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A Nonlinear Observer for Rigid Body Attitude Estimation using Vector Observations *

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Abstract: This work proposes a nonlinear observer for attitude estimation on SO(3), exploiting the information of vector observations and biased angular rate measurements. It is shown that the attitude and bias estimation errors converge exponentially fast to the origin, for arbitrary angular velocity trajectories. The proposed attitude feedback law is an explicit function of the vector measurements and observer estimates, and convergence rate bounds are obtained using recent results for parametrized linear time-varying systems. The stability and convergence properties of the estimation errors are evidenced in simulation for time-varying angular velocities.

1. INTRODUCTION

Attitude estimation is a classical problem with a rich and fascinating historical background, still subject of intensive research and advances in the present times [Crassidis et al., 2007]. Recent insights on the problem of nonlinear attitude estimation [Chaturvedi and McClamroch, 2006, Bhat and Bernstein, 2000] present guidelines for observer design on SO(3) and evidence the topological obstacles for global stabilization on the non-Euclidean spaces used in attitude representation.

Nonlinear attitude estimation and compensation algorithms have been proposed in recent literature. In [Lee et al., 2007], a deterministic attitude filter is derived using single direction measurements, where the attitude estimate and the vector observation are merged by intersecting uncertainty ellipsoids. In [Rehbinder and Ghosh, 2003], the observability of a locally exponentially convergent attitude observer is studied, using a monocular camera and inertial sensors. A symmetry-preserving observer for velocity-aided inertial navigation is presented in [Bonnabel et al., 2006]. In [Chaturvedi and McClamroch, 2006], an almost globally stabilizable attitude controller is obtained, guaranteeing input torque levels below the saturation limits. An eventually globally exponentially convergent angular velocity observer, expressed in the Euler quaternion representation, is derived in the reference work [Salcudean, 1991] by exploiting attitude and torque measurements.

In many applications it is desired to construct an attitude observer based only on the rotation kinematics. In [Pflimlin et al., 2007], an asymptotically stable attitude observer on SO(3) is derived using attitude and biased angular velocity readings. The nonlinear attitude observer proposed in [Thienel and Sanner, 2003] is formulated using the quaternion representation, to obtain global exponential convergence to the origin given attitude measurements and biased inertial readings. In these references, the observer is derived assuming that a perfect rotation matrix/quaternion attitude reading is available, obtained by batch processing information such as landmark measurements, image based features, and vector readings. However, it is desirable to exploit the sensor readings directly in the observer and to analyze how the estimation results are influenced by the characteristics of the vector measurements.

This work derives an exponentially convergent attitude observer on SO(3), exploiting directly vector observations and biased inertial readings. The feedback law, derived constructively using the Lyapunov's stability theory, is an explicit function of the vector measurements. Exponential convergence is obtained, and convergence bounds for the biased angular rate sensor case are given resorting to the recent results for parameterized linear time-varying (LTV) systems [Loría and Panteley, 2002].

The paper is organized as follows. In Section 2, the attitude estimation problem is described. Section 3 introduces the tools adopted for attitude observer design. A Lyapunov function based on the error of the vector observations is formulated, a coordinate transformation of the vector readings is proposed and the necessary sensor setup for attitude determination is discussed. In Section 4, the attitude observer is derived for the cases of unbiased and biased angular velocity readings. Exponential convergence of the attitude and bias estimation errors to the origin is demonstrated. In Section 5, the feedback law is written as an explicit function of the sensor readings, and exponential convergence bounds are obtained using the recent results for parametrized LTVs. In Section 6, the convergence of the estimation errors is illustrated in simulation for distinct initial conditions and feedback gain values. Section 7 presents concluding remarks and comments on future work.

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NOMENCLATURE

The notation adopted is fairly standard. The set of $n \times m$ matrices with real entries is denoted as $\mathrm{M}(n,m)$ and $\mathrm{M}(n) := \mathrm{M}(n,n)$. The sets of skew-symmetric, orthogonal, and special orthogonal matrices are respectively denoted as $\mathrm{K}(n) := \{\mathbf{K} \in \mathrm{M}(n) : \mathbf{K} = -\mathbf{K}'\}$, $\mathrm{O}(n) := \{\mathbf{U} \in \mathrm{M}(n) : \mathbf{U}'\mathbf{U} = \mathbf{I}\}$, $\mathrm{SO}(n) := \{\mathbf{R} \in \mathrm{O}(n) : \det(\mathbf{R}) = 1\}$, and the n-dimensional sphere and ball are described by $\mathrm{S}(n) := \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}'\mathbf{x} = 1\}$ and $\mathrm{B}(n) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}'\mathbf{x} \leq 1\}$, respectively.

2. PROBLEM FORMULATION

In this section, the attitude estimation problem is introduced. The vector observations are a function of the rigid body's orientation, and the inertial sensors measure the angular velocities of the body. While vector observations yield a snapshot attitude estimate for each time instant, inertial sensors allow for the propagation of the attitude in time. The attitude estimator combines the inertial measurements with the vector observations, hence exploiting both information sources.

The rigid body kinematics are described by

$$\dot{\bar{\mathcal{R}}} = \bar{\mathcal{R}}\left[\bar{\boldsymbol{\omega}}\times\right],$$

where $\bar{\mathcal{R}}$ is the shorthand notation for the rotation matrix ${}^L_B\mathbf{R}$ from body frame $\{B\}$ to local frame $\{L\}$ coordinates, $\bar{\boldsymbol{\omega}}$ is the body angular velocity expressed in $\{B\}$, and $[\mathbf{a}\times]$ is the skew symmetric matrix defined by the vector $\mathbf{a} \in \mathbb{R}^3$ such that $[\mathbf{a}\times]\,\mathbf{b} = \mathbf{a} \times \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^3$.

The body angular velocity is measured by a rate gyro sensor triad

$$\boldsymbol{\omega}_r = \bar{\boldsymbol{\omega}}. \tag{1}$$

On-board sensors such as magnetometers, star trackers and pendulums, among others, provide vector observations expressed in body frame coordinates

$$\mathbf{h}_{r\,i} = {}^B\bar{\mathbf{h}}_i := \bar{\mathcal{R}}'^L\mathbf{h}_i, \tag{2}$$

where $i = 1..n$ is the vector index, n is the number of vector measuring sensors and the vector representation in the local coordinate frame $\{L\}$, denoted by ${}^L\mathbf{h}_i$, is known.

