Open Problems in Topology
J. van Mill and G.M. Reed (Editors)

Chapter 11

Small uncountable cardinals and topology

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Contents

[1] Research supported by NSF Grant DMS–8900064

1. Definitions and set-theoretic problems

Let ω denote the set of natural numbers, $[\omega]^\omega$ the set of all infinite subsets of ω and ${}^\omega\omega$ the set of all functions from ω into ω. We give a brief discussion of some problems which involve cardinal numbers defined from various properties on these and related sets such as the real and the irrational numbers. For background on small cardinals, we refer to the article by Eric van Douwen, van Douwen [1984]. In that article, van Douwen defined eight small cardinals, and studied six of them in detail. We take these eight cardinals as our starting point, and begin with some preliminary definitions.

Two countable, infinite sets are *almost disjoint* provided their intersection is finite. A family of pairwise almost disjoint subsets of a set X is *maximal* provided it is not properly contained in another pairwise almost disjoint family of subsets of X. Let $\mathcal{P}(X)$ denote the power set of X, $[X]^\omega$ the set of all countably infinite subsets of X, and $[X]^{<\omega}$ the set of all finite subsets of X.

For A, B in $[\omega]^\omega$, we say A is *almost included* in B (denoted $A \subset^* B$) provided $A - B$ is finite.

For a family $\mathcal{F} \subset [\omega]^\omega$, we say that $\mathcal{F}$ has the *strong finite intersection property* provided every finite subfamily has an infinite intersection, and an infinite set A is called a *pseudointersection* of $\mathcal{F}$ provided $A \subset^* F$ for all $F \in \mathcal{F}$.

A family $\mathcal{T} \subset [\omega]^\omega$ is a (decreasing) *tower* provided there exist an ordinal α and a bijection $f\colon \alpha \to \mathcal{T}$ such that $\beta < \gamma < \alpha$ imlies that $f(\gamma) \subset^* f(\beta)$, and no infinite set A is a pseudointersection of $\mathcal{T}$.

A family $\mathcal{U} \subset [\omega]^\omega$ *generates an ultrafilter* (or is an *ultrafilter base*) provided every finite intersection of elements of $\mathcal{U}$ contains an element of $\mathcal{U}$, and for every $A \in [\omega]^\omega$ either there exists $U \in \mathcal{U}$ such that $U \subset A$, or there exists $U \in \mathcal{U}$ such that $U \subset \omega - A$. An ultrafilter base $\mathcal{U}$ is called *free* (or *non-principal*) provided $\bigcap \mathcal{U} = \emptyset$.

A family $\mathcal{I} \subset [\omega]^\omega$ is an *independent family* provided for every $A, B \in [\mathcal{I}]^{<\omega}$, if $A \neq \emptyset$ and $A \cap B = \emptyset$ then $\bigcap A - \bigcup B \neq \emptyset$.

A family $\mathcal{S} \subset [\omega]^\omega$ is a *splitting family* provided for every $A \in [\omega]^\omega$ there exists $S \in \mathcal{S}$ such that $|A \cap S| = |A - S| = \omega$.

We define the *mod finite order* $\leq^*$ (a reflexive transitive order) on the set ${}^\omega\omega$ as follows: for $f, g \in {}^\omega\omega$ we say $f \leq^* g$ provided there exists $N \in \omega$ such that for all $n \geq N$, $f(n) \leq g(n)$.

A set $X \subset {}^\omega\omega$ is *dominating* (in the mod finite order) provided for every $f \in {}^\omega\omega$ there exists $g \in X$ such that $f \leq^* g$, and X is *bounded* (in the mod finite order) if there exists $g \in {}^\omega\omega$ such that $f \leq^* g$ for all $f \in X$.

We denote the *cardinality of the continuum* by $\mathfrak{c}$. Since German type is often used for this cardinal, van Douwen and I used this same convention in the mnemonic notation for the following cardinals which have been studied under a variety of names (see Hechler [1972], van Douwen [1984, p.123], and Vaughan [1979b]). When German type is not available, we use ordinary letters for these cardinals (including the cardinality of the continuum). The mnemonic is derived from the key concept in the definition of a cardinal. The cardinal $\mathfrak{b}$ is called the (un)$\mathfrak{b}$ounding number, $\mathfrak{d}$ is the $\mathfrak{d}$ominating number, $\mathfrak{s}$ is the $\mathfrak{s}$plitting number, $\mathfrak{p}$ is derived from the notion of $\mathfrak{P}$-points, $\mathfrak{t}$ is the $\mathfrak{t}$ower number, $\mathfrak{i}$ is the $\mathfrak{i}$ndependent family number, and $\mathfrak{u}$ the $\mathfrak{u}$ltrafilter character number. The cardinal $\mathfrak{a}$ can be called the $\mathfrak{a}$lmost disjointness number (unfortunately, mnemonic does not necessarily imply euphonic).

$$\mathfrak{a} = \min\{\,|A| : A \subset [\omega]^\omega \text{ is an infinite, maximal almost disjoint family in } \omega\}.$$

$$\mathfrak{b} = \min\{\,|B| : B \subset {}^\omega\omega \text{ is unbounded in the mod finite order }\}.$$

$$\mathfrak{d} = \min\{\,|D| : D \subset {}^\omega\omega \text{ is dominating in the mod finite order }\}.$$

$$\mathfrak{s} = \min\{\,|S| : S \subset [\omega]^\omega \text{ is a splitting family on } \omega\}.$$

$$\mathfrak{p} = \min\{\,|P| : P \subset [\omega]^\omega \text{ has the strong finite intersection property but no } X \in [\omega]^\omega \text{ is a pseudointersection for } P\,\}.$$

$$\mathfrak{t} = \min\{\,|T| : T \subset [\omega]^\omega \text{ is a tower on } \omega\}.$$

$$\mathfrak{i} = \min\{\,|I\,| : I \subset [\omega]^\omega \text{ is a maximal independent family on } \omega\}.$$

$$\mathfrak{u} = \min\{\,|U| : U \subset [\omega]^\omega \text{ is a base for an ultrafilter on } \omega\}.$$

Diagram 1 below, the shape of which is based on a similar diagram of BLASS [1989], is intended to display the basic relations among these cardinals. A line connecting two cardinals indicates that the cardinal lower on the diagram is less than or equal to the cardinal higher on the diagram (in **ZFC**). It would be nice if the line also indicated that it is consistent that the two cardinals are different. For example, Rothberger proved $\mathfrak{t} \leq \mathfrak{b}$, and there is a model where $\mathfrak{t} < \mathfrak{b}$ (see VAN DOUWEN [1984, 3.1 and 5.3]). This aspect of the diagram, however, is not completely settled (see 1.1).

The proofs of the results implied by Diagram 1, or references to them, can be found in VAN DOUWEN [1984], except for the recent result of SHELAH [1990], who proved in **ZFC** that $\mathfrak{d} \leq \mathfrak{i}$ (he also mentions a model in which the inequality "$\mathfrak{d} < \mathfrak{i}$" holds). These results of Shelah are included with his permission as an appendix to this paper.

