Mechanization for solving SPP by reducing order method

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Abstract

In this paper, by using the theories and methods of mathematical analysis and computer algebra, a reliable algorithm of reduction of order approximation method for solving singular perturbation problems was established, a new Maple procedure redordproc was established, too. The procedure redordproc give not only the approximate analytic solutions but also the numerical solutions of the kinetics problems. Some examples are presented to illustrate the implementation of the program.

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Keywords: Singular perturbation problems (SPP); Reduction of order method; Algorithm; Mechanization; Maple

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1. Introduction

Many physical processes connected with nonuniform transitions are described by differential equations with large or small parameters. If, in problems arising in this manner, the role of the perturbation is played by leading terms of the differential operator (or part of them), then the problem is called a singular perturbation problem (SPP) [1]. Singular perturbation problems occur very frequently in fluid mechanics and other branches of applied mathematics [2]. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts [3]. So, SPP have been received a significant amount of attention in past and recent years. And some valid methods for solving SPP have been developed in resent years, such as Taylor series, spline in compression, least-square spectral collection, Paè approximation method, etc. [1–11].

But it is a difficult point in the field of solving such problems because of huge size of calculations and the high complexity in practice. It is well known that the rapid development of computer science and computer algebra system has a profound effect on the concept and the methods of mathematical researches [12–18]. The objective of this paper is to establish a promising algorithm that can be easily solved singular perturbation problems in Maple with mechanization.

2. Basic methods

Let’s first recall the basic principles of the reducing order method for solving singular perturbation problems.

Consider a singular perturbed two-point boundary value problem of the form:

$$
e y''(x) + f(x)y(x) + g(x)y(x) = h(x), \quad x \in [a, b],$$

(2.1)

with conditions

$$y(a) = \alpha, \quad y(b) = \beta,$$

where $$\varepsilon$$ is a small positive parameter $$(0 < \varepsilon \ll 1)$$, $$\alpha$$ and $$\beta$$ are given constants, $$f(x)$$ and $$g(x)$$ are assumed to be sufficiently continuously differentiable functions on $$[a, b]$$, and $$f(x) \gg M > 0$$ throughout the interval $$[a, b]$$, where $$M$$ is some positive constant. Under these assumptions, (2.1) has a unique solution $$y(x)$$ which in general, displays a boundary layer of width $$O(\varepsilon)$$ at $$x = a$$ for some small values of $$\varepsilon$$.

**Theorem [2]** 1. The singular perturbation problem (2.1) with conditions $$y(a) = \alpha$$, $$y(b) = \beta$$ could be convert into the following form:
\[
\begin{aligned}
&\frac{d}{dx}[f(x)q(x)] = g(x)q(x), \quad q(a) = z - p(a), \\
&r'(x) = f(x), \quad r(a) = 0.
\end{aligned}
\]

Solving the above equations (2.2) to find \(p(x), q(x)\) and \(r(x)\), then we can get the solution of (2.1) as follows:

\[
y(x) = p(x) + q(x)e^{-r(x)/\varepsilon}.
\]

Based on this, when we input the values of \(x\) and \(\varepsilon\) into (2.3), we can get the numerical solutions of (2.1) rapidly.

3. Algorithm for solving SPP by reducing order method with mechanization

It is a new development orientation in the field of mathematics and computer to conduct science calculation by computer [12,13].

Maple, an international mathematical software, is an exchanged computer algebra system with great ability of symbolic operation, numerical calculation, coping with graphics, etc. Its powerful functions library and unique interior programming language provide scientific calculation and programming with friendly platform [14–18].

In Maple, the process of solving equations (2.3) can be well established a procedure which can calculate the approximate analytic solutions and numerical solutions of problem (2.1) easily. Now if we want to solve SPP by using the method of reduction of order, everything we have to do is just to input information about the equations, then the program will give out the approximate analytic solution of the problem. Based on this, when we input the values of \(x\) and \(\varepsilon\), we can get the numerical solutions of the equations. The main algorithm of redordproc is as follows:

```maple
redordproc:=proc(expr,init)
local f,g,h,p,q,r,px,qx,rx,eqn1,eqn2,eqn3,ini1,ini2,ini3;
f:=coefproc(expr,1);
g:=coefproc(expr,0);
h:=-coefproc(expr,-1);
f:=unapply(f,x);
g:=unapply(g,x);
h:=unapply(h,x);
eqn1:=f(x)*diff(p(x),x)+g(x)*p(x)=h(x);
ini1:=p(op(2,op(1,init)))=op(2,op(2,init));
px:=dsolve(eqn1,ini1,p(x));
p:=unapply(rhs(px),x);
end proc;
```

eqn2:=diff(f(x),x)*q(x)+f(x)*diff(q(x),x)=g(x)*q(x);
ini2:=q(op(1,op(1,init)))=op(1,op(2,init))-p(op(1,op(1,init)));
qx:=dsolve(eqn2,ini2,q(x));
q:=unapply(rhs(qx),x);
eqn3:=diff(r(x),x)=f(x);
ini3:=r(op(1,op(1,init)))=0;
rx:=dsolve(eqn3,ini3,r(x));
lprint('The singular perturbation problem can be converted into:');
print(eqn1,ini1);
print(eqn2,ini2);
print(eqn3,ini3);
lprint('Solving above equations, we can get:');
print(px, qx, rx);
lprint('Then we can get the approximate analytic solution of the SPP is
as follows:');
print(y(x)=rhs(px)+rhs(qx)*exp(-rhs(rx)/epsilon));
end proc:

The use of sub-procedure coefproc we established is extracting the coefficients of a differential-function polynomial.

Easy to read, in this paper, we established the form of readable outputting for the solving results by use of the commands lprint and print. So, we can get the whole processes of solving the singular perturbation problems just like we do them with our hands. For a detailed discussion on readable outputting we may refer the readers to the article [15] of ours.

For convenience, in the program redordproc, we always let \( x \) represent the independent variable, and set the parameters as following:

expr: the equation which to be solved
init: list of initial conditions of (2.1) in the form \([a, b], [x, \beta]\)

For example, if we consider the singular perturbation problems:

\[ \varepsilon y''(x) + y(x)y'(x) - e^{\varepsilon x} = x + 3, \quad y(0) = e^2, \quad y(1) = \ln 3. \]

One just sets:

eqns:=\varepsilon*diff(y(x),x$2)+y(x)*diff(y(x),x)-exp(y(x))=x+3;
init:=[y(0)=exp(2), y(1)=ln3];
redordproc(eqns,[[0,1],[exp(2),ln3]]);
4. Examples

In this section, there are some examples to illustrate the mechanized process.

Example 1. Consider the following singular perturbation problem:

\[ \varepsilon y''(x) + y'(x) - y(x) = 0, \quad x \in [0, 1], \]
\[ y(0) = 1, \quad y(1) = 1. \]  \hspace{1cm} (4.1)

The exact solution is given by

\[ y(x) = \frac{(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}}{e^{m_2} - e^{m_1}}, \]

where \( m_1 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon} \) and \( m_2 = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon} \).

To solve this problem by using procedure `redordproc`, everything we have to do is just inputting the commands as follows:

\[
\begin{align*}
\text{eqns} & := \varepsilon \cdot \text{diff}(y(x), x^2) + \text{diff}(y(x), x) - y(x) = 0; \\
\text{ini} & := y(0) = 1, \ y(1) = 1; \\
\text{redordproc}(\text{eqns}, [[0,1],[1,1]]);
\end{align*}
\]

Then we can get the following results:

The singular perturbation problem can be converted into:

\[ \frac{d}{dx} p(x) - p(x) = 0, \quad p(1) = 1, \]
\[ \frac{d}{dx} q(x) = -q(x), \quad q(0) = 1 - \frac{1}{e}, \]
\[ \frac{d}{dx} r(x) 1, \quad r(0) = 0. \]

Solving above equations, we can get:

\[ p(x) = \frac{e^x}{e}, \quad q(x) = \frac{(e - 1)e^{-x}}{e}, \quad r(x) = x. \]

Then we can get the approximate analytic solution of the SPP is as follows:

\[ y(x) = \frac{e^x}{e} + \frac{(e - 1)e^{-x}}{e}. \]

If we want to obtain the numerical solutions of the equations, we need to input the value of \( x \) and \( \varepsilon \). The numerical results of Example 1 are given in Table 1.

Apparently, our results in Table 1 are more approximate with the exact solutions.
In Maple, it is a simple thing for inputing the values of $x$ and $\varepsilon$ to calculate $y(x)$ as Table 1. So, we do not want to repeat this process in this paper.

**Example 2.** Consider the equations:

$$
\varepsilon y''(x) + \left(1 - \frac{x}{2}\right)y'(x) - \frac{1}{2}y(x) = 0, \quad x \in [0, 1],
$$

$$
y(0) = 0, \quad y(1) = 1.
$$

(4.2)

One just sets:

```maple
eqns := epsilon*diff(y(x),x$2)+(1-x/2)*diff(y(x),x)-y(x)/2=0;
ini := y(0)=0,y(1)=1;
reordproc(eqns,[[0,1],[0,1]]);
```

We can get:

The singular perturbation problem can be converted into:

$$
\left(1 - \frac{x}{2}\right) \frac{dp(x)}{dx} - \frac{1}{2}p(x) = 0, \quad p(1) = 1,
$$

$$
- \frac{1}{2}q(x) + \left(1 - \frac{1}{2}x\right) \frac{dq(x)}{dx} = - \frac{1}{2}q(x), \quad q(0) = - \frac{1}{2},
$$

$$
\frac{dr(x)}{dx} = 1 - \frac{1}{2}x, \quad r(0) = 0.
$$

Solving above equations, we can get:

$$
p(x) = - \frac{1}{2 + x}, \quad q(x) = - \frac{1}{2}, \quad r(x) = - \frac{1}{4}x^2 + x.
$$