The proposed observer estimates the orientation of the rigid body by computing the kinematics

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}}\left[\hat{\boldsymbol{\omega}}\times\right],$$

where $\hat{\mathcal{R}}$ is the estimated attitude and $\hat{\boldsymbol{\omega}}$ is the feedback term constructed to compensate for the attitude estimation error.

The attitude error is defined as $\tilde{\mathcal{R}} := \hat{\mathcal{R}}'\bar{\mathcal{R}}$, and the Euler angle-axis parametrization of $\tilde{\mathcal{R}}$ is described by the rotation vector $\boldsymbol{\lambda} \in \mathrm{S}(2)$ and by the rotation angle $\theta \in [0\ \pi]$, yielding $\tilde{\mathcal{R}} = \mathrm{rot}(\theta,\boldsymbol{\lambda}) := \cos(\theta)\mathbf{I} + \sin(\theta)\,[\boldsymbol{\lambda}\times] + (1-\cos(\theta))\boldsymbol{\lambda}\boldsymbol{\lambda}'$. While the observer results are formulated directly in the SO(3) manifold, the rotation angle θ is adopted to characterize some of the convergence properties of the observer.

The attitude error kinematics are a function of the angular velocity estimates and given by $\dot{\tilde{\mathcal{R}}} = -\tilde{\mathcal{R}}\left[\tilde{\mathcal{R}}'\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}\times\right]$. The attitude feedback law $\hat{\boldsymbol{\omega}}$ is defined as function of the velocity readings (1) and vector observations (2), so that the closed loop attitude estimation errors converge to the origin, i.e., $\tilde{\mathcal{R}} \to \mathbf{I}$, as $t \to \infty$.

3. OBSERVER CONFIGURATION

The attitude feedback law is derived resorting to the Lyapunov's stability theory and to a conveniently defined transformation of the vector observations. Define the linear combination of the sensed vector ${}^L\mathbf{h}_i$ expressed in the local coordinate frame, given by

$${}^L\mathbf{u}_j := \sum_{i=1}^{n} a_{ij}\,{}^L\mathbf{h}_i, \quad j = 1..n. \tag{3}$$

The vector transformation (3) is represented in matrix form by $\mathbf{U}_H = \mathbf{H}\mathbf{A}_H$, where $\mathbf{U}_H := \left[{}^L\mathbf{u}_1\ \ldots\ {}^L\mathbf{u}_n\right]$, $\mathbf{H} := \left[{}^L\mathbf{h}_1\ \ldots\ {}^L\mathbf{h}_n\right]$, $\mathbf{U}_H, \mathbf{H} \in \mathrm{M}(3,n)$ and $\mathbf{A}_H := [a_{ij}] \in \mathrm{M}(n)$ is invertible.

Let ${}^B\hat{\mathbf{u}}_i := \hat{\mathcal{R}}'^L\mathbf{u}_i$ and ${}^B\bar{\mathbf{u}}_i := \bar{\mathcal{R}}'^L\mathbf{u}_i$ be the estimated and the nominal representation of ${}^L\mathbf{u}_j$ in Body frame coordinates, respectively. The corresponding matrix representation is ${}^B\bar{\mathbf{U}}_H = \bar{\mathcal{R}}'\mathbf{U}_H$, ${}^B\hat{\mathbf{U}}_H = \hat{\mathcal{R}}'\mathbf{U}_H$, where ${}^B\hat{\mathbf{U}}_H := \left[{}^B\hat{\mathbf{u}}_1\ \ldots\ {}^B\hat{\mathbf{u}}_n\right]$ and ${}^B\bar{\mathbf{U}}_H := \left[{}^B\bar{\mathbf{u}}_1\ \ldots\ {}^B\bar{\mathbf{u}}_n\right]$, ${}^B\hat{\mathbf{U}}_H, {}^B\bar{\mathbf{U}}_H \in \mathrm{M}(3,n)$.

The candidate Lyapunov function is defined by the estimation error of the transformed vectors

$$V = \tfrac{1}{2}\sum_{i=1}^{n}\|{}^B\hat{\mathbf{u}}_i - {}^B\bar{\mathbf{u}}_i\|^2 = \tfrac{1}{2}\|{}^B\hat{\mathbf{U}}_H - {}^B\bar{\mathbf{U}}_H\|^2. \tag{4}$$

Algebraic manipulation produces the equivalent Lyapunov formulation and time derivative

$$\begin{aligned} V &= \mathrm{tr}\left[(\mathbf{I}-\tilde{\mathcal{R}})\mathbf{U}_H\mathbf{U}_H'\right] = (1-\cos(\theta))\,\boldsymbol{\lambda}'\mathbf{P}\boldsymbol{\lambda}, \\ \dot{V} &= \left[\tilde{\mathcal{R}}'\mathbf{U}_H\mathbf{U}_H' - \mathbf{U}_H\mathbf{U}_H'\tilde{\mathcal{R}}\otimes\right]'(\tilde{\mathcal{R}}'\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}) \end{aligned} \tag{5}$$

where $\mathbf{P} := \mathrm{tr}(\mathbf{U}_H\mathbf{U}_H')\mathbf{I} - \mathbf{U}_H\mathbf{U}_H'$, $\mathbf{P} \in \mathrm{M}(3)$ and $\otimes$ is the unskew operator such that $[[\mathbf{a}\times]\otimes] = \mathbf{a}, \mathbf{a} \in \mathbb{R}^3$.

3.1 Vector Measurement Configuration

The proposed Lyapunov function measures the error of the vector observations. To guarantee that $V = 0$ if and only if the attitude is correctly estimated, i.e. $\tilde{\mathcal{R}} = \mathbf{I}$, the geometric configuration of the measured vectors is required to satisfy the following assumption.

Assumption 1. There are at least two noncollinear vectors ${}^L\mathbf{h}_i$, that is, $\mathrm{rank}(\mathbf{H}) \geq 2$.