? 333. **Problem 1.1.** *Are any of the following inequalities consistent with* **ZFC***?*
(a) $\mathfrak{p} < \mathfrak{t}$
(b) $\mathfrak{d} < \mathfrak{a}$
(c) $\mathfrak{i} < \mathfrak{a}$
(d) $\mathfrak{u} < \mathfrak{a}$
(e) $\mathfrak{i} < \mathfrak{u}$

The first two of these inequalities are of interest because they concern four of the six cardinals studied by VAN DOUWEN in [1984]. We believe that (a) is the most interesting. Concerning (a): Rothberger proved that $\mathfrak{p} > \omega_1$ if and

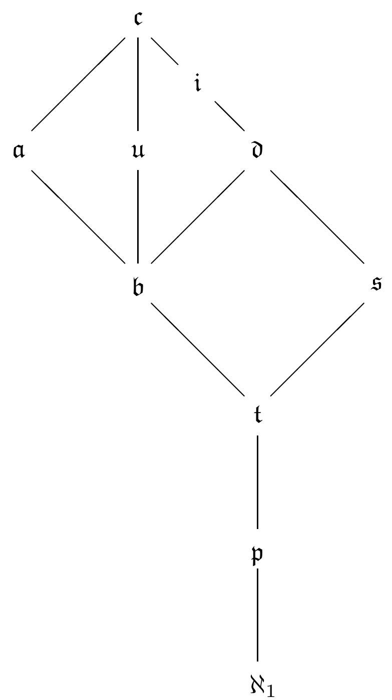

Diagram 1.

only if $\mathfrak{t} > \omega_1$ (FREMLIN [1984, 14D]), and Szymański proved that $\mathfrak{p}$ is regular (VAN DOUWEN [1984, 3.1(e)]) and that $\mathfrak{t}$ cannot be real-valued measurable (SZYMAŃSKI [1988]). Given 8 cardinals there are $64 - 8 = 56$ questions of the form "is $\kappa < \lambda$". For the above 8 cardinals, the questions in 1.1 are the only ones of this form which remain open (see VAN DOUWEN [1984], BLASS and SHELAH [1987, 1989], SHELAH [1984, 1990]).

There are many small cardinals which have been studied. Some of them are probably of more interest than some of those defined above. One interesting cardinal that has been discovered in several contexts is the cardinal $\mathfrak{h}$, called the *distributivity number* (the reason for the letter "$\mathfrak{h}$" is given below). A family $D \subset [\omega]^\omega$ is called a *dense* family provided for every $X \in [\omega]^\omega$ there exists $Y \in D$ such that $Y \subset^* X$, and D is called an *open* family provided for every $Y \in D$ and every $X \in [\omega]^\omega$, if $X \subset^* Y$ then $X \in D$. The ordered set $([\omega]^\omega, \subset^*)$ is called *κ-distributive* if every set of less than κ dense open families has non-empty intersection. The distributivity number is defined by

$$\mathfrak{h} = \min\{\mathcal{D} : \mathcal{D} \text{ is a set of dense open families in } [\omega]^\omega \text{ with } \bigcap \mathcal{D} = \emptyset\}.$$

Now $\mathfrak{t} \le \mathfrak{h} \le \mathfrak{b}$, and $\mathfrak{h} \le \mathfrak{s}$; see BALCAR, PELANT and SIMON [1980]. The letter $\mathfrak{h}$ comes from the word "height" in the interesting result, proved by Balcar, Pelant and Simon, that in **ZFC** one can prove that there exists a tree π-base for ω^*, and further

$$\mathfrak{h} = \min\{\kappa : \text{there exists a tree } \pi\text{-base for } \omega^* \text{ of } \mathfrak{h}\text{eight } \kappa\}$$

where, as usual, $\omega^* = \beta\omega - \omega$ and a family $\mathcal{B}$ of non-empty open sets is called a *π-base* for a space X provided every nonempty open set contains a member of $\mathcal{B}$. A *tree π-base* T is a π-base which is a tree when considered as a partially ordered set under reverse inclusion (i.e., for every $t \in T$ the set $\{s \in T : t \subset s\}$ is well-ordered by "$\supset$"). The height of an element $t \in T$ is the ordinal α such that $\{s \in T : t \subset s \text{ and } s \ne t\}$ is of order type α, and the *height* of a tree T is the smallest ordinal α such that no element of T has height α.

For any topological space, the *Novák number* (BALCAR, PELANT and SIMON [1980]) (resp. *weak Novák number* (VAN MILL and WILLIAMS [1983])) of X, denoted $n(X)$ (resp. $wn(X)$), is the smallest number of nowhere dense subsets of X needed to cover X (resp. to cover a dense subset of X).

We consider here only the case $X = \omega^*$ and write $\mathfrak{n} = n(\omega^*)$.

It is easy to see that $\omega_2 \le \mathfrak{t}^+ \le n(\omega^*) \le 2^{\mathfrak{c}}$ (HECHLER [1978, 4.11]). The equality $wn(\omega^*) = \mathfrak{h}$ was proved in NYIKOS, PELANT and SIMON [19∞], and gives one way to see that $\mathfrak{h} \le \mathfrak{n}$. In BALCAR, PELANT and SIMON [1980] it is proved that $\mathfrak{h} \le \min\{\mathfrak{b}, \mathrm{cf}(\mathfrak{c})\}$. In Cohen's original models of not-**CH** one has $\mathfrak{h} = \mathfrak{b} = \aleph_1 < \aleph_2 \le \mathfrak{n}$. Blass pointed out to us that to get a model where $\mathfrak{n} < \mathfrak{b}$, start with a model of **MA** in which $\mathfrak{c}$ is at least $\aleph_3$, and add $\aleph_1$ random reals (giving $\mathfrak{h} = \aleph_1$ and $\mathfrak{b} \ge \aleph_3$), and apply BALCAR, PELANT and SIMON [1980, 3.5(i)] which says that if $\mathfrak{h} < \mathfrak{c}$ then $\mathfrak{n} \le \mathfrak{h}^+$. In [1984] SHELAH gave a model where $\mathfrak{h} < \mathfrak{s} = \mathfrak{b}$. Dow has determined the value of $\mathfrak{h}$ in a number of models. For example, he has a model where $\mathfrak{h} = \mathfrak{s} < \mathfrak{b}$ (DOW [1989]).

A family $\mathcal{G} \subset [\omega]^\omega$ is said to be *groupwise dense* (BLASS [1989]) provided

(a) $\mathcal{G}$ is an open family, and

(b) for every family Π of infinitely many pairwise disjoint finite subsets of ω, the union of some (necessarily infinite) subfamily of Π is in $\mathcal{G}$.

