)^n(x)}{J_{\mathcal{H}}(T^R)^n(y)} - 1 \right| \leq C\alpha^{s(x,y)-n)} \tag{26}$$

for every $x, y \in \Delta_0$. Indeed, by (23)

$$\left| \frac{(f_i^n)'(x)}{(f_i^n)'(y)} \right|^d \leq \exp(dCd(f_i^n(x), f_i^n(y)) \ .$$

Therefore, we can find a constant $C' > 1$ such that

$$\left| \frac{J_{\mathcal{H}}(T^R)^n(x)}{J_{\mathcal{H}}(T^R)^n(y)} - 1 \right| \leq C'\lambda^{-(s(x,y)-n)}d((T^R)^{s(x,y)}(x), (T^R)^{s(x,y)}(y)) \ ,$$

using the definitions of $J_{\mathcal{H}}(T^R)$, the separation time and expansion properties of $T^R$. Inequality (26) follows inmediatly from the above estimative.

**Proof of Theorem A**

By Lemma 5.1 $\dim_{\mathcal{H}}(\Delta_0^n) \longrightarrow \dim_{\mathcal{H}}(\Delta_0)$. Now we let $\Lambda_n = \bigcup_{j=0}^{+\infty} f^j(\Delta_0^n)$ be the f-invariant saturated of $\Delta_0^n$. For every $n > 0$ there exists $m_n > 0$ such that $\Lambda_n = \bigcup_{i=1}^{m_n} \bigcup_{j=0}^{n_i} f^j(\Delta_0^n)$, therefore $\dim_{\mathcal{H}}(\Lambda_n) = \dim_{\mathcal{H}}(\Delta_0^n)$ and $\mathcal{H}_{\alpha(n)}(\Lambda_n)$ is uniformly bounded away from zero since $\mathcal{H}_{\alpha(n)}(\Delta_0^n) \geq C^{-1}$ for every $n > 0$ and a suitable constant $C > 1$. Hence, by (2) and (5) $\dim_{\mathcal{H}}(\Lambda_n) \longrightarrow \dim_{\mathcal{H}}(\mu)$.

The existence of a sequence of hyperbolic measures $\mu_n$ of maximal dimension supported on $\Lambda_n$ converging to $\mu$ follows using arguments similar to those used in the proof of Lemma 5.1. See [7] or [17, Lemma 6.1]. This completes the proof of Theorem A.

**Proof of Corollary C**

By Theorem B, $\dim_{\mathcal{H}}(\Delta_0) = 1$ iff $\Delta_0$ has positive Lebesgue measure, since $\Delta_0$ is a regular Cantor set. Therefore $\mu$ is an absolutely continuous invariant measure iff it satisfies the Rokhlin formula. Moreover, if $\mu$ satisfies the Rokhlin entropy formula, then

$$\overline{\mu}_z(B) = \frac{\int_{\xi(z)\cap B} \Delta(z,w)dLeb_z(w)}{\int_{\xi(z)} \Delta(z,w)dLeb_z(w)}, \tag{27}$$

where

$$\Delta(z,w) = \prod_{i=0}^{+\infty} \frac{|F'(F^{-i}z))|}{|F'(F^{-i}w))|}.$$

See [9]. Now, recall that $\overline{\mu}_z(B) \asymp Leb_z(B)$ for every Borelian $B \subseteq Z$ for every $z \in Z_\delta$, bounded by some constant $C = C(\delta) > 1$, using Lemma 3.4 and Lemma 3.5. In particular, $\overline{\mu}_z$ is equivalent to $\overline{Leb}_z$ and then it is equivalent to $Leb_z$ for every $z$ in a hyperbolic Pesin set. So, arguing as in Theorem A's proof we find $\Sigma \subset Z$ with positive measure and a collection of non-overlapping s-rectangles $S_i \subset \Sigma$ and u-sets $U_i$ such that $\tau$ maps each $S_i$ hyperbolically onto $U_i$ and such that $Leb_z(\Sigma - \bigcup_i S_i) = 0$ almost surely. So, $Leb(J - \bigcup_i J_i) = 0$, $\Delta_0 = J$ and $\sum_{i=0}^{+\infty} n_i|J_i| < \infty$ since $Leb$ is equivalent to $Leb_z$ by Lemma 3.4, concluding the proof.

6. **Proof of Lemma 5.1.** Let $T : \bigcup_i J_i \longrightarrow J$ be the Markov map defined by the expanding branches $f^{n_i} : J_i \longrightarrow J$. Given $0 < \delta < 1$ we can find $N = N(\delta) > 0$, the first integer such that $\sum_{i=1}^{N} \operatorname{diam}(J_i) \geq (1-\delta)\operatorname{diam}(\bigcup_{i=1}^{+\infty} J_i)$. Then, choose $J(\delta) \subseteq J$ a closed interval such that $\bigcup_{i=1}^{N} J_i \subseteq J$ and $\operatorname{diam}(J(\delta)) \geq (1-\delta/2)|J|$.