Lemma 2. The Lyapunov function V has a unique global minimum at $\tilde{\mathcal{R}} = \mathbf{I}$ if and only if Assumption 1 is verified

$$\forall_{\tilde{\mathcal{R}}\neq\mathbf{I}} V > 0 \quad \text{if and only if} \quad \exists_{i\neq j}\forall_{\alpha\in\mathbb{R}} : {}^L\mathbf{h}_i \neq \alpha^L\mathbf{h}_j.$$

Proof. The result can be obtained by following the proof of [Vasconcelos et al., 2007, Lemma 1] where the properties of a similar Lyapunov function are derived. □

To illustrate the necessity of Assumption 1, assume that $\mathrm{rank}(\mathbf{H}) = 1$, i.e. all ${}^L\mathbf{h}_i$ (and ${}^L\mathbf{u}_i$) are collinear. Any attitude error $\tilde{\mathcal{R}}$ represented by an arbitrary rotation θ about the vector $\boldsymbol{\lambda} = {}^L\mathbf{u}_i/\|{}^L\mathbf{u}_i\|$ satisfies $V = 0$. For a detailed insight on the limitations of attitude estimation using single direction measurements see [Lee et al., 2007] and references therein.

3.2 Vector Measurement Directionality

The asymptotic stability and the region of attraction of the origin are derived by analyzing the level sets $V \leq c$. For c large enough, the level sets contain multiple critical points due to the directionality of $\mathbf{P}$, see [Vasconcelos et al., 2007, Lemma 2] for a motivation. In the present work, the directionality of $\mathbf{P}$ is made uniform by means of transformation $\mathbf{A}_H$.

Proposition 3. Assume that $\mathbf{H}$ is full rank, then there is a nonsingular $\mathbf{A}_H \in \mathrm{M}(n)$ such that $\mathbf{U}_H\mathbf{U}_H' = \mathbf{I}$.

Proof. Take the SVD decomposition of $\mathbf{H} = \mathbf{USV'}$ where $\mathbf{U} \in \mathrm{O}(3)$, $\mathbf{V} \in \mathrm{O}(n)$, $\mathbf{S} = \left[\mathrm{diag}(s_1, s_2, s_3)\ \mathbf{0}_{3\times(n-3)}\right] \in \mathrm{M}(3, n)$, and $s_1 > s_2 > s_3 > 0$ are the singular values of $\mathbf{H}$. Any $\mathbf{A}_H = \mathbf{V}\,\mathrm{blkdiag}(s_1^{-1}, s_2^{-1}, s_3^{-1}, \mathbf{B})\mathbf{V}_A'$, where $\mathbf{B} \in \mathrm{M}(n-3)$ is nonsingular and $\mathbf{V}_A \in \mathrm{O}(n)$, verifies $\mathbf{U}_H\mathbf{U}_H' = \mathbf{H}\mathbf{A}_H\mathbf{A}_H'\mathbf{H} = \mathbf{U}\mathbf{V}_A'\mathbf{V}_A\mathbf{U}' = \mathbf{I}$. □

Using the transformation $\mathbf{A}_H$ defined in Proposition 3, the Lyapunov function (5) is expressed by

$$V = \tfrac{1}{2}\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 = 2(1 - \cos(\theta)), \tag{6}$$
$$\dot{V} = \left[\tilde{\mathcal{R}}' - \tilde{\mathcal{R}}\otimes\right]'(\tilde{\mathcal{R}}'\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}) = -2\sin(\theta)\boldsymbol{\lambda}'(\tilde{\mathcal{R}}'\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}).$$

Note that the conditions of Proposition 3 are not satisfied directly by Assumption 1. In Appendix A it is shown that in case $\mathrm{rank}(\mathbf{H}) = 2$, the direction orthogonal to the columns of $\mathbf{H}$ can be generated, producing a full rank matrix $\mathbf{H}_a$ that is used in the observer equations in the place of $\mathbf{H}$.

4. OBSERVER SYNTHESIS

In this section, the feedback law for attitude estimation in the presence of rate gyro bias is derived. The formulation of a feedback law for unbiased angular velocity readings is presented to illustrate the topological limitations to global stabilization on SO(3).

4.1 Unbiased Angular Velocity Measurements

Under Assumption 1 and given the Lyapunov function time derivative (6), a feedback law is proposed to drive the attitude error to zero,

$$\hat{\boldsymbol{\omega}} = \tilde{\mathcal{R}}\bar{\boldsymbol{\omega}} - K_\omega\mathbf{s}_\omega, \tag{7}$$

where the feedback term is given by

$$\mathbf{s}_\omega := \left[\tilde{\mathcal{R}}' - \tilde{\mathcal{R}}\otimes\right] = -2\sin(\theta)\boldsymbol{\lambda}, \tag{8}$$

and $K_\omega > 0$ is a positive scalar. The attitude feedback yields the autonomous closed loop attitude kinematics

$$\dot{\tilde{\mathcal{R}}} = K_\omega\tilde{\mathcal{R}}(\tilde{\mathcal{R}}' - \tilde{\mathcal{R}}), \tag{9}$$

and the closed loop Lyapunov function time derivative is given by $\dot{V} = -K_\omega\mathbf{s}_\omega'\mathbf{s}_\omega = -4K_\omega\sin^2(\theta) \leq 0$, so it is immediate that the attitude feedback law produces a Lyapunov function that decreases along the system trajectories. Under Assumption 1, the set of points where $\dot{V} = 0$ is given by

$$C_\mathcal{R} = \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \tilde{\mathcal{R}} = \mathbf{I} \vee \tilde{\mathcal{R}} = \mathrm{rot}(\pi, \boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathrm{S}(2)\}$$

By direct substitution in the closed loop system (9), it is easy to see that $\tilde{\mathcal{R}} = \mathrm{rot}(\pi, \lambda) \Rightarrow \dot{\tilde{\mathcal{R}}} = 0$. Convergence to the origin by LaSalle's invariance principle is inconclusive given that $C_\mathcal{R}$ is invariant.

The equilibrium points $\theta = \pi$ are a consequence of the fact that the region of attraction of a stable equilibrium point is homeomorphic to some Euclidean vector space and that SO(3) is only locally homeomorphic to $\mathbb{R}^3$ [Bhat and Bernstein, 2000]. However, the set $\theta = \pi$ has zero measure and the following theorem establishes that the trajectories emanating from the set $\theta < \pi$, i.e. almost everywhere, converge exponentially fast to the origin. The proof is obtained by adaptation of the results presented in [Vasconcelos et al., 2007, Theorem 3] for a similar error kinematics and Lyapunov function.