Clearly every groupwise dense family is a dense open family. Define

$$\mathfrak{g} = \min\{\,|\mathcal{G}| : \mathcal{G} \text{ is a set of } \mathfrak{g}\text{roupwise dense families in } [\omega]^\omega \text{ with } \textstyle\bigcap \mathcal{G} = \emptyset\}.$$

Obviously $\mathfrak{h} \le \mathfrak{g}$ and it is known that $\mathfrak{g} \le \mathfrak{d}$ (BLASS [1989]). There is a model of Blass and Shelah where $\mathfrak{u} < \mathfrak{g}$ and $\mathfrak{h} < \mathfrak{g}$ (BLASS and LAFLAMME [1989]). Blass has proved that if $\mathfrak{u} < \mathfrak{g}$ then $\mathfrak{b} = \mathfrak{u}$ and $\mathfrak{g} = \mathfrak{d} = \mathfrak{c}$ (BLASS [19∞]).

The next number requires no further definitions:

$$\mathfrak{a}_{\mathrm{s}} = \min\{\,|A| : A \text{ is a maximal family of almost disjoint subsets of } \omega\times\omega \text{ that are graphs of functions from subsets of } \omega \text{ to } \omega\}$$

Balcar and Simon proved that $\mathfrak{s} \le \mathfrak{a}_{\mathrm{s}}$, and $\mathfrak{a} \le \mathfrak{a}_{\mathrm{s}} \le \mathfrak{c}$. Shelah has a model where $\mathfrak{a} < \mathfrak{a}_{\mathrm{s}}$, and another where $\mathfrak{s} < \mathfrak{b} \le \mathfrak{a}$ (Shelah [1984]).

Let $(P,\le)$ be a partially ordered set (poset). A set $D \subset P$ is said to be *dense* provided for every $p \in P$ there exists $d \in D$ such that $d \le p$. A set $G \subset P$ is a *filter* provided (a) for every $g \in G$ and every $p \in P$, if $g \le p$ then $p \in G$, and (b) for every g and $g' \in G$ there exists $r \in G$ such that $r \le g$ and $r \le g'$. The set $(P \le)$ satisfies the **ccc** provided every antichain is countable (i.e., if $A \subset P$, and A is uncountable, then there exist distinct a and $a' \in A$ and $r \in P$ such that $r \le a$ and $r \le a'$). Let "**MA**(κ) for **ccc** posets" (or "**MA**(κ)" for short) be the statement: for every **ccc** partially ordered set and every family $\mathcal{D}$ of no more than κ dense subsets of P, there exists a generic filter G for $\mathcal{D}$ (i.e., G is a filter and $G \cap D \ne \emptyset$ for all $D \in \mathcal{D}$). Define

$$\mathfrak{m} = \min\{\kappa : \mathbf{MA}(\kappa) \text{ for } \mathbf{ccc} \text{ posets fails }\}$$

Of course, Martin's Axiom is the statement "$\mathfrak{m}=\mathfrak{c}$".

Fremlin [1984] has given a proper class of definitions $\mathfrak{m}_\Phi$, where Φ is a class of partially ordered sets, similar to the definition of $\mathfrak{m}$. We mention two of these here. A poset P is called *σ-centered* if there exists a partition $\{P_i : i \in \omega\}$ of P such that each P_i is *centered* (i.e., if $p, q \in P_i$ then there exists $r \in P_i$ such that $r \le p$, and $r \le q$). Define

$$\mathfrak{m}_{\sigma-\text{centered}} = \min\{\kappa : \text{“}\mathbf{MA}(\kappa) \text{ for } \sigma\text{-centered posets” fails }\},$$
$$\mathfrak{m}_{\text{countable}} = \min\{\kappa : \text{“}\mathbf{MA}(\kappa) \text{ for countable posets” fails }\}.$$

Bell [1981] proved that $\mathfrak{p} = \mathfrak{m}_{\sigma-\text{centered}}$, thus $\mathfrak{m} \le \mathfrak{p} \le \mathfrak{m}_{\text{countable}}$.

Let $L(\kappa)$ be the statement: If P is a **ccc** partially ordered set of cardinality $\le \kappa$, then P is σ-centered (Fremlin [1984, 41L]). Define

$$\mathfrak{l} = \min\{\kappa : L(\kappa) \text{ is false }\}.$$

In [1987] Todorčević and Veličković have proved that $\mathfrak{m} = \mathfrak{l}$, and as a corollary, the result of Fremlin [1984, 41C(d)]: $\mathrm{cf}(\mathfrak{m}) > \omega$ (also see Kunen [1988] and Problem 1.3).

A family $\mathcal{B}$ is called a *π-base* for a free ultrafilter u on ω provided for every $U \in u$ there exists $B \in \mathcal{B}$ such that $B \subset U$. Define

$$\pi\mathfrak{u} = \min\{\,|\mathcal{B}| : \mathcal{B} \subset [\omega]^\omega \text{ is a } \pi\text{-base for a free ultrafilter on } \omega\}.$$

The $\mathfrak{r}$*efinement number* is defined by

$$\mathfrak{r} = \min\{\,|\mathcal{R}| : \mathcal{R} \subset [\omega]^\omega \text{ for every } X \in [\omega]^\omega \text{ there exists } R \in \mathcal{R} \text{ such that } R \subset^* X \text{ or } R \subset^* \omega - X\}.$$

PRICE [1982] was the first to (implicitly) consider this cardinal, and it was discovered independently by VOJTÁŠ [19∞], J. Cichoń, and BEŠLAGIĆ and VAN DOUWEN [19∞]. The last two looked at $\mathfrak{r}$ from the following point of view: A set $R \in [\omega]^\omega$ is said to *reap* a family $\mathcal{F} \subset [\omega]^\omega$ provided for every $F \in \mathcal{F}$, $|F \cap R| = |F - R| = \omega$. Thus, $\mathfrak{r}$ is the smallest cardinality of a family $\mathcal{F}$ such that no set $R \in [\omega]^\omega$ reaps $\mathcal{F}$.

Balcar (unpublished) has shown that $\mathfrak{r} = \pi\mathfrak{u}$. Clearly no set reaps the Boolean algebra generated by a maximal independent family; so $\pi\mathfrak{u} = \mathfrak{r} \leq \mathfrak{i}$, and clearly $\mathfrak{r} = \pi\mathfrak{u} \leq \mathfrak{u}$. GOLDSTERN and SHELAH [19∞] have a model where $\mathfrak{r} < \mathfrak{u}$ (thus this is also a model where $\pi\mathfrak{u} < \mathfrak{u}$). Also see JUST [19∞]. In [1982] PRICE noted a model where $\mathfrak{r} = \mathfrak{c}$ (for another such model, see BEŠLAGIĆ and VAN DOUWEN [19∞]). In this context, a plausible definition of a small cardinal is the smallest cardinality of a family $\mathcal{F}$ such that every infinite set $A \subset \omega$ contains a member of $\mathcal{F}$. ROTHBERGER [1948] pointed out, however, that since there exist almost disjoint families of cardinality $\mathfrak{c}$, every such family $\mathcal{F}$ has cardinality $\mathfrak{c}$.