Now, we choose closed intervals $J_i(\delta) \subseteq J_i$ such that $T_i : J_i(\delta) \longrightarrow J(\delta)$ is 1-to-1 and onto and define $T_\delta \mid J_i(\delta) = T_i \mid J_i(\delta)$. Denote $\wp(\delta) = \{J_i(\delta)\}_{1 \leq i \leq N}$. $T_\delta : \bigcup_i J_i(\delta) \longrightarrow J$ is expanding and has bounded distortion with the same coefficients as T and $\wp(\delta)$ is a Markov partition for $T_\delta$. As the atoms of $\wp(\delta)$ are pairwise disjoint, so $\Lambda_\delta$, the maximal invariant subset of $T_\delta$ is a topologically mixing hyperbolic Cantor set conjugated to a full-shift in $N = N(\delta)$ symbols. Notice that the non linear distortion and the expansion coefficients of $\Lambda_\delta$ are independent of $\delta$.

Now, $0 < \mathcal{H}_\alpha(\Lambda_\delta) < +\infty$ and there is a unique ergodic Gibbs measure $\mu_\delta$ equivalent to $\mathcal{H}_\alpha \mid \Lambda_\delta$ such that

$$C^{-1} \inf_i \operatorname{diam}(T_i(J_i)) \leq \frac{\mu_\delta(B(x,r))}{r^{\dim_{\mathcal{H}}(\Lambda)}} \leq C \sup_i \operatorname{diam}(T_i(J_i)) \tag{28}$$

for every $x \in \Lambda$ and $0 < r < 1$. $C > 1$ is a universal constant which only depends on the distortion and the expansion coefficient so it is constant and independent of $\delta$. $\mu_\delta$ is the measure of maximal dimension of $\Lambda$,m that is, $\dim_{\mathcal{H}}(\mu_\delta) = \dim_{\mathcal{H}}(\Lambda)$. See [14]. Now, we fix $0 < \delta_0 < 1$ such that for every $0 < \delta < \delta_0$ and $N = N(\delta)$ it holds

$$(1-\delta_0/2)\operatorname{diam}(J) \leq \min_{P\ in\wp(\delta)} \operatorname{diam}(T_\delta P) \leq \max_{P \in \wp(\delta)} \operatorname{diam}(T_\delta P) \leq \operatorname{diam}(J).$$

We get a constant $C > 1$ such that for every $n > 0$, $\mu_\delta(B(x,r)) / r^{\dim_{\mathcal{H}}(\Lambda(\delta))}$ is bounded in $[C^{-1}, C]$ for every $x \in \Lambda_\delta$ and $0 < r < 1$ and for every $0 < \delta < \delta_0$.

Let $\{\phi_i\}_{i>0}$ be a dense countable subset of $C^0(J)$. We fix first an $\epsilon(n) > 0$ such that

$$|x-y| < \epsilon(n) \Longrightarrow |\phi_i(x) - \phi_i(y)| < 1/4n, \text{for } \ i = 1, \cdots, n.$$

Let $\wp = \{J_i\}_{i>0}$ the initial Markov partition of $T$. We can find for every $n > 0$ an integer $N_n > n$ such that

$$\operatorname{diam}(T^{-N_n+k}\wp) < \epsilon(n) \quad for \ \ k = 0, \cdots, n.$$

The following property of our construction will be used below. Given a sequence $\beta(n) \longrightarrow 1$ we may choose $\delta_n > 0$ small enough in the above construction such that $\mu^*(\bigcup T^{-N_n}\wp(n)) \geq \beta(n)$ where $\wp(n) = \wp(\delta_n)$ is the initial Markov partition of $\Lambda(\delta_n)$. This follows since $\mu \mid \Delta_0$ is ergodic and $T$-invariant.

With the above choice of constants we define $\Delta_0^n = \Lambda(\delta(n))$ and $\mu_n^*$ be the unique (Gibbs) measure of maximal dimension of $\Delta_0^n$. Clearly, $\mu_n^*$ satisfies the uniform distribution property (24) and $\Delta_0 = \overline{\bigcup_n \Delta_0^n}$ as stated in Lemma 5.1. We will prove below that $\mu_n^* \longrightarrow \mu^*$.