Theorem 4. The closed-loop system (9) has an exponentially stable point at $\tilde{\mathcal{R}} = \mathbf{I}$. For any initial condition in the region of attraction $\tilde{\mathcal{R}}(t_0) \in \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \tilde{\mathcal{R}} = \mathrm{rot}(\theta, \boldsymbol{\lambda}), |\theta| < \pi, \boldsymbol{\lambda} \in \mathrm{S}(2)\}$ the trajectory satisfies

$$\|\tilde{\mathcal{R}}(t) - \mathbf{I}\| \leq k_\mathcal{R}\|\tilde{\mathcal{R}}(t_0) - \mathbf{I}\|e^{-\frac{1}{2}\gamma_\mathcal{R}(t-t_0)}, \tag{10}$$

where $k_\mathcal{R} = 1$ and $\gamma_\mathcal{R} = 2K_\omega(1 + \cos(\theta(t_0)))$.

4.2 Biased Angular Velocity Measurements

In this section, asymptotic stabilization of the attitude error in the presence of angular velocity bias is derived and exponential convergence to the origin is obtained. The rate gyro readings are corrupted by a bias term

$$\boldsymbol{\omega}_r = \bar{\boldsymbol{\omega}} + \bar{\mathbf{b}}_\omega$$

where the nominal bias is considered constant, $\dot{\bar{\mathbf{b}}}_\omega = \mathbf{0}$. The proposed Lyapunov function (6) is augmented to account for the effect of the rate gyro bias

$$V_b = 2\left(1 - \cos(\theta)\right) + \tfrac{1}{2}\tilde{\mathbf{b}}_\omega'\mathbf{W}_{b_\omega}\tilde{\mathbf{b}}_\omega$$

where $\tilde{\mathbf{b}}_\omega = \hat{\mathbf{b}}_\omega - \bar{\mathbf{b}}_\omega$ is the bias compensation error, $\hat{\mathbf{b}}_\omega$ is the estimated bias and $\mathbf{W}_{b_\omega}$ is a positive definite matrix. Under Assumption 1 and given Lemma 2, the Lyapunov function V_b has an unique global minimum at $(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, 0)$.

The feedback law for the angular velocity is obtained by compensating the bias of the angular velocity reading in (7), producing

$$\hat{\boldsymbol{\omega}} = \tilde{\mathcal{R}}(\bar{\boldsymbol{\omega}} + \bar{\mathbf{b}}_\omega - \hat{\mathbf{b}}_\omega) - K_\omega\mathbf{s}_\omega = \tilde{\mathcal{R}}(\bar{\boldsymbol{\omega}} - \tilde{\mathbf{b}}_\omega) - K_\omega\mathbf{s}_\omega.$$

The time derivative of the augmented Lyapunov function is described by $\dot{V}_b = -K_\omega\mathbf{s}_\omega'\mathbf{s}_\omega + \tilde{\mathbf{b}}_\omega'(\mathbf{W}_b\dot{\tilde{\mathbf{b}}}_\omega - \mathbf{s}_\omega)$. Noting that $\dot{\tilde{\mathbf{b}}}_\omega = \dot{\hat{\mathbf{b}}}_\omega$, the bias feedback law is defined as $\dot{\hat{\mathbf{b}}}_\omega = K_{b_\omega}\mathbf{s}_\omega$, and $\mathbf{W}_b = K_{b_\omega}^{-1}\mathbf{I}$ where K_{b_ω} is a positive scalar. The closed loop kinematics are given by

$$\dot{\tilde{\mathcal{R}}} = K_\omega\tilde{\mathcal{R}}(\tilde{\mathcal{R}}' - \tilde{\mathcal{R}}) + \tilde{\mathcal{R}}\left[\tilde{\mathbf{b}}_\omega\times\right],\ \dot{\tilde{\mathbf{b}}}_\omega = K_{b_\omega}\left[\tilde{\mathcal{R}}' - \tilde{\mathcal{R}}\otimes\right], \tag{11}$$

and the time derivative of the Lyapunov function is described by $\dot{V}_b = -K_\omega\tilde{\mathbf{s}}_\omega'\tilde{\mathbf{s}}_\omega = -4K_\omega\sin^2(\theta)$.

The set of points where $\dot{V}_{b_\omega} = 0$ is characterized by $C_{b_\omega} = \{(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) \in \mathrm{SO}(3) \times \mathbb{R}^3 : \tilde{\mathcal{R}} \in C_\mathcal{R}\}$. The invariant subsets of C_{b_ω} where $\theta = \pi$ are a consequence of the topological limitation to global stabilization on SO(3) discussed in Section 4.1. By analyzing the level sets of V_b, the next lemma shows that the attitude and bias estimation errors are bounded, providing sufficient conditions that exclude convergence to the attitude error $\theta = \pi$.

Lemma 5. The attitude and bias estimation errors, $\tilde{\mathcal{R}}$ and $\tilde{\mathbf{b}}_\omega$ respectively, are bounded. For any initial condition such that

$$K_{b_\omega} > \frac{\|\tilde{\mathbf{b}}_\omega(t_0)\|^2}{4(1+\cos(\theta(t_0)))}, \tag{12}$$

the attitude error is bounded by $\theta(t) \leq \theta_{\max} < \pi$ for all $t \geq t_0$.

Proof. Let $\mathbf{x} := (\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega)$ and define the set $\Omega_\rho = \{\mathbf{x} \in \mathrm{D} : V_b \leq \rho\}$ where $\mathrm{D} = \mathrm{SO}(3) \times \mathbb{R}^3$. The Lyapunov function is given by the weighted distance of the state to the origin $V_b = \frac{1}{2}(\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 + \frac{1}{K_{b_\omega}}\|\tilde{\mathbf{b}}_\omega\|^2)$, so the set Ω_ρ is compact. The Lyapunov function decreases along the system trajectories, $\dot{V}_b \leq 0$, so any trajectory starting in Ω_ρ will remain in Ω_ρ. Consequently, $\forall_{t\geq t_0} \frac{1}{2}(\|\mathbf{I} - \tilde{\mathcal{R}}(t)\|^2 + \frac{1}{K_{b_\omega}}\|\tilde{\mathbf{b}}_\omega(t)\|^2) \leq V_b(\mathbf{x}(t_0))$ and the state is bounded.