For sets A, B in $[\omega]^\omega$ we say that A *splits* B provided the partition of B, $\{B \cap A, B - A\}$, consists of two infinite sets (i.e., $|B \cap A| = |B - A| = \omega$). Using this term, the cardinal $\mathfrak{s}$ is the minimal cardinality of a family $\mathcal{S}$ of subsets of ω such that every infinite subset of ω is split by some member of $\mathcal{S}$, and $\mathfrak{r}$ is the minimal cardinality of a family $\mathcal{R}$ of subsets of ω such that no infinite subset of ω splits every member of $\mathcal{R}$. We now think of characteristic functions of subsets of ω. For $f \in 2^\omega$, and $A \in [\omega]^\omega$, we say $\lim_A f = i$ provided $i \in 2$ and the subsequence $f|A$ converges to i in the discrete space $\{0, 1\}$ (thus, $f^{-1}(1)$ splits A iff $f^{-1}(0)$ splits A iff $\lim_A f$ does not exist). The cardinals $\mathfrak{r}$ and $\mathfrak{s}$ can be formulated in terms of sequences of zeroes and ones as follows: $\mathfrak{s}$ is the minimal cardinality of a family $S \subset 2^\omega$ such that for every $A \in [\omega]^\omega$ there exists f in S such that $\lim_A f$ does not exist. Also, $\mathfrak{r}$ is the minimal cardinality of a family R of subsets of ω such that for every $f \in 2^\omega$ there exists A in R such that $\lim_A f$ exists. This leads to the following two cardinals defined by VOJTÁŠ [1988, 19∞]. Let l^∞ denote the set of all bounded real valued sequences. Here, of course, for $f \in l^\infty$ and $A \in [\omega]^\omega$, we define $\lim_A f = x$ provided the sequence $f|A$ converges to x in the usual topology on $\mathbb{R}$.

$$\mathfrak{s}_\sigma = \min\{\ |S| \subset l^\infty\ : (\forall A \in [\omega]^\omega)(\exists f \in S)\ \lim_A f \text{ does not exist }\},$$
$$\mathfrak{r}_\sigma = \min\{|R| \subset [\omega]^\omega : (\forall f \in l^\infty)(\exists A \in R)\ \lim_A f \text{ exists }\}.$$

It is easily seen that $\mathfrak{s} = \mathfrak{s}_\sigma$, and $\mathfrak{r}_\sigma \geq \mathfrak{r}$, but it is not known if $\mathfrak{r} = \mathfrak{r}_\sigma$.

Some other cardinals which have been considered are the Ramsey number (which also is denoted by $\mathfrak{r}$) $\mathfrak{K}_\mathfrak{c}$, $\mathfrak{u}_\mathrm{p}$, $\mathfrak{q}$ and $\mathfrak{q}_0$. The $\mathfrak{R}$*amsey number* is the smallest cardinality κ of a family of functions $\pi_\alpha\colon [\omega]^2 \to 2$ such that for every $X \in [\omega]^\omega$ there exists $\alpha < \kappa$ such that for every $i \in \omega$, $|\pi_\alpha(X - i)| = 2$ (IHODA [1988]). Blass has recently proved (unpublished) that the

Ramsey number equals $\min\{\mathfrak{b}, \mathfrak{s}\}$. The cardinal $\mathfrak{K}_\mathfrak{c}$ is defined to be the smallest cardinality of a family $F \subset {}^\omega\omega$ such that for every $A \in [\omega]^\omega$ there exists $f \in F$ such that $f(A) = \omega$. In Cohen's original models of not-**CH** $\mathfrak{K}_\mathfrak{c} = \aleph_1$ (Hechler [1973]), and Nyikos recently has proved (unpublished) $\mathfrak{s} \leq \mathfrak{K}_\mathfrak{c} \leq \mathfrak{d}$. The cardinal $\mathfrak{u}_\mathrm{p}$ is defined as the smallest cardinality of a base for a P-point in ω^* if there exists a P-point, and is defined to be $\mathfrak{c}$ if there do not exist any P-points (there are models of set theory in which P-points do not exist (Wimmers [1982]). It is easy to see that $\mathfrak{r}_\sigma \leq \mathfrak{u}_\mathrm{p}$ (Vojtáš [19∞]). The cardinal $\mathfrak{q}$ is defined as the smallest cardinal such that no set of reals of this size or larger is a $\mathfrak{Q}$-set (Gruenhage and Nyikos [19∞]) (a set $X \subset \mathbb{R}$ is a *Q-set* provided every subset of X is a G_δ-set in the subspace topology of X). In [1948] Rothberger proved that $\mathfrak{p} \leq \mathfrak{q}$, and it is consistent that $\mathfrak{p} < \mathfrak{q}$ (Fleissner and Miller [1980]). The cardinal $\mathfrak{q}_0$ is defined the supremum of the set of cardinals κ such that every subset of $\mathbb{R}$ of cardinality strictly less than κ is a Q-set (Gruenhage and Nyikos [19∞]).

We now define some cardinals related to measure and category. Let $\mathcal{I}$ be an ideal of subsets of a set. Define

$$\begin{aligned}
\mathrm{add}(\mathcal{I}) &= \min\{\, |\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \textstyle\bigcup \mathcal{J} \notin \mathcal{I}\},\\
\mathrm{cov}(\mathcal{I}) &= \min\{\, |\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \textstyle\bigcup \mathcal{J} = \mathbb{R}\},\\
\mathrm{non}(\mathcal{I}) &= \min\{\, |Y| : Y \subset \mathbb{R} \text{ and } Y \notin \mathcal{I}\}, \text{ and}\\
\mathrm{cf}(\mathcal{I}) &= \min\{\, |\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \mathcal{I} = \textstyle\bigcup\{\mathcal{P}(E) : E \in \mathcal{J}\}\,\}.
\end{aligned}$$

These cardinals have been studied mainly for the set $\mathbb{R}$ of real numbers, and the ideals of meager (= first category) sets and Lebesgue null sets. There seems to be no standard notation for these two important ideals. We will denote the ideal of meager sets by $\mathbb{K}$ and the ideal of Lebesgue null sets by $\mathbb{L}$.

The cardinal $\mathrm{cov}(\mathbb{K})$ has been considered under several names (Miller [1981, 1982b]), and it is known that $\mathrm{cov}(\mathbb{K}) = \mathfrak{m}_{\mathrm{countable}}$ (the key idea is in Grigorieff [1975], and an explicit proof is in Fremlin and Shelah [1979]). Bartoszynski [1987] proved that $\mathrm{cov}(\mathbb{K})$ is the least cardinal of any $F \subset {}^\omega\omega$ such that for every $g \in {}^\omega\omega$ there exists $f \in F$ such that $f(n) \neq g(n)$ for all $n \in \omega$. The ideal $\mathcal{F}$ of nowhere dense sets of $\mathbb{R}$ has also been considered; for example, Fremlin [19∞b, 3B(b), 1J(b)] has proved $\mathrm{cf}(\mathcal{F}) = \mathrm{cf}(\mathbb{K})$. The relations among these cardinals can be displayed in the following diagram (called Cichoń's diagram, see Fremlin [1983/84])). We have redrawn Cichoń's diagram to follow the conventions of Diagram 1 (in addition, the dotted enclosures indicate the following two results: $\mathrm{add}(\mathbb{K}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathbb{K})\}$, and $\mathrm{cf}(\mathbb{K}) = \max\{\mathfrak{d}, \mathrm{non}(\mathbb{K})\}$). In the case $\mathfrak{c} = \omega_2$, a number of people have combined to give models for all cases which the diagram allows for assignment of the values ω_1, and ω_2; see Bartoszynski, Judah and Shelah [19∞]. Thus, the shape of Cichoń's diagram is settled.