For this, let us define

$$\Gamma_n = \{x \in J : |\frac{1}{k}\sum_{j=0}^{k-1} \phi_i(T^j(x)) - \int \phi_i(y) d\mu^*(y)| < \frac{1}{4n}, \quad \forall\, k \geq n \ i = 1, \ldots, n\}$$

$\mu^*(\Gamma_n) \longrightarrow 1$ by ergodicity. Now we choose $\delta(n) \longrightarrow 0^+$ above such that there is an integer $n_0 > 0$ with

$$\mu^*(\bigcup\{P \in T^{-N_n}\wp(\delta(n)) : \mu^*(P \cap \Gamma_n) > 0\}) \geq \mu^*(\Gamma_n)$$

for every $n \geq n_0$. In particular $\wp(n) \equiv \{P \in T^{-N_n}\wp(\delta(n)) : P \cap \Gamma_n \neq \emptyset\}$ is non-empty and we define $\wp(n) = \{T^{N_n}(P) : P \in \wp(n)\}$ and define

$$\Delta_0^n = \bigcap_{k \geq 0} T^{-k}(\bigcup \wp(n))$$

as the maximal invariant subset of T in $\bigcup \wp(n)$. $\Delta_0^n$ is a topologically mixing, hyperbolic Cantor set, conjugated to some shift with $m_n \leq \nu(\delta(n))$ symbols. We also have

$$\mu^*(\bigcup \wp(n)) \longrightarrow 1 \quad \text{ and } \quad Leb(\bigcup \wp(n)) \longrightarrow Leb(\bigcup_i J_i),$$

since $\mu^*(\Gamma_n) \longrightarrow 1$, where $Leb$ is the Lebesgue measure.

Now, take a point $x_P \in P \cap \Gamma_n$ for every $P \in \wp(n) \subseteq T^{-N_n}\wp(n)$ which intersects $\Gamma_n$ to form a subset $D_n \subseteq \Gamma_n$. Thus, given $x \in \Delta_0^n$, $N > 0$ and a piece of the orbit

$$\{T^j(x)\, j = 0, \cdots, nN\}$$

we find an ordered sequence $x_k \in D$, $j = 1, \cdots, N$ such that

$$|T^j(x) - T^j(x_k)| < \epsilon \qquad j = (k-1)n, \cdots, kn-1$$

for every $k = 1, \cdots, N$. Then

$$\begin{aligned} &\left| \frac{1}{nN} \sum_{j=0}^{nN-1} \phi_i(T^j(x)) - mu(\phi_i) \right| \\ \leq &\left| \frac{1}{N} \sum_{k=1}^{N} \{ \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(T^{(k-1)n+j}(x)) - \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(T^j(x_k)) \} \right| + \\ &\left| \frac{1}{N} \sum_{k=1}^{N} \{ \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(T^j(x_k)) - \mu^*(\phi) \} \right| \leq \frac{1}{2n} \end{aligned}$$

for $i = 1, \cdots, n$. For every $n > 0$ and a generic point $x \in \Delta_0^n$ for $\mu_n^*$ we can find $K_n > 0$ such that

$$\left| \frac{1}{k} \sum_{j=0}^{k-1} \phi_i(T^j(x)) - \mu_n^*(\phi_i) \right| < \frac{1}{2n} \qquad \forall\; k \geq K_n$$

and $K_n \longrightarrow +\infty$. Now for each $n \geq n_0$ take $N_n > 0$ such that $N_n \geq K_n$ and $x_n \in \Delta_0^n$ as above then

$$|\mu_n^*(\phi_i) - \mu^*(\phi_i)| < \frac{1}{n} \qquad \forall\; i = 1, \cdots n$$

and we are done.

7. **Proof of Lemma 3.4.** Let $\mu$ be any Borel probability in $[0,1]$, $\overline{\mu}$ its lift to Z and let $\mu_z$ be the family of conditional measures obtaining by pulling-back to $\xi(z)$, via $\pi_z = \pi \mid \xi(z)$, the Borel measure $\mu \mid \pi(\xi(z))$, normalized to a probability. This family $\{\mu_z\}$ integrates to a new $\sigma$-finite measure $\mu^*$ on Z defined by

$$\mu^*(B) = \int_{Z_\xi} \mu_z(B \cap \xi(z))\, d\overline{\mu}_\xi(\xi(z)),$$

where $(Z_\xi, \mathcal{B}_\xi, \overline{\mu}_\xi)$ is the quotient space of $Z$ defined by the measurable partition $\xi$. Cf. [15]. We first prove that

$$\overline{\mu}_z \ll \quad \mu_z \quad \overline{\mu} - a.e. \;\; z \in Z \tag{29}$$

For this we notice that, for every Borel measure $\mu$ on $[0,1]$, it holds $\mu(\pi(B)) \geq \overline{\mu}(B)$ for every Borelian $B \subseteq Z$. This can be seen from the definition of the lift measure (see below). Now, by definition, $\mu^*(B) = 0$ if and only if $\mu_z(B \cap \xi(z)) = 0$ for $\overline{\mu}_\xi$-a.e. $\xi(z) \in Z_\xi$ so that $\mu(\pi(B \cap \xi(z)) = 0$ for $\overline{\mu}$-a.e. $z \in Z$. Therefore, $\mu(\pi(B)) = 0$ and then $\overline{\mu}(B) = 0$, proving that $\overline{\mu}$ absolutely continuous with respect to $\mu^*$. Hence, $\overline{\mu}_z \ll \mu_z$ almost surely. Indeed, $\overline{\mu}$ has a density w.r.t. $\mu^*$ and then

$$\frac{d\overline{\mu}_{\xi(z)}}{d\mu_z}(w) = \frac{d\overline{\mu}}{d\mu^*}(w) \quad \text{almost surely.}$$

Cf. [11, Proposition 4.1].

