The Join of the Varieties of R-trivial and L-trivial Monoids via Combinatorics on Words

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The join of two varieties is the smallest variety containing both. In finite semigroup theory, the varieties of R-trivial and L-trivial monoids are two of the most prominent classes of finite monoids. Their join is known to be decidable due to a result of Almeida and Azevedo. In this paper, we give a new proof for Almeida and Azevedo’s effective characterization of the join of R-trivial and L-trivial monoids. This characterization is a single identity of ω-terms using three variables.

Keywords: finite semigroup theory, join of pseudovarieties, Green’s relations, combinatorics on words

1 Introduction

Green’s relations R and L are a standard tool in the study of semigroups [5]. In the context of finite monoids, among other results, they have been used to give effective characterizations of language classes such as star-free languages [3, 11] and piecewise testable languages [6, 12]. A deterministic extension of piecewise testable languages yields the class of languages corresponding to R-trivial monoids, and a codeterministic extension corresponds to L-trivial monoids [4, 9].

Almeida and Azevedo gave an effective characterization for the least variety of finite monoids containing all R-trivial and all L-trivial monoids [2], i.e., for the join of the two varieties. Their proof is based on sophisticated algebraic techniques, on Reiterman’s Theorem [10], and on a combinatorial result of König [7]. In this paper, we give a new proof of Almeida and Azevedo’s Theorem. The current proof was inspired by another proof of the authors [8], which in turn uses ideas of Klíma [6]. The main ingredient is a system of congruences which relies on simple combinatorics on words.
2 Preliminaries

Let $A$ be a finite alphabet. The set of finite words over $A$ is denoted by $A^*$. It is the free monoid over $A$. The empty word is 1. The content of a word $u = a_1 \cdots a_n$ with $a_i \in A$ is $\alpha(u) = \{a_1, \ldots, a_n\}$, and its length is $|u| = n$. The length of the empty word is 0. A word $u$ is a prefix (respectively suffix) of $v$ if there exists $x \in A^*$ such that $ux = v$ (respectively $xu = v$); if $x \neq 1$, then $u$ is a proper prefix.

For more details concerning the algebraic concepts introduced in the remainder of this section, we refer the reader to textbooks such as [1, 4, 9]. Green’s relations $\mathcal{R}$ and $\mathcal{L}$ are important tools in the study of finite monoids. Let $M$ be a finite monoid. We set $u \mathcal{R} v$ for $u, v \in M$ if $uM = vM$, and the latter condition is equivalent to the existence of $x, y \in M$ with $u = vx$ and $v = uy$. Symmetrically, $u \mathcal{L} v$ if $Mu = Mv$. The monoid $M$ is $\mathcal{R}$-trivial (respectively $\mathcal{L}$-trivial) if $\mathcal{R}$ (respectively $\mathcal{L}$) is the identity relation on $M$. We write $u \not\mathcal{R} v$ if $uM \not\subseteq vM$, and we write $u < \not\mathcal{R} v$ if $Mu \not\subseteq Mv$.

A variety of finite monoids is a class of monoids closed under finite direct products, submonoids, and quotients. A variety of finite monoids is often called a pseudovariety in order to distinguish from varieties in Birkhoff’s sense. Since we do not need this distinction in the current paper, whenever we use the term variety we mean a variety of finite monoids. The join $V_1 \lor V_2$ of two varieties $V_1$ and $V_2$ is the smallest variety containing $V_1 \cup V_2$. A monoid $M$ is in $V_1 \lor V_2$ if and only if there exist $M_1 \in V_1$ and $M_2 \in V_2$ such that $M$ is a quotient of a submonoid of $M_1 \times M_2$. If $M$ is a finite monoid, then there exists an integer $\omega_M \geq 1$ such that, for all $u \in M$, the element $u^{\omega_M}$ is idempotent. Moreover, the element $u^{\omega_M}$ is the unique idempotent generated by $u$. Usually, the monoid $M$ is clear from the context and thus, we simply write $\omega$ instead of $\omega_M$. This leads to the following definition. An $\omega$-term over a finite alphabet $X$ is either a word in $X^\omega$, or of the form $t^\omega$ for some $\omega$-term $t$, or the concatenation $t_1 t_2$ of two $\omega$-terms $t_1, t_2$. A homomorphism $\varphi : X^* \to M$ uniquely extends to $\omega$-terms over $X$ by setting $\varphi(t^\omega) = \varphi(t)^{\omega_M}$. Let $u, v$ be two $\omega$-terms over $X$. A finite monoid $M satisfies the identity $u = v$ if $\varphi(u) = \varphi(v)$ for all homomorphisms $\varphi : X^* \to M$. The class of finite monoids satisfying the identity $u = v$ is denoted by $[u = v]$. For all $\omega$-terms $u, v$, the class $[u = v]$ forms a variety. We need the following three varieties in this paper:

$$
\begin{align*}
\mathcal{R} &= \langle (xy)^\omega x = (xy)^\omega \rangle, \\
\mathcal{L} &= \langle x(zx)^\omega = (zx)^\omega \rangle, \\
\mathcal{W} &= \langle (xy)^\omega x(zx)^\omega = (xy)^\omega (zx)^\omega \rangle.
\end{align*}
$$

A monoid is in $\mathcal{R}$ if and only if it is $\mathcal{R}$-trivial. Symmetrically, a monoid is in $\mathcal{L}$ if and only if it is $\mathcal{L}$-trivial.

The aim of this paper is to give a new proof of Almeida and Azevedo’s result $\mathcal{R} \lor \mathcal{L} = \mathcal{W}$. The inclusion $\mathcal{R} \lor \mathcal{L} \subseteq \mathcal{W}$ is trivial since $\mathcal{R} \lor \mathcal{L} \subseteq \mathcal{W}$ and $\mathcal{W}$ is a variety.

3 Congruences

In this section, we introduce the main combinatorial tool for our proof. It is a family of congruences $\equiv_n$ on $A^*$ for some finite alphabet $A$ such that $A^*/\equiv_n \in \mathcal{R} \lor \mathcal{L}$ for all integers $n \geq 0$, see Lemma 2 below. As a first step towards the definition of $\equiv_n$ we need to introduce an asymmetric, weaker congruence $\equiv_n^\delta$. Let $u, v \in A^*$. We let $u \equiv_0^\delta v$ if $\alpha(u) = \alpha(v)$. For $n \geq 0$, we let $u \equiv_n^\delta v$ if the following conditions hold:

1. $\alpha(u) = \alpha(v)$,
2. for all factorizations $u = u_1 a u_2$ and $v = v_1 a v_2$ with $a \in A \setminus (\alpha(u_1) \cup \alpha(v_1))$ we have $u_1 \equiv_n^\delta v_1$ and $u_2 \equiv_n^\delta v_2$, and
3. for all factorizations \( u = u_1 u_2 v = v_1 a v_2 \) with \( a \in A \setminus (\alpha(u_1) \cup \alpha(v_2)) \) we have \( u_1 \equiv_n^R v_1 \).

By a straightforward verification we see that \( \equiv_n^R \) is an equivalence relation. The factorization \( u_1 u_2 v \) with \( a \in A \setminus \alpha(u_1) \) is unique. Therefore, induction on \( n \) shows that the index of \( \equiv_n^R \) is finite. If \( u \equiv_n^R v \), then \( u \equiv_n^R v \). Moreover, if \( u \equiv_n^R v \) and \( a \in A \), then \( au \equiv_n^R av \) and \( ua \equiv_n^R va \). Therefore, the relation \( \equiv_n^R \) is a finite index congruence on \( A^* \).

**Lemma 1** For every finite alphabet \( A \) and every integer \( n \geq 0 \) we have \( A^*/\equiv_n^R \in \mathbb{R} \).

**Proof:** It suffices to show \((xy)^{n+1}x \equiv_n^R (xy)^{n+1} \) for all words \( x, y \in A^* \). We note that for \( y = 1 \) this yields \( x^{n+2} \equiv_n^R x^{n+1} \). The proof is by induction on \( n \). For \( n = 0 \), the claim is true since \( \alpha(xyy) = \alpha(xy) \).