Not much is known about the relations among the cardinals in Cichoń's diagram and the other cardinals above. The following diagram of Vojtáš indicates

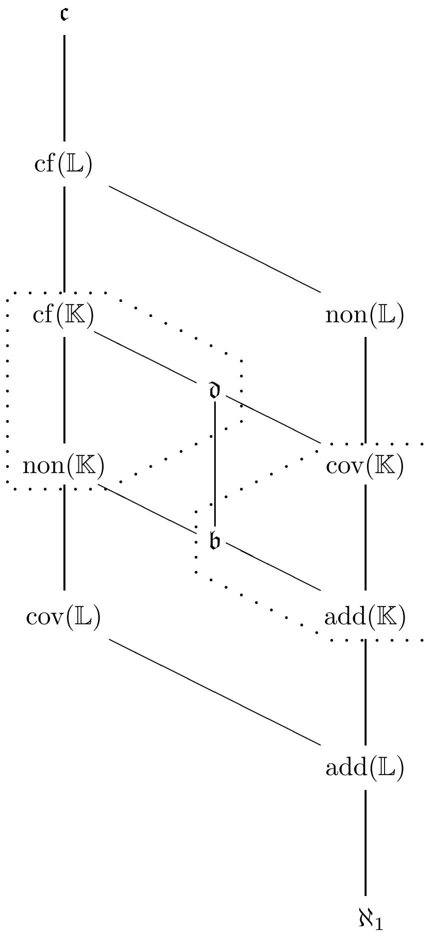

Diagram 2.

some of the known results. Among them are $\text{cov}(\mathbb{K}) \leq \mathfrak{r}$ (Vojtáš [19∞]), $\mathfrak{t} \leq \text{add}(\mathbb{K})$ (Piotrowski and Szymański [1987]), $\mathfrak{b} \leq \text{non}(\mathbb{K})$ (Rothberger [1941]). Also, $\mathfrak{s} \leq \text{non}(\mathbb{L})$ and $\mathfrak{s} \leq \text{non}(\mathbb{K})$ are attributed to J. Brzuchowski in Cichoń [1981]. Note that $\text{cov}(\mathbb{K}) \leq \mathfrak{d}$ follows at once from Bartoszynski's characterization of $\text{cov}(\mathbb{K})$ mentioned in the preceeding paragraph.

Given the above cardinals, it is natural to consider their exponentiations (i.e., 2^κ, 2^{2^κ}, and possibly further exponentiations), and their cofinalities (if not regular in **ZFC**). It is natural to ask how these cardinals are related to each other. Furthermore, there are occasions when one wants to look at

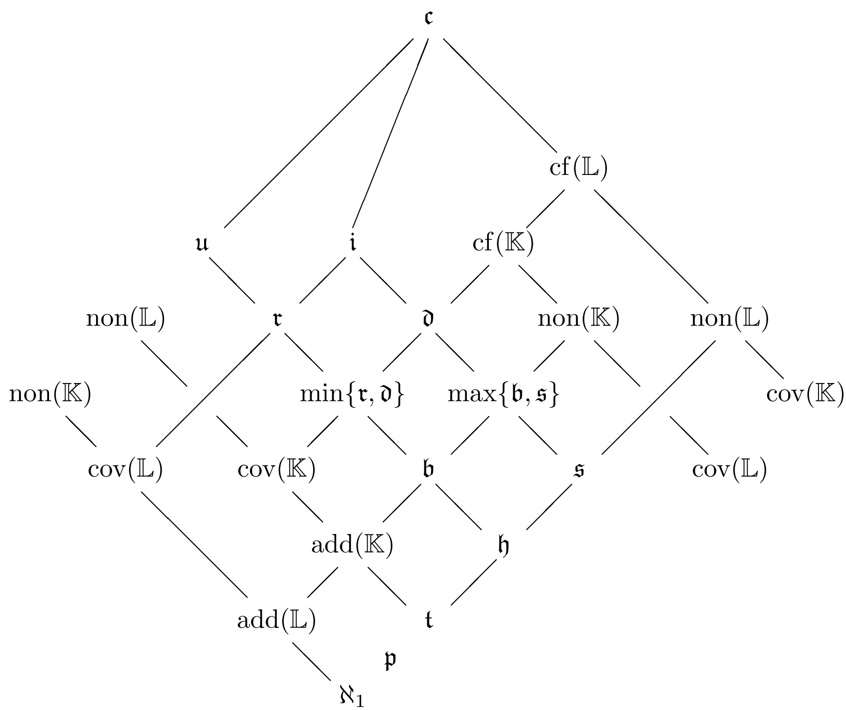

Diagram 3.

three or more of these cardinals at the same time. Indeed there are so many "obvious" open question about these cardinals that it is not possible to list them all here (an auxiliary problem is to find the most interesting among these questions). We mention several specific questions.

Problem 1.2. *Can $\mathfrak{a}$ and $\mathfrak{s}$ be singular?* **334. ?**

Problem 1.3. *Can $\mathfrak{m}$ be a singular cardinal of cofinality greater than ω_1?* **335. ?**

See Kunen [1988].

Problem 1.4. (Vojtáš [1988]) *Is $\mathfrak{r} = \mathfrak{r}_\sigma$?* **336. ?**

In a letter of October 1989, W. Just notes that if $\mathfrak{r} < \mathfrak{r}_\sigma$, then either $\mathfrak{r} < \mathfrak{u}$, or $\mathrm{cf}([\mathfrak{u}]^{\aleph_0}) > \mathfrak{u}$, and in the latter case $\mathfrak{u} \geq \aleph_\omega$ and there is an inner model where the covering lemma fails.

? **337.** **Problem 1.5.** *Can* $\mathrm{cf}(\mathrm{cov}(\mathbb{K})) = \omega$*?*

None of the other have countable cofinality; see BARTOSZYNSKI [1988], BARTOSZYNSKI and IHODA [19∞], BARTOSZYNSKI, IHODA and SHELAH [19∞], BARTOSZYNSKI and JUDAH [19∞], FREMLIN [1983/84], IHODA and SHELAH [19∞], and MILLER [1982a].

? **338.** **Problem 1.6.** *Can* $\mathrm{cf}(\mathrm{cov}(\mathbb{K})) < \mathrm{cf}(\mathrm{add}(\mathbb{K}))$*?*

The same question is open for measure zero sets; see BARTOSZYNSKI and IHODA [19∞] and BARTOSZYNSKI, IHODA and SHELAH [19∞].