The gain condition (12) is equivalent to $V_b(\mathbf{x}(t_0)) < 4$. Given that $V_b(\mathbf{x}(t)) \leq V_b(\mathbf{x}(t_0))$, then $2(1-\cos(\theta(t))) \leq V_b(\mathbf{x}(t_0)) < 4$ which implies that $\theta(t) < \pi$ for all $t > t_0$. □

Although asymptotic stability of the origin can be obtained by LaSalle's invariance principle and Lemma 5, the exponential properties of LTV systems [Khalil, 1996, Theorem 3.9] motivate the stability analysis of the attitude observer kinematics (11), in the form $\dot{\mathbf{x}} = f(t, \mathbf{x})\mathbf{x}$, as a parameterized LTV system $\dot{\mathbf{x}}_\star = \mathbf{A}(\lambda, t)\mathbf{x}_\star$, where the parameter λ is associated with the initial conditions of the nonlinear system [Loría and Panteley, 2002].

A seminal approach to the LTV formulation can be found in [Khalil, 1996, Section 13.4] for a model reference adaptive controller, and exploited in [Thienel and Sanner, 2003] for a quaternion based attitude observer. However, as shown in [Khalil, 1996, p. 629], the inference of exponential stability must be addressed properly given that $\mathbf{A}(\lambda, t)$ depends on the parameter λ. The validity of the parameterized LTV approach is demonstrated in [Loría and Panteley, 2002], which generalizes the exponential stability results of nonparameterized to parameterized systems, under uniformity conditions with respect to the parameter λ. The tools presented therein allow for the derivation of exponential stability of the present attitude observer.

Theorem 6. For any initial condition such that $\theta(t_0) < \pi$, let the feedback gain satisfy (12). Then the attitude and bias estimation errors converge exponentially fast to the stable equilibrium point $(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, 0)$.

Proof. Let the attitude error vector be given by $\tilde{\mathbf{q}}_q = \sin(\frac{\theta}{2})\boldsymbol{\lambda}$, the closed loop attitude and bias compensation errors kinematics are described by

$$\dot{\tilde{\mathbf{q}}}_q = \tfrac{1}{2}\mathbf{Q}(\tilde{\mathbf{q}})(\tilde{\mathbf{b}}_\omega - 4K_\omega\tilde{\mathbf{q}}_q\tilde{q}_s), \quad \dot{\tilde{\mathbf{b}}}_\omega = -4K_{b_\omega}\mathbf{Q}'(\tilde{\mathbf{q}})\tilde{\mathbf{q}}_q, \tag{13}$$

where $\mathbf{Q}(\tilde{\mathbf{q}}) := \tilde{q}_s\mathbf{I} + [\tilde{\mathbf{q}}_q\times]$, $\tilde{q}_s = \cos(\frac{\theta}{2})$, $\dot{\tilde{q}}_s = 2K_\omega\tilde{\mathbf{q}}_q'\tilde{\mathbf{q}}_q\tilde{q}_s - \frac{1}{2}\mathbf{q}_q'\tilde{\mathbf{b}}_\omega$, and $\tilde{\mathbf{q}} = \begin{bmatrix}\tilde{\mathbf{q}}_q' & \tilde{q}_s\end{bmatrix}'$ is the Euler quaternion representation [Wertz, 1978]. Using $\|\tilde{\mathbf{q}}_q\|^2 = \frac{1}{8}\|\tilde{\mathcal{R}} - \mathbf{I}\|^2$, the Lyapunov function in quaternion coordinates is described by $V_b = 4\|\tilde{\mathbf{q}}_q\|^2 + \frac{1}{2K_{b_\omega}}\|\tilde{\mathbf{b}}_\omega\|^2$.

Define the system (13) in the domain $\mathcal{D}_q = \{(\tilde{\mathbf{q}}_q, \tilde{\mathbf{b}}_\omega) \in \mathrm{B}(3) \times \mathbb{R}^3 : V_b \leq 4 - \varepsilon_q\}$, $0 < \varepsilon_q < 4$. The set $\mathcal{D}_q$ is given by the interior of the Lyapunov surface, so it is positively invariant and well defined. The feedback gain (12) implies that the initial condition is in the set $\mathcal{D}_q$ for ε_q small enough.

Define the parameterized LTV system

$$\begin{bmatrix}\dot{\tilde{\mathbf{q}}}_{q\star} \\ \dot{\tilde{\mathbf{b}}}_{\omega\star}\end{bmatrix} = \begin{bmatrix}\mathcal{A}(t,\lambda) & \mathcal{B}'(t,\lambda) \\ -\mathcal{C}(t,\lambda) & \mathbf{0}_{3\times3}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{q}}_{q\star} \\ \tilde{\mathbf{b}}_{\omega\star}\end{bmatrix} \tag{14}$$

where $(\tilde{\mathbf{q}}_{q\star}, \tilde{\mathbf{b}}_{\omega\star}) \in \mathbb{R}^3 \times \mathbb{R}^3$, $\lambda \in \mathbb{R}_{\geq 0} \times \mathcal{D}_q$. The matrices $\mathcal{A}(t,\lambda) := -2K_\omega\tilde{q}_s(t,\lambda)\mathbf{Q}(\tilde{\mathbf{q}}(t,\lambda))$, $\mathcal{B}(t,\lambda) := \frac{1}{2}\mathbf{Q}'(\tilde{\mathbf{q}}(t,\lambda))$ and $\mathcal{C}(t,\lambda) := 4K_{b_\omega}\mathbf{Q}'(\tilde{\mathbf{q}}(t,\lambda))$ are bounded, so the system is well defined [Khalil, 1996, p. 626]. The quaternion $\tilde{\mathbf{q}}(t,\lambda)$ represents the solution of (13) with initial condition $\lambda = (t_0, \tilde{\mathbf{q}}_q(t_0), \tilde{\mathbf{b}}_\omega(t_0))$. If the parameterized LTV system (14) is λ-UGES, then the nonlinear system (13) is uniformly exponentially stable in the domain $\mathcal{D}_q$, see Appendix B for details. The parameterized LTV system verifies the assumptions of [Loría and Panteley, 2002, Theorem 1]:

1) The elements of $\mathcal{B}(t,\lambda)$ and $\frac{\partial\mathcal{B}(t,\lambda)}{\partial t} = \frac{1}{2}\mathbf{Q}'(\dot{\tilde{\mathbf{q}}}(t,\lambda))$ are bounded, so there exits b_M such that

$$\max_{\lambda\in\mathbb{R}_{\geq 0}\times\mathcal{D}_q,\ t\geq 0}\left\{|\mathcal{B}(t,\lambda)|, \left|\frac{\partial\mathcal{B}(t,\lambda)}{\partial t}\right|\right\} \leq b_M$$

where $|\cdot|$ is the induced Euclidean norm of matrices.