Let now \( n > 0 \). As before, \( \alpha((xy)^{n+1}x) = \alpha((xy)^{n+1}) \).

1. Suppose \((xy)^{n+1}x = u_1 u_2 v \) and \((xy)^{n+1} = v_1 a v_2 \) for \( a \in A \setminus (\alpha(u_1) \cup \alpha(v_2)) \). Then \( u_1 = v_1 \) and both are proper prefixes of \( y \). Thus \( u_2 = p(xy)^n x \) and \( v_2 = p(xy)^n \) for some \( p \in A^* \).

By induction \((xy)^n x \equiv_n^R (xy)^n \) and hence, \( u_2 \equiv_n^R v_2 \).

2. Suppose now \((xy)^{n+1}x = u_1 u_2 v \) and \((xy)^{n+1} = v_1 a v_2 \) for \( a \in A \setminus (\alpha(u_1) \cup \alpha(v_2)) \). Then \( v_2 \) is a suffix of \( y \) and \( au_2 \) is a suffix of \( yv_2 \). We can therefore write \( v_1 = (xy)^p a' \) for some prefix \( p \) of \( y \). Similarly, \( u_1 = (xy)^p \) for some \( k \in \{n, n+1\} \) and some prefix \( p \) of \( xy \), i.e., we have \( pq = xy \) for some \( q \in A^* \). By induction, we have \((xy)^{n+1} \equiv_n^R (xy)^n \) and thus \((xy)^{n+1}p = (xy)^n p \). We can therefore assume \( k = n \).

Without loss of generality, let \( |p| \leq |p'| \), i.e., \( p' = ps \) for some \( s \in A^* \). It follows

\[
u_1 = (pq)^n p \quad \text{and} \quad v_1 = (pq)^n s.
\]

Since \( p' = ps \) is a prefix of \( xy = pq \), the word \( s \) is a prefix of \( q \). In particular, there exists \( t \in A^* \) such that \( qp = st \). This yields

\[
u_1 = p(st)^n \quad \text{and} \quad v_1 = p(st)^n s.
\]

By induction, \((st)^n \equiv_n^R (st)^n s \) and \( u_1 \equiv_n^R v_1 \). This shows \((xy)^{n+1}x \equiv_n^R (xy)^{n+1} \) which concludes the proof. \( \square \)

There is a left-right symmetric congruence \( \equiv_n^L \) on \( A^* \). It can be defined by setting \( u \equiv_n^L v \) if and only if \( u \equiv_n^R v \). Here, \( u \equiv_n^L v \) is the reversal of the word \( u = a_1 \cdots a_n \) with \( a_i \in A \). It satisfies \( A^*/\equiv_n^L \in \mathbb{L} \) for every \( n \geq 0 \). We define \( u \equiv_n^L v \) if and only if both \( u \equiv_n^R v \) and \( u \equiv_n^L v \). The following lemma puts together some properties of the finite index congruence \( \equiv_n \).

**Lemma 2** For every finite alphabet \( A \) and every integer \( n \geq 0 \) the following properties hold:

1. \( A^*/\equiv_n \in \mathbb{R} \lor \mathbb{L} \)
2. If \( u_1 u_2 \equiv_n^L v_1 a v_2 \) for \( a \in A \setminus (\alpha(u_1) \cup \alpha(v_2)) \), then \( u_1 \equiv_n^L v_1 \) and \( u_2 \equiv_n^L v_2 \).
3. If \( u_1 u_2 \equiv_n^R v_1 a v_2 \) for \( a \in A \setminus (\alpha(u_2) \cup \alpha(v_2)) \), then \( u_1 \equiv_n^R v_1 \) and \( u_2 \equiv_n^R v_2 \).