? **339.** **Problem 1.7.** *Is* $\mathfrak{t} \leq \mathrm{add}(\mathbb{K})$*?*

PIOTROWSKI and SZYMAŃSKI [1987] proved that $\mathfrak{t} \leq \mathrm{add}(\mathbb{K})$.

2. Problems in topology

With many problems in topology, it is not immediately obvious whether small cardinals are involved or not. An example of this is the following rather old problem, raised in [1963] by MICHAEL:

? **340.** **Problem 2.1.** *Is there a Lindelöf space whose product with the space of irrational numbers is not normal?*

Consistent examples are known: any Lindelöf subspace of the Michael line has non-normal product with the irrationals, but such spaces exist iff $\mathfrak{b} = \omega_1$ (VAN DOUWEN [1984, 10.2]). ALSTER [19∞] has proved that under **MA**, there is a space that shows the answer is "yes". In Alster's result, however, W. Fleissner has pointed out that the part of **MA** that Alster used can be stated in the notation of small cardinals as "$\mathfrak{b} \leq \mathrm{cov}(\mathbb{K})$". Fleissner also noted that a model considered by MILLER [1982b] satisfies "$\mathrm{cov}(\mathbb{K}) < \mathfrak{b}$". LAWRENCE [19∞b] has proved that if X is Lindelöf and has non-normal product with the irrationals, then both the weight and cardinality of X are at least $\min\{\mathfrak{b}, \aleph_\omega\}$. Thus, the possibility is raised that the answer to Michael's question might be equivalent to some statement involving small cardinals.

Another old question, raised in 1966, is the Scarborough-Stone problem (SCARBOROUGH and STONE [1966]). A space is called *sequentially compact* (resp. *countably compact*) provided every sequence in the space has a convergent subsequence (resp. cluster point). The problem asks:

? **341.** **Problem 2.2.** *Is every product of sequentially compact spaces countably compact?*

It was recently solved in the negative by Nyikos for the class of T_2-spaces, but for the classes of T_3-spaces or $T_{3\frac{1}{2}}$-spaces, the problem has been solved (in the negative) so far only by assuming some extra axiom such as $\mathfrak{b} = \mathfrak{c}$ (van Douwen [1980a, 13.1]). It has also been solved in the negative in some models where $\mathfrak{b} < \mathfrak{c}$ (Nyikos and Vaughan [1987]). Also see Vaughan[1984].

A related problem raised by Comfort [1977] comes from the theorem of Ginsburg and Saks that states: if $X^{2^{\mathfrak{c}}}$ is countably compact then X^α is countably compact for all α (as was noted by Comfort, the proof of Ginsburg and Saks yields that if $\{X_i : i \in I\}$ is a family of spaces such that for all $J \subset I$ with $|J| \leq 2^{\mathfrak{c}}$, we have $\prod\{X_i : i \in J\}$ is countably compact, then $\prod\{X_i : i \in I\}$ is countably compact). The problem asks:

Problem 2.3. *Can $2^{\mathfrak{c}}$ be replaced by a smaller cardinal in this result, i.e., is there a cardinal $\kappa < 2^{\mathfrak{c}}$ such that for every space X, if X^κ is countably compact then every power of X is countable compact?* **342. ?**

Examples of Z. Frolík show that the answer is in the negative under the assumption of the generalized continuum hypothesis, and examples of V. Saks (assuming **MA** + ¬**CH** and weaker statements) show the same thing (cf. Vaughan [1984]).

Problem 2.4. *Is every product of $\mathfrak{h}$ sequentially compact spaces countably compact?* **343. ?**

See Nyikos, Pelant and Simon [19∞].

Problem 2.5. (Nyikos [19∞b]) *Does there exist a compact space which can be mapped continuously onto $[0,1]^{\mathfrak{s}}$ and has the following property: there exists a countable dense subset D such that every sequence in D has a subsequence that converges to some point in the space?* **344. ?**

Problem 2.6. *Is there a compact T_2-space X with no non-trivial convergent sequences and $|X| < 2^{\mathfrak{s}}$?* **345. ?**

Nyikos pointed out to me that a construction of Fedorchuk [1977] can be adapted to show that there exists such an X of cardinality $2^{\mathfrak{s}}$.

Problem 2.7. *Are there two countably compact topological groups whose product is not countably compact?* **346. ?**

Under **MA** (i.e., $\mathfrak{m} = \mathfrak{c}$) the answer was given in the affirmative by van Douwen [1980b], and using a different technique Hart and van Mill [19∞] proved that if $\mathfrak{m}_{\text{countable}} = \mathfrak{c}$, then there exists a countably compact group H such that $H \times H$ is not countably compact. The problem is still open in **ZFC**.

As far I know, the following variation of the Scarborough-Stone problem is also open:

? 347. Problem 2.8. *Is every product of sequentially compact, topological groups countably compact?*

A space is called *Fréchet* (or *Fréchet-Urysohn*) if every point in the closure of a set is the limit of a convergent sequence in the set.

? 348. Problem 2.9. *Is there a* countable *Fréchet topological group that is not metrizable?*

Such groups have been constructed assuming $\omega_1 < \mathfrak{p}$, or $\mathfrak{p} = \mathfrak{b}$ (NYIKOS [19∞a]). A Σ-product of uncountably many copies of $\{0, 1\}$ is a countably compact, non-compact Fréchet topological group, hence not metrizable (see ENGELKING [1989, 3.10.D]).

? 349. Problem 2.10. *Is there a separable, first countable, countably compact, non-normal T_2-space? Is there one, which is also almost compact (a space X is called* almost compact *provided $|\beta X - X| = 1$)?*

If "separable" is not required, then such examples exist which are ω-bounded (VAUGHAN [1979a, 1988]).

? 350. Problem 2.11. *Is there a separable, first countable, countably compact, non-compact T_2-space?*

This problem is discussed by Nyikos in this book.

Let μ_{cc} (resp. μ_{sc}) denote the cardinality of the smallest separable Hausdorff space with no isolated points which is countably (resp. sequentially) compact.

? 351. Problem 2.12. *Is $\mu_{cc} = \mathfrak{p}$? Is $\mu_{sc} = \mathfrak{a}$? is it consistent that $\mu_{cc} < \mu_{sc}$?*

See BEŠLAGIĆ, VAN DOUWEN, MERRILL and WATSON [1987].

? 352. Problem 2.13. *Is the box product of countably many copies of the rational numbers paracompact (normal)?*

LAWRENCE [19∞a] proved that the answer is in the affirmative under either assumption $\mathfrak{b} = \mathfrak{d}$, or $\mathfrak{d} = \mathfrak{c}$.

? 353. Problem 2.14. *Is the box product of countably many copies of the convergent sequence $\omega + 1$ normal?*

See van Douwen [1980a] and Williams [1984].

Many of the concepts considered in this paper can be extended to higher cardinals. To illustrate this, we list two such questions.