Now, let us recall the lift a measure to the canonical extension. For each $n \geq 0$ we let $\pi_n : Z \longrightarrow [0,1]$ be the projection onto the n-th factor: $\pi_n(z) = x_n$. Given $\mu \in \mathcal{M}$ we define $\overline{\mu}_n(\pi_n^{-1}(B)) = \mu(B)$ for $B \in \mathcal{B}$ any Borel subset in [0,1]. The measure

$$\overline{\mu}_n : \mathcal{B}_n \longrightarrow [0,1]$$

is a countably additive, non-negative function and $\overline{\mu}_n(\emptyset) = 0$ defined over the $\sigma$-algebra $\mathcal{B}_n = \pi_n^{-1}\mathcal{B}$. Clearly $\bigvee_{n=0}^{+\infty} \mathcal{B}_n = \overline{\mathcal{B}}$. Hence,

$$\overline{\mu} \equiv \bigotimes_{n=0}^{+\infty} \overline{\mu}_n : \overline{\mathcal{B}} \longrightarrow [0,1]$$

is a Borel probability on $Z$ with $\pi_n^* \overline{\mu} = \mu$ for every $n \geq 0$. Let $B = \bigcap_{k=0}^{n} \pi_k^{-1}(B_k)$. Then, $\overline{Leb}(B) = \prod_{k=0}^{n} Leb(B_k)$ where $B_k = \pi_k(B)$ for $k = 0, \cdots, n$. Now, for $\overline{Leb}$-a.e. $z \in Z$ we get $\pi_k(B \cap \xi(z)) = g_z^k(\pi(B \cap \xi(z))) \subseteq B_k$ for $k = 0, \cdots, n$, where $g_z^k$ are the contractive inverse branches of $f^k$ associated to the pre-orbit $z \in Z$. $\overline{Leb}(B) = 0$ iff there is some $0 \leq k \leq n$ such that $Leb(B_k) = 0$. In particular, $Leb(\pi_k(B \cap \xi(z))) = 0$ for $\overline{Leb}$-a.e. $z \in Z$. Thus, $Leb(\pi(B \cap \xi(z))) = 0$ for $\overline{Leb}$-a.e. since $g_z^k$ are non singular maps. We conclude that $Leb^*(B) = 0$. Therefore

$$Leb^* \mid \mathcal{B}_n \ll \overline{Leb} \mid \mathcal{B}_n \quad \forall\, n \geq 0,$$

where $\mathcal{B}_n = \bigvee_{k=0}^{n} \pi_k^{-1}\mathcal{B}$ and then $Leb^*$ will be absolutely continuous w.r.t. $\overline{Leb}$, since the sequence $\mathcal{B}_n$ generates the Borel subsets of $Z$. We conclude $Leb_z \ll \overline{Leb}_{\xi(z)}$ for $\overline{Leb}$-a.e. $z \in Z$. Thus, by (29), $\overline{Leb}_z$ is equivalent to $\overline{Leb}_{\xi(z)}$ almost surely. Let us consider now an $f$ invariant Borel probability $\mu$ and let $B \in \mathcal{B}_n$. $\overline{\mu}(B) = 0$ iff there exists $0 \leq k \leq n$ such that $\mu(B_k) = 0$. But,

$$\pi_k(B \cap \xi(z)) = g_z^k(\pi(B \cap \xi(z))) \subseteq B_k.$$

Thus, $\mu(g_z^k(\pi(B \cap \xi(z)))) = 0$ for $\overline{\mu}$-a.e. $z \in Z$. Therefore, $\mu(f^{-k}(\pi(B))) = 0$, since

$$f^{-k}(\pi(B)) = \bigcup \{g_z^k(\pi(B \cap \xi(z)) : z \in \pi^{-1}\pi(B)\}$$

and then $\mu(\pi(B)) = 0$, by the invariance of $\mu$. We conclude that $\overline{\mu}_z$ is absolutely continuous w.r.t. to $\overline{\mu}_{\xi(z)}$ for $\overline{\mu}$-a.e. $z \in Z$ so they are equivalent almost surely. **QED**

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