**Proof:** We have \( A^*/\equiv_n \in \mathbb{R} \lor \mathbb{L} \) since it is a submonoid of \( (A^*/\equiv_n^R) \times (A^*/\equiv_n^L) \), and \( A^*/\equiv_n \in \mathbb{R} \) and \( A^*/\equiv_n \in \mathbb{L} \) by Lemma 1 and its left-right dual. The properties 2 and 3 trivially follow from the definition of \( \equiv_n \). \( \square \)
4 An Equation for the Join

The goal of this section is to prove \( W \subseteq R \lor L \). By Lemma\(^2\) it suffices to show that for every \( A \)-generated monoid \( M \in W \) there exists an integer \( n \geq 0 \) such that \( M \) is a quotient of \( A^*/\equiv_n \). The outline of the proof is as follows. First, in Lemma\(^5\) we give a substitution rule valid in \( W \). Then, in Lemma\(^3\) we show that \( \equiv_n \)-equivalence allows a factorization satisfying the premise for applying this substitution rule; this relies on a property of \( W \) shown in Lemma\(^4\). Finally, in Theorem\(^6\) all the ingredients are put together.

**Lemma 3** Let \( M \in W \) and let \( u, v, x \in M \). If \( u \ R \ u x \) and \( v \ L \ x v \), then \( u x v = u v \).

**Proof:** Since \( u \ u x \) and \( v \ L \ x v \), there exist \( y, z \in M \) with \( u = u x y \) and \( v = z v y \). In particular, we have \( u = u(xy)^\omega \) and \( v = (zx)^\omega v \). By \( M \in W \) we conclude \( u x v = u(xy)^\omega x(zx)^\omega v = u(xy)^\omega (zx)^\omega v = u v \). \( \square \)

We will apply the previous lemma as follows. Let \( M \in W \) and \( u, v, s, t \in M \) such that \( u \ u s \ u t \) and \( v \ L \ s v \ L \ t v \). Then \( u s v = u t v \) since \( u s v = u v \) and \( u t v = u v \) by Lemma\(^3\). The \( \equiv \)-equivalences and \( \equiv \)-equivalences for being able to apply this substitution rule are established in Lemma\(^5\). Before, we give a simple property of \( W \). It is the link between Green’s relations and the congruence \( \equiv_n \).

**Lemma 4** Let \( M \in W \) and let \( u, v, a \in M \). If \( u \ L \ v \ R \ a \) then \( u a v \). If \( u \ L \ v \ L \ a u \).

**Proof:** Since \( u \ R \ v \) and \( u \ R \ va \), there exist \( x, y \in M \) with \( v = u x \) and \( v = v y \). Now, \( u = u x y = u(x y)^\omega + 1 = u(x y)^\omega x(y x)^\omega y = u(x y)^\omega (ay x)^\omega y = u(ay x)^\omega y \in u a M \) where the fourth equality uses \( M \in W \). This shows \( u a M \subseteq u a M \) and thus \( u R \ u a \). The second implication is left-right symmetric. \( \square \)

The intuitive interpretation of the algebraic statement in Lemma\(^4\) is the following: For \( M \in W \) it only depends on the element \( a \) and the \( \equiv \)-class of \( u \) whether \( u R \ u a \) or not (but not on the element \( a \) itself). The statement for \( \equiv \)-classes is analogous.

**Lemma 5** Let \( M \in W \) and let \( \phi : A^* \to M \) be a homomorphism. If \( u \equiv_n v \) for \( n \geq 2 |M| \), then there exist factorizations \( u = a_1 s_1 \cdots a_{\ell-1} s_{\ell-1} a_\ell \) and \( v = a_1 t_1 \cdots a_{\ell-1} t_{\ell-1} a_\ell \) with \( a_i \in A \) and \( s_i, t_i \in A^* \) with \( \ell \leq 2 |M| \) such that for all \( i \in \{1, \ldots, \ell-1\} \) we have:

\[
\phi(a_1 s_1 \cdots a_{i-1} s_{i-1} a_i) \ R \ \phi(a_1 s_1 \cdots a_i s_i) \ R \ \phi(a_1 s_1 \cdots a_{i-1} s_{i-1} a_i t_i),
\]

\[
\phi(a_1 t_1 \cdots a_{i-1} t_{i-1} a_i) \ L \ \phi(t_1 a_1 \cdots t_{i-1} a_i) \ L \ \phi(s_1 a_1 t_1 \cdots a_{i-1} t_{i-1} a_i).
\]