Problem 2.15. *Does there exist a first countable, initially $\aleph_1$-compact T_2-space which is not compact?* **354. ?**

A space is called initially $\aleph_1$-compact provided every open cover of cardinality $\leq \aleph_1$ has a finite subcover. This question was raised by Dow, who showed that the answer is in the negative under **CH** and several other conditions (Dow [1985]). Fremlin showed the answer is in the negative under **PFA** (see Balogh, Dow, Fremlin and Nyikos [1988]).

Problem 2.16. (Comfort [1988]) *Consider ${}^{\omega_1}\omega$ as a product of $\aleph_1$ countable, discrete spaces with the product topology. What is the smallest number of compact sets needed to cover ${}^{\omega_1}\omega$? In particular, $(*)$ can ${}^{\omega_1}\omega$ be covered by fewer than $2^{\aleph_1}$ compact sets?* **355. ?**

This is equivalent to asking if there exists a dominating (= cofinal) family $\mathcal{F} \subset {}^{\omega_1}\omega$ with $|\mathcal{F}| < 2^{\aleph_1}$ with respect to the product order: $f \leq g$ iff $f(\alpha) \leq g(\alpha)$ for all $\alpha \in \omega_1$ (Tall [1989]). It is the same question if we work with respect to the mod countable order (Comfort [1977]), and therefore it is known that the problem involves large cardinals: Jech and Prikry [1984] have proved that if $\mathfrak{c}$ is real-valued measurable, then the answer to $(*)$ is "no", and if the answer is "yes", and $2^{\aleph_1}$ has a certain property, then there are models with large cardinals.

3. Questions raised by van Douwen in his Handbook article

In his article van Douwen [1984], van Douwen raised ten questions related to small cardinals. For the convenience of the reader we will state all of them here, and use his enumeration of these problems. The ones that have been solved so far are van Douwen [1984, 6.6, 6.10, part of 6.11, 8.11, and 8.14].

Problem 6.6. *Is there a compact space of cardinality $2^{\mathfrak{t}}$ which is not sequentially compact?*

Solution: Alan Dow has answered the above question in the positive by noting that if X is compact Hausdorff and not sequentially compact, then $\mathfrak{n} \leq |X|$, and by constructing a model (a variation on model V in Balcar, Pelant and Simon [1980]) where $2^{\mathfrak{t}} < \mathfrak{n}$.

Van Douwen's question can be revived by asking: how can the cardinal, which is defined as the smallest cardinality of a compact, non-sequentially compact space, be expressed as a set-theoretically defined cardinal?

? **356.** **Problem 6.7.** *For compact X, in* **ZFC** *does "every countable compact subspace of X is closed" imply "X is a sequential space"?*

It is known that the answer is "yes" if $\mathfrak{c} < 2^{\mathfrak{t}}$ (VAN DOUWEN [1984, 6.4]). Let

$$\mu = \min\{\kappa : \text{some product of } \kappa \text{ sequentially compact spaces is not sequentially compact}\}.$$

Problem 6.10. *Can the cardinal μ be expressed as a set-theoretically defined cardinal?*

SOLUTION: $\mu = \mathfrak{h}$ (NYIKOS, PELANT and SIMON [19∞] and FRIČ and VOJTÁŠ [1985] independently).

? **357.** **Problem 6.11.** (Restatement) *Is every product of sequentially compact spaces countably compact (i.e., is $\{\kappa :$ some product of κ sequentially compact spaces is not countably compact$\}$ non-empty)?*

This is the Scarborough-Stone problem 2.2. If this set is non-empty, then let

$$\mu_1 = \min\{\kappa : \text{some product of } \kappa \text{ sequentially compact spaces is not countably compact}\}.$$

Can the cardinal μ_1 be expressed as a set-theoretically defined cardinal? It is known that $\mu_1 \geq \mathfrak{n}$ (NYIKOS, PELANT and SIMON [19∞] and FRIČ and VOJTÁŠ [1985]).

Partial solution to the Scarborough-Stone problem: NYIKOS [1988] has shown that in **ZFC** there exists a family of Hausdorff (non-regular) sequentially compact spaces whose product is not countably compact. This answers the first part of the above question. For regular (or $T_{3\frac{1}{2}}$-spaces) all of VAN DOUWEN [1984, 6.11] is still open in **ZFC**. Some consistency results are discussed in 2.2.

Problem 8.11. *If X is a separable metric space and (a) X is analytic or (b) absolutely Borel, then is $\mathrm{cf}(\mathcal{K}(X)) = k(X) = \mathfrak{d}$?*

SOLUTION: By VAN DOUWEN [1984, 8.10] this question is clearly intended for X that are not σ-compact, and for them, $\mathfrak{d} \leq k(X) \leq \mathrm{cf}(\mathcal{K}(X))$. Thus, the question reduces to: is $\mathrm{cf}(\mathcal{K}(X)) \leq \mathfrak{d}$? Here, $\mathrm{cf}(\mathcal{K}(X))$ denotes the smallest cardinality of a family $\mathcal{L}$ of compact subsets of X such that for every compact set $K \subset X$, there exists $L \in \mathcal{L}$ with $K \subset L$. A space is called *analytic* if it is the continuous image of the space of irrational numbers, and *absolutely Borel* if it is a Borel set in any of its metrizable compactifications. The answer to (b) is in the affirmative, but the answer to (a) is independent of the axioms of **ZFC**.

Concerning (a): Becker [19∞] has constructed a model in which there is an analytic space $X \subset 2^\omega$ with $\mathrm{cf}(\mathcal{K}(X)) > \mathfrak{d}$. On the other hand, under **CH**, $\mathrm{cf}(\mathcal{K}(X)) = \mathfrak{d} = \omega_1$.

Concerning (b): van Engelen [19∞] proved that if X is co-analytic (absolutely Borel sets are both analytic and co-analytic), then $\mathrm{cf}(\mathcal{K}(X)) \leq \mathfrak{d}$. The same follows from Fremlin's theory of Tukey's ordering (Fremlin [19∞a, 4, 15, 16]). Also see Fremlin [19∞b].

Problem 8.14. *Let S be a subset of a separable metric space X and assume that S is absolutely Borel. Is it true that if $S \cap \mathrm{cl}_X(X - S)$ is noncompact, then $\chi(S, X) = \mathfrak{d}$?*

Solution: Van Engelen and Becker have observed (independently) that the answer is "yes". It follows from van Engelen's result "$\mathrm{cf}(\mathcal{K}(X)) \leq \mathfrak{d}$" and the method of proof of van Douwen [1984, 8.10(c), 8.13(c)].

Problem 8.17. *Is there a (preferably metrizable) not locally compact space X with $\mathrm{Exp}_{\mathbb{R}}(X) < \mathrm{Exp}_\omega < \infty$?* **358. ?**

Here ∞ is defined to be larger than any cardinal, and

$\mathrm{Exp}_{\mathbb{R}}(X) = \min\{\kappa : X \text{ embeds as a closed subspace in } {}^\kappa\mathbb{R}\}$, and
$\mathrm{Exp}_\omega(X) = \min\{\kappa : X \text{ embeds as a closed subspace in } {}^\kappa\omega\}$.