2) The positive definite matrices $P(t,\lambda) = 8K_{b_\omega}\mathbf{I}$ and $Q(t,\lambda) = 32K_{b_\omega}K_\omega\tilde{q}_s^2(t,\lambda)$ satisfy $-Q(t,\lambda) = \mathcal{A}'(t,\lambda)P(t,\lambda) + P(t,\lambda)\mathcal{A}(t,\lambda) + \dot{P}(t,\lambda)$, $P(t,\lambda)\mathcal{B}'(t,\lambda) = \mathcal{C}'(t,\lambda)$, $p_m\mathbf{I} \leq P(t,\lambda) \leq p_M\mathbf{I}$, $q_m\mathbf{I} \leq Q(t,\lambda) \leq q_M\mathbf{I}$, with $p_m = p_M = 8K_{b_\omega}$, $q_m = q_M\cos^2(\frac{\theta_{\max}}{2})$ and $q_M = 32K_\omega K_{b_\omega}$.

The system (14) is λ-UGES if and only if $\mathcal{B}(t,\lambda)$ is λ-uniformly persistently exciting (λ-uPE) [Loría and Panteley, 2002]. For any unitary norm vector $\mathbf{y}$

$$\|\mathcal{B}'(\tau,\lambda)\mathbf{y}\|^2 = \frac{1-(\mathbf{y}'\tilde{\mathbf{q}}_q)^2}{4} \geq \frac{1-\|\tilde{\mathbf{q}}_q\|^2}{4} \geq \frac{1-\sin^2\left(\frac{\theta_{\max}}{2}\right)}{4},$$

hence the persistency of excitation condition is satisfied, $\mathbf{y}'\int_t^{t+T}\mathcal{B}(\tau,\lambda)\mathcal{B}'(\tau,\lambda)d\tau\mathbf{y} \geq \frac{T}{4}\cos^2\left(\frac{\theta_{\max}}{2}\right)$. Consequently, the parameterized LTV (14) is λ-UGES, and the nonlinear system (13) is exponentially stable in the domain $\mathcal{D}_q$. □

Theorem 6 guarantees that the trajectories emanating from the initial conditions in the set $\{(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) \in \mathrm{SO}(3) \times \mathbb{R}^3 : \frac{1}{2}\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 + \frac{1}{2K_{b_\omega}}\|\tilde{\mathbf{b}}_\omega\|^2 < 4\}$ converge exponentially fast to the origin if K_{b_ω} satisfies (12). The following corollary establishes sufficient conditions in K_{b_ω} for uniform exponential stability, i.e. the exponential convergence rate bounds are independent of the initial condition $\mathbf{x}(t_0)$.

Corollary 7. Assume that the initial estimation errors are bounded

$$\theta(t_0) \leq \theta_{0\,\max} < \pi, \quad \|\tilde{\mathbf{b}}_\omega(t_0)\| \leq \tilde{b}_{0\,\max}. \tag{15}$$

Let $K_{b_\omega} > \frac{\tilde{b}_{0\,\max}^2}{4(1+\cos(\theta_{0\,\max}))}$, the origin $(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, 0)$ is exponentially stable, uniformly in the set defined by (15).

5. OBSERVER PROPERTIES

This section evidences important characteristics of the observer. Namely, it is shown that the attitude feedback law is an explicit function of the sensor readings and the state estimates. Convergence bounds for the estimation error of the attitude observer are also presented.

Theorem 8. The feedback laws are explicit functions of the sensor readings and state estimates

$$\hat{\boldsymbol{\omega}} = \hat{\mathbf{U}}_H \mathbf{A}'_H \mathbf{H}'_r \left(\boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega\right) - K_\omega \mathbf{s}_\omega, \quad \dot{\hat{\mathbf{b}}}_\omega = K_{b_\omega} \mathbf{s}_\omega,$$

$$\mathbf{s}_\omega = \sum_{i=1}^{n} ({}^B\hat{\mathbf{U}}_H \mathbf{e}_i) \times (\mathbf{H}_r \mathbf{A}_H \mathbf{e}_i),$$

where $\mathbf{H}_r := [\mathbf{h}_{r\,1} \cdots \mathbf{h}_{r\,n}]$ is the concatenation of the vector readings, ${}^B\hat{\mathbf{U}}_H := \hat{\mathcal{R}}'\mathbf{H}\mathbf{A}_H$, and $\mathbf{e}_i$ is the unit vector where $e_i = 1$.

Proof. Using ${}^B\hat{\mathbf{U}}_H {}^B\bar{\mathbf{U}}'_H = \hat{\mathcal{R}}'\mathbf{U}_H\mathbf{U}'_H\bar{\mathcal{R}} = \tilde{\mathcal{R}}$ in (8) yields $\mathbf{s}_\omega = \left[{}^B\bar{\mathbf{U}}_H {}^B\hat{\mathbf{U}}'_H - {}^B\hat{\mathbf{U}}_H {}^B\bar{\mathbf{U}}'_H \otimes\right]$. Using ${}^B\bar{\mathbf{U}}_H {}^B\hat{\mathbf{U}}'_H = \sum_{i=1}^n {}^B\bar{\mathbf{u}}_i {}^B\hat{\mathbf{u}}'_i$ and ${}^B\bar{\mathbf{u}}_i {}^B\hat{\mathbf{u}}'_i - {}^B\hat{\mathbf{u}}_i^{B}\bar{\mathbf{u}}'_i = [({}^B\hat{\mathbf{u}}_i \times {}^B\bar{\mathbf{u}}_i)\times]$, bears $\mathbf{s}_\omega = \sum_{i=1}^n \left({}^B\hat{\mathbf{u}}_i \times {}^B\bar{\mathbf{u}}_i\right) = \sum_{i=1}^n ({}^B\hat{\mathbf{U}}_H \mathbf{e}_i) \times ({}^B\bar{\mathbf{U}}_H \mathbf{e}_i)$. Applying ${}^B\bar{\mathbf{U}}_H = \bar{\mathcal{R}}'\mathbf{H}\mathbf{A}_H$ and $\mathbf{H}_r = \bar{\mathcal{R}}'\mathbf{H}$ produces the desired results. □

The convergence rate bounds are a direct consequence of the results presented in [Loría, 2004, Theorem 1 and Remark 2]. The analytical derivation of the constants for the attitude observer is omitted due to space constraints.