**Proof:** To simplify notation, for some relation \( \equiv \) on \( M \) we write \( u \equiv v \) for words \( u, v \in A^* \) if \( \phi(u) \equiv \phi(v) \). Consider the \( \equiv \)-factorization of \( u \), i.e., let \( u = b_1 u_1 \cdots b_k u_k \) with \( b_i \in A \) such that

\[
b_1 u_1 \cdots b_i R b_1 u_1 \cdots b_i u_i \quad \text{for all } i \in \{1, \ldots, k\},
\]

\[
b_1 u_1 \cdots b_i u_i \equiv b_1 u_1 \cdots b_i u_{i+1} \quad \text{for all } i \in \{1, \ldots, k-1\}.
\]

Similarly, let \( v = v_1 c_1 \cdots v_k c_k \) be the \( \equiv \)-factorization of \( v \), i.e., we have \( c_i \in A \) and

\[
c_1 \cdots v_k c_k \ L \ v_1 c_1 \cdots v_k c_k \quad \text{for all } i \in \{1, \ldots, k\},
\]

\[
v_1 c_1 \cdots v_k c_k \equiv v_1 c_1 \cdots v_k c_{i+1} \quad \text{for all } i \in \{1, \ldots, k\}.
\]
We have $k, k' \leq |M|$ because neither the number of $R$-classes nor the number of $L$-classes can exceed $|M|$. By Lemma 4 we have $b_i \notin \alpha(u_{i-1})$ for all $i \in \{2, \ldots, k\}$ and $c_i \notin \alpha(v_{i+1})$ for all $i \in \{1, \ldots, k' - 1\}$. We use these properties to convert the $R$-factorization of $u$ to $v$ and to convert the $L$-factorization of $v$ to $u$: Let $v = b_1v'_1 \cdots b_kv'_k$ such that $b_i \notin \alpha(v'_{i-1})$, and let $u = u'_1c_1 \cdots u'_kc'_k$ with $c_i \notin \alpha(u'_{i+1})$. These factorizations exist because $u \equiv_n v$; in particular, by Lemma 2

$$u_i b_{i+1} u_{i+1} \cdots b_k u_k = v_i b_{i+1} v_{i+1} \cdots b_k v_k$$

$$v_i c_1 \cdots v_{j-1} c_{j-1} v_j = n_{k' - 1 + j} u'_1 c_1 \cdots u'_{j-1} c_{j-1} u'_j$$

for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, k'\}$. Moreover, we see that $\alpha(u_i) = \alpha(v'_i)$ and $\alpha(v_j) = \alpha(u'_j)$.

We now show that the relative positions of the $b_i$’s and $c_j$’s in the above factorizations are the same in $u$ and $v$. Let $p$ be the position of $b_1$ in the $R$-factorization of $u$ and let $q$ be the position of $c_1$ in the above factorization of $u$. Similarly, let $p'$ be the position of $b_1$ in $v$ and let $q'$ be the position of $c_1$ in $v$. First, suppose $p < q$. Let

$$u = b_1 u_1 \cdots b_{i-1} u_{i-1} b_i u'_{i+1} c_j u'_{j+1} c_{j+1} \cdots u'_{k'} c'_{k'}.$$ 

By an $i$-fold application of property (3) in Lemma 2 with $a \in \{b_1, \ldots, b_i\}$ (which is possible for $u$) we obtain $v = b_1 v'_{j} \cdots b_{i-1} v'_{j-1} b_j z$ with $z = a_{i-1} c_j u'_{j+1} c_{j+1} \cdots u'_{k'} c'_{k'}$. By a $(k' + 1 - j)$-fold application of property (3) in Lemma 2 with $a \in \{c_{k'}, \ldots, c_j\}$ (which is possible for the word $u'_{i+1} c_j u'_{j+1} c_{j+1} \cdots u'_{k'} c'_{k'}$) we obtain $z = v' c_j v_{j+1} c_{j+1} \cdots v_{k'} c'_{k'}$. Thus