Let

$\mathfrak{a}_{\mathrm{p}} = \min\{\,|X| : X$ is first countable and pseudocompact but not countably compact$\}$.

It is trivial that $\mathfrak{b} \leq \mathfrak{a}_{\mathrm{p}} \leq \mathfrak{a}$.

Problem 12.5. *What is $\mathfrak{a}_{\mathrm{p}}$?* **359. ?**

Problem 12.6. *Is there in* **ZFC** *a first countable (preferably separable and locally compact) pseudocompact space that is not countably compact and that has no uncountable closed discrete subset?* **360. ?**

Problem 13.4. *Is the following true in* **ZFC***: Each first countable space of* **361. ?**

cardinality at most $\mathfrak{c}$ is a quasi-perfect image of some locally compact space.

It is true under "$\mathfrak{b} = \mathfrak{c}$" (van Douwen [1984, 13.4]). Is the condition "of cardinality at most $\mathfrak{c}$" essential?

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Appendix

Remarks on some cardinal invariants of the continuum

by

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A.1. Theorem. **ZFC** $\vdash \mathfrak{d} \leq \mathfrak{i}$.

A.2. Notation. Let $\mathcal{A} \subseteq [\omega]^\omega$ and $\mathrm{Fn}(\mathcal{A}) = \{ f : f \text{ finite}, \mathrm{dom}(f) \subseteq \mathcal{A}, \mathrm{rng}(f) \subseteq \{0,1\} \}$. For the following f, g and h will always range over $\mathrm{Fn}(\mathcal{A})$.

For $f \in \mathrm{Fn}(\mathcal{A})$, let

$$X_f = \bigcap_{a \in \mathrm{dom}(f)} a^{f(a)},$$

where $a^1 = a$, $a^0 = \omega - a$.

From now on let $\mathcal{A}$ be independent i.e., X_f is infinite for all f.

Let $I = I_{\mathcal{A}} = \{ A \subseteq \omega : \forall f \exists g \supseteq f \quad (X_g \cap A \text{ is finite}) \}$.

Clearly, I is an ideal containing all finite sets and $X_f \notin I_{\mathcal{A}}$ for all f.

A.3. Lemma (Assuming $|\mathcal{A}| < \mathfrak{d}$). *Let $E \in I_{\mathcal{A}}$ and assume that $f \in \mathrm{Fn}(\mathcal{A})$ and $A_0, A_1, \ldots, A_n, \ldots \in \mathcal{A}$ $(n \in \omega)$ are such that* $\mathrm{dom}(f) \subseteq \mathcal{A}' =^{def} \mathcal{A} - \{A_0, \ldots\}$, *then there is a set E' such that:*

(α) $E' \in I_{\mathcal{A}}$.

(β) $E' \cap E = \emptyset$

(γ) $\forall g \in \mathrm{Fn}(\mathcal{A}')$: If $g \supseteq f$ then $X_g \cap E'$ is infinite.

Proof. For any $H\colon \omega \to \omega$ let

$$E'_H = \bigcup_n \left[(A_n - \bigcup_{i<n} A_i) \cap H(n) \right] - E.$$

Then clearly $E'_H \in I_{\mathcal{A}}$ and $E'_H \cap E = \emptyset$, so any E'_H satisfies (α) and (β). We have to find a suitable H such that (γ) is satisfied.

Note that if $g \supseteq f$ and $\mathrm{dom}(g) \subset \mathcal{A}'$ then $X_g \cap (A_n - \bigcup_{i<n} A_i) - E \notin I_{\mathcal{A}}$, so in particular it is infinite. (Since $(A_n - \bigcup_{i<n} A_i)$ is of the form X_h for some h with $\mathrm{dom}(h) \cap \mathrm{dom}(g) = \emptyset$, it is not in $I_{\mathcal{A}}$.)

For each $g \in \mathrm{Fn}(\mathcal{A}')$ extending f, let

$$H_g(n) = \min(X_g \cap (A_n - \bigcup_{i<n} A_i) - E).$$

Clearly H_g is a 1-to-1 function, and if $H_g(n) < H(n)$, then $H_g(n) \in X_g \cap E'_H$. Hence if for infinitely many n,

$$H_g(n) < H(n)$$

then

$$X_g \cap E'_H \text{ is infinite.}$$

Since $|\mathcal{A}| < \mathfrak{d}$, we can find H such that for all $g \in \mathrm{Fn}(\mathcal{A})$ with $g \supseteq f$ there are infinitely many n for which $H_g(n) < H(n)$.

Then $E' = E'_H$ satisfies the requirements of the lemma. □

PROOF OF THE THEOREM: Assume $\mathcal{A}$ is an independent family of size $< \mathfrak{d}$. We will show that $\mathcal{A}$ is not maximal.

Let $N \prec \langle H(\lambda), \in \rangle$ for sufficiently large λ with N countable and $\mathcal{A} \in N$. Let $\{f_n : n \in \omega\}$ list $\mathrm{Fn}(\mathcal{A}) \cap N$, such that each element of $\mathrm{Fn}(\mathcal{A}) \cap N$ appears with even and with odd index. By induction choose $E_n \in N$ such that:

(A) $E_n \in I_{\mathcal{A}}$

(B) $E_n \cap (\bigcup_{l<n} E_l) = \emptyset$

(C) If $f_n \subseteq g \in \mathrm{Fn}(\mathcal{A})$ and $\mathrm{dom}(g) \cap N = \mathrm{dom}(f_n)$ then $X_g \cap E_n$ is infinite

We can do this by the previous lemma, letting $E = \bigcup_{l<n} E_l$, $f = f_n$ and $\{A_0, A_1, \ldots\} \in N$ be some family disjoint from $\mathrm{dom}(f_n)$. (we can have $E_n \in N$ by elementarity of N).

Now let $Y = \bigcup_n E_{2n}$. Then $\mathcal{A} \cup \{Y\}$ is independent: Let $g \in \mathrm{Fn}(\mathcal{A})$. Find n such that $f_{2n} = g \cap N$. Then $X_g \cap Y$ contains $X_g \cap E_{2n}$ which is infinite. If $g \cap N = f_{2k+1}$ for some k then $X_g \cap (\omega - Y)$ contains $X_g \cap E_{2k+1}$ which is also infinite. This finishes the proof of the theorem. □

A.4. REMARK. $\mathfrak{d} < \mathfrak{i}$ is consistent: e.g., take a model of **CH** and add $\aleph_2$ many random reals with countable support. Then the old reals still form a dominating family. But an independent family of size ω_1 must be in an intermediate model, so it cannot be maximal, since the next random real will be independent from it. We can understand this argument more generally: if the set of reals is not the union of fewer than λ sets of measure zero, then any independent family of subsets of ω has cardinality at least λ. So if P is the forcing of the measure algebra of dimension $\lambda > \aleph_0$ then in V^P one has $\mathfrak{i} \geq \lambda$, whereas $\mathfrak{d}$ is not changed by forcing with P.