Corollary 9. Under the conditions described in Corollary 7, the trajectories of the system (11) satisfy

$$\|\mathbf{x}(t)\| \le k_{b_\omega} \|\mathbf{x}(t_0)\| e^{-\frac{1}{2}\gamma_{b_\omega}(t-t_0)}, \forall t \ge t_0$$

where

$$\mathbf{x}(t) := \left(\frac{\tilde{\mathcal{R}}(t) - \mathbf{I}}{\sqrt{8}}, \tilde{\mathbf{b}}_\omega(t)\right), k_{b_\omega} = t_M t_M^{\mathrm{inv}} e^{\frac{1}{2}} \gamma_{b_\omega}^{-\frac{1}{2}}, \gamma_{b_\omega} = \frac{\rho}{\pi_c},$$

$$t_M = \left(\frac{5}{4} + \frac{2}{1+\sqrt{17}}\right)^{\frac{1}{2}}, t_M^{\mathrm{inv}} = \left(\frac{5}{4} + \frac{2}{1-\sqrt{17}}\right)^{-\frac{1}{2}},$$

and the constants ρ and π_c are defined as in [Loría, 2004, Theorem 1], with

$$a_M = 2K_\omega, b_M = \max\left\{\tfrac{1}{2}, \tfrac{1}{2}\left(16K_\omega^2 + \tilde{b}_{\max}^2\right)^{\frac{1}{2}}\right\},$$

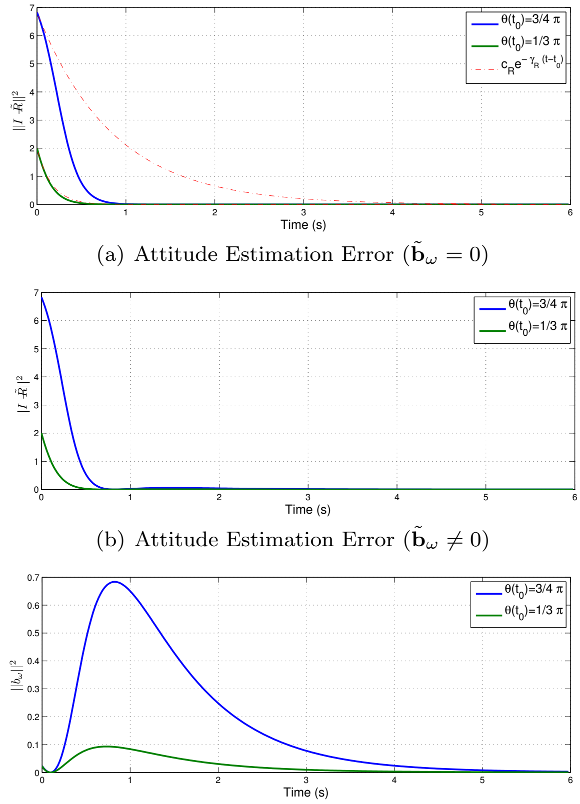
$$K_{b_\omega} = \rho_\omega \frac{\tilde{b}_{0\,\max}^2}{4(1+\cos(\theta_{0\,\max}))}, \rho_\omega > 1, \gamma_x = \frac{\cos^2(\frac{\theta_{\max}}{2})}{4(1+b_M^2 T)}, T > 0,$$

$$\tilde{b}_{\max} = \tilde{b}_{0\,\max}\left(\rho_\omega \tan^2\left(\frac{\theta_{0\,\max}}{2}\right) + 1\right)^{\frac{1}{2}},$$

$$\theta_{\max} = \arccos\left((1-\rho_\omega^{-1})\cos(\theta_{0\,\max}) - \rho_\omega^{-1}\right).$$

6. SIMULATIONS

In this section, simulation results for the proposed attitude observer are presented. The directions of the sensed vectors are given by ${}^L\mathbf{h}_1 = [1\ 0\ 0]'$ and ${}^L\mathbf{h}_2 = [0\ 0\ 1]'$, which are a simple representation of the vectors that are measured in the body coordinates by a magnetic compass and a pendulum, respectively. Under strong accelerations or magnetic distortions, other vector measurements such as star trackers or image based feature detection can be adopted. The matrix $\mathbf{H}$ satisfies the conditions of Assumption 1 and corresponds to the case discussed in Section 3.2 and Appendix A. The attitude feedback gain is given by $K_\omega = 2$ and the rigid body trajectory is computed using oscillatory angular rates of 1 Hz.



(a) Attitude Estimation Error ($\tilde{\mathbf{b}}_\omega = 0$)

(b) Attitude Estimation Error ($\tilde{\mathbf{b}}_\omega \neq 0$)

(c) Bias Estimation Error

Fig. 1. Attitude and Bias Estimation Errors.

For the case of biased velocity readings, the bias is identical on each rate gyro channel $\bar{\mathbf{b}}_\omega = [5\ 5\ 5]\,^\circ/s$, and the initial bias estimate is $\hat{\mathbf{b}}_\omega(t_0) = \mathbf{0}\,^\circ/s$. The feedback gain is $K_{b_\omega} = 1$, the bounds of Proposition 7 are defined by $\theta_{0\,\max} = \frac{3}{4}\pi$, $\tilde{b}_{0\,\max} = 5\sqrt{3}\,^\circ/s$, which correspond to the minimum gain $K_{b_\omega\,\min} = 1.95 \times 10^{-2}$.

The attitude estimation error, depicted in Fig. 1(a) for the case of unbiased angular rate readings, converges exponentially fast to the origin. As expected, the convergence bound (10) is more conservative as $\theta(t_0)$ is closer to π.