$$v = b_1 v'_1 \cdots b_{i-1} v'_{j-1} b_i v'_j c_j v_{j+1} c_{j+1} \cdots v_{k'} c'_{k'}$$

showing that $p' < q'$. Symmetrically, one shows that $p' < q'$ implies $p < q$. We conclude $p < q$ if and only if $p' < q'$. Similarly, we have $p = q$ if and only if $p' = q'$. It follows that the relative order of the $b_i$’s and $c_j$’s in $u$ and $v$ is the same. By factoring $u$ and $v$ at all $b_i$’s and $c_j$’s, we obtain $u = a_1 s_1 \cdots a_{k-1} s_{k-1} a_k$ and $v = a_1 t_1 \cdots a_{k'} t_{k'-1} a_{k'}$ with $a_i \in A$ and $\ell \leq k + k' \leq 2 |M|$.

We have $a_1 s_1 \cdots a_{k-1} s_{k-1} a_k R a_1 s_1 \cdots a_{k-1} s_{k-1} a_k s_1$ since the factorization $u = a_1 s_1 \cdots a_{k-1} s_{k-1} a_k$ is a refinement of the $R$-factorization. Note that we cannot assume $\alpha(s_i) = \alpha(t_i)$. But each $t_i$ is a factor of some $v'_j$, and at the same time $s_i$ is a factor of $u_j$. More precisely, there exists $m \leq i$ such that

$$b_1 v'_1 \cdots b_{j-1} v'_{j-1} b_j = a_1 t_1 \cdots a_{m-1} t_{m-1} a_m$$

and

$$t_m a_{m+1} \cdots t_{i-1} a_i t_i$$

is a prefix of $v'_j$.

Furthermore, $s_m a_{m+1} \cdots s_{i-1} a_i s_i$ is a prefix of $u_j$. Now, $\alpha(t) \subseteq \alpha(v'_j) = \alpha(u_j)$ and, by Lemma 4 for all words $z$ with $\alpha(z) \subseteq \alpha(u_j)$ we have $a_1 s_1 \cdots a_{k-1} s_{k-1} a_k R a_1 s_1 \cdots a_{k-1} s_{k-1} a_k z$. Symmetrically we see $a_{i+1} t_{i+1} \cdots a_{i-1} t_{i-1} a_i \not\subseteq a_1 t_1 \cdots a_{i-1} t_{i-1} a_i L s_m a_{m+1} \cdots a_{i-1} t_{i-1} a_i$. 

\begin{proof}

The inclusion $R \lor L \subseteq W$ is trivial since $R \cup L \subseteq W$ and $W$ is a variety of finite monoids. Let $M \in W$ be generated by $A$, and let $\phi : A^* \rightarrow M$ be the homomorphism induced by $A \subseteq M$. Let $n = 2 |M|$ and

$$R \lor L = \{(x y)^{2n} x (x y)^{2n} = (x y)^{2n} (x y)^{2n}\}$$

\end{proof}
suppose \( u \equiv_n v \). Let \( u = a_1s_1 \cdots a_{\ell-1}s_{\ell-1}a_{\ell} \) and \( v = a_1t_1 \cdots a_{\ell-1}t_{\ell-1}a_{\ell} \) be the factorizations from Lemma 5. Applying Lemma 5 repeatedly, we get

\[
\phi(v) = \phi(a_1t_1a_2t_2 \cdots a_{\ell-2}t_{\ell-2}a_{\ell-1}t_{\ell-1}a_{\ell}) \\
= \phi(a_1s_1a_2s_2 \cdots a_{\ell-2}s_{\ell-2}a_{\ell-1}s_{\ell-1}a_{\ell}) \\
= \vdots \\
= \phi(a_1s_1a_2s_2 \cdots a_{\ell-2}s_{\ell-2}a_{\ell-1}s_{\ell-1}a_{\ell}) = \phi(u).
\]

Note that the substitution rules \( t_i \rightarrow s_i \) are \( \phi \)-invariant only when applied from left to right. This shows that \( M \) is a quotient of \( A^*/\equiv_n \), and the latter is in \( R \lor L \) by Lemma 2. Thus \( M \in R \lor L \).

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References