The attitude and bias errors converge exponentially fast, as shown in Figs. 1(b) and 1(c). The peak of the bias estimation error is explained by the level set $V_b \le c$ containing points with small attitude error $2(1-\cos(\theta)) \approx 0$, but with bias error $\|\tilde{\mathbf{b}}_\omega\|^2 \approx 2K_{b_\omega}c$. In Fig. 2, the exponential convergence of the Lyapunov function (and of the estimation error) is illustrated using a logarithmic scale. Large feedback gains K_{b_ω} are beneficial for tackling the estimation errors. The convergence bounds of Corollary 9 are very conservative, $\gamma_{b_\omega} = 6.50 \times 10^{-18}$ and $k_{b_\omega} = 7.50 \times 10^8$ with optimized ρ, and should be subject to further study. Interestingly enough, the estimation error convergence rate obtained in simulation are satisfactory for practical applications.

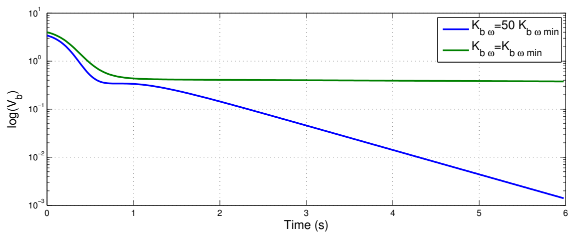


Fig. 2. Exponential Convergence of V_b ($\theta_0 = \frac{3\pi}{4}$,$\tilde{\mathbf{b}}_\omega \neq 0$).

7. CONCLUSIONS

A nonlinear observer for attitude estimation on SO(3) exploiting vector measurements and biased angular velocity readings was derived. Using a parameterized linear time-varying formulation of the system, exponential convergence of the estimation errors to the origin was shown. The feedback terms of the proposed observer were expressed as an explicit function of the vector measurements and state estimates. Simulation results depicted the convergence properties of the estimation errors, however the theoretical convergence rate bounds were too conservative for the case of biased velocity sensors. Future work will focus on obtaining a more accurate estimate of the convergence rate bounds and on the implementation of the algorithm for practical applications, where the presence of sensor noise and time-varying bias must be considered.

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Appendix A. AUGMENTED VECTOR OBSERVATION

Assumption 1 establishes that rank($\mathbf{H}$) $\geq$ 2 whereas the coordinate transformation described in Proposition 3 assumes that $\mathbf{H}$ is full rank. If rank($\mathbf{H}$) = 2, it is possible to construct an augmented matrix $\mathbf{H}_a$ with the columns of $\mathbf{H}$ such that rank($\mathbf{H}_a$) = 3. Taking two linearly independent columns of $\mathbf{H}$, ${}^L\mathbf{h}_i$ and ${}^L\mathbf{h}_j$, the augmented matrices are given by $\mathbf{H}_a := \left[\mathbf{H}\ {}^L\mathbf{h}_i \times {}^L\mathbf{h}_j\right]$, $\mathbf{U}_{Ha} = \mathbf{H}_a\mathbf{A}_{Ha}$, where $\mathbf{H}_a, \mathbf{U}_{Ha} \in \mathrm{M}(3, n+1)$, $\mathbf{A}_{Ha} \in \mathrm{M}(n+1)$ is a nonsingular matrix such that $\mathbf{U}_{Ha}\mathbf{U}'_{Ha} = \mathbf{I}$, which exists by the proof of Proposition 3. Using the fact that the cross product is commutable with coordinate transformations, $(\mathcal{R}'{}^L\mathbf{h}_i) \times (\mathcal{R}'{}^L\mathbf{h}_j) = \mathcal{R}'({}^L\mathbf{h}_i \times {}^L\mathbf{h}_j)$, the representation of the vector measurements in body coordinates is given by ${}^B\bar{\mathbf{U}}_{Ha} = \bar{\mathcal{R}}'\mathbf{U}_{Ha}$ and ${}^B\hat{\mathbf{U}}_{Ha} = \hat{\mathcal{R}}'\mathbf{U}_{Ha}$. The modified observer is obtained by replacing the matrices $\mathbf{U}_H$ and $\mathbf{H}$ by $\mathbf{U}_{Ha}$ and $\mathbf{H}_a$, respectively, yielding the same desired observer properties, namely the convergence results of Theorem 8.

Appendix B. UNIFORM EXPONENTIAL STABILITY

The following result from [Loría and Panteley, 2002] establishes that if the parameterized nonlinear system is exponentially stable uniformly in λ, then uniform exponential stability (independent of the initial conditions) of the associated nonlinear system can be inferred. This result is presented here for the sake of clarity.

Lemma 10. (λ-UGES and UES Loría and Panteley [2002]). Consider i) the nonautonomous system $\dot{y} = f(t, y)$ where $f : \mathbb{R}_{\geq 0} \times \mathcal{D}_y \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in y uniformly in t, and $\mathcal{D}_y \subset \mathbb{R}^n$ is a domain that contains the origin, ii) the parameterized nonautonomous system $\dot{x} = f_\lambda(t, \lambda, x)$, where $f_\lambda : \mathbb{R}_{\geq 0} \times \mathcal{D}_p \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, locally Lipschitz uniformly in t and λ, $\mathcal{D}_p = \mathbb{R}_{\geq 0} \times \mathcal{D}_\lambda$ and $\mathcal{D}_\lambda \subset \mathbb{R}^n$ is a closed not necessarily compact set. Let $\mathcal{D}_y \subset \mathcal{D}_\lambda$ and assume that $x(t) = 0$ is λ-UGES, i.e. there exist k_e and $\gamma_e > 0$ such that, for all $t \geq t_0$, $\lambda \in \mathcal{D}_p$ and $x_0 \in \mathbb{R}^n$, the solution of the system verifies $\|x(t, \lambda, t_0, x_0)\| \leq k_e\|x_0\|e^{-\gamma_e(t-t_0)}$. If the solution of both systems coincide, $y(t, y_0, t_0) = x(t, \lambda, x_0, t_0)$, for $\lambda = (t_0, y_0)$ and $x_0 = y_0$, then $y(t) = 0$ is exponentially stable in $\mathcal{D}_y$.

Proof. Let $x_0 = y_0$ and $\lambda = (t_0, y_0)$, then $x(t, \lambda, t_0, x_0) = y(t, t_0, y_0)$ and by change of variables, the solution satisfies $\|y(t, t_0, y_0)\| \leq k_e\|y_0\|e^{-\gamma_e(t-t_0)}$, and uniform exponential stability of $y(t) = 0$ in $\mathcal{D}_y$ is immediate. □