Table 1

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</table>

In Maple, it is a simple thing for inputing the values of $x$ and $\varepsilon$ to calculate $y(x)$ as Table 1. So, we do not want to repeat this process in this paper.
Then we can get the approximate analytic solution of the SPP is as follows:

\[
y(x) = -\frac{1}{2+x} - \frac{1}{2} e^{-(1/4)x^2}.
\]

**Example 3.** Solving the following singular perturbation problems:

\[
\begin{align*}
\epsilon y''(x) - y'(x) - (1 + \epsilon) y(x) &= 0, \quad x \in [0, 1], \\
y(0) &= 1 + \exp(-(1 + \epsilon)/\epsilon), \quad y(1) = 1 + \frac{1}{e}.
\end{align*}
\]

(4.3)

Input the following commands:

\[
\begin{align*}
eqns&:=\text{epsilon}\times\text{diff}(y(x),x^2)\text{-diff}(y(x),x)-(1+\text{epsilon}) \* y(x)=0; \\
\text{ini}&:=y(0)=1+\exp(-(1+\text{epsilon})/\text{epsilon}),y(1)=1+1/\exp(1); \\
\text{reordprocon}(\text{eqns},[[0,1],[1+\exp(-(1+\text{epsilon})/\text{epsilon}),1+1/\exp(1)]]);
\end{align*}
\]

We can get:

The singular perturbation problem can be converted into:

\[
\begin{align*}
-\frac{d}{dx}p(x) + (-1 - \epsilon)p(x) &= 0, \quad p(1) = 1 + \frac{1}{e}, \\
-\frac{d}{dx}q(x) &= (-1 - \epsilon)q(x), \quad q(0) = 1 + e^{-\frac{1+\epsilon}{\epsilon}} - \frac{e + 1}{e^{-\epsilon}}, \\
\frac{d}{dx}r(x) &= -1, \quad r(0) = 0.
\end{align*}
\]

Solving above equations, we can get:

\[
\begin{align*}
p(x) &= \frac{(e + 1)e^{-(1+\epsilon)x}}{e^{-\epsilon}}, \quad q(x) = \frac{(e^{-\epsilon} + e^{-\frac{1+\epsilon}{\epsilon}} - e - 1)e^{(1+\epsilon)x}}{e^{-\epsilon}}, \quad r(x) = -x.
\end{align*}
\]

Then we can get the approximate analytic solution of the SPP is as follows:

\[
y(x) = \frac{(e + 1)e^{-(1+\epsilon)x}}{e^{-\epsilon}} + \frac{(e^{-\epsilon} + e^{-\frac{1+\epsilon}{\epsilon}} - e - 1)e^{(1+\epsilon)x}}{e^{-\epsilon}} e^{-x/\epsilon}.
\]

**Example 4**

\[
\begin{align*}
\epsilon y''(x) + x^2 y'(x) + 2x^3 y(x) &= 0, \quad x \in [1, 2], \\
y(1) &= e, \quad y(2) = \ln 2.
\end{align*}
\]

(4.4)
Input the following commands:

\[
eqns := \epsilon \frac{\partial^2 y(x)}{\partial x^2} + \frac{\partial y(x)}{\partial x} + 2 x^3 y(x) = 0; \\
ini := y(1) = \exp(1), y(2) = \ln(2); \\
\text{redordproc}(\eqns, \{[1, 2], [\exp(1), \ln(2)]\});
\]

The approximate solution is as follows:
The singular perturbation problem can be converted into:

\[
x^2 \frac{d}{dx} p(x) + 2x^3 p(x) = 0, \quad p(2) = \ln 2,
\]

\[
2xq(x) + x^2 \frac{d}{dx} q(x) = 2x^3 q(x), \quad q(1) = e^1 - \frac{\ln 2e^{-1}}{e^{-4}},
\]

\[
\frac{d}{dx} r(x) = x^2, \quad r(1) = 0.
\]

Solving above equations, we can get:

\[
p(x) = \frac{\ln(2)e^{-x^2}}{e^{-4}}, \quad q(x) = -\frac{(-ee^{-4} + \ln(2)e^{-1})e^{x^2}}{ee^{-4}x^2}, \quad r(x) = \frac{1}{3} x^3 - \frac{1}{3}.
\]

Then we can get the approximate analytic solution of the SPP is as follows:

\[
y(x) = \frac{\ln(2)e^{-x^2}}{e^{-4}} - \frac{(-ee^{-4} + \ln(2)e^{-1})e^{x^2}}{ee^{-4}x^2} e^{(1/3)x^3-(1/3)/\epsilon}.
\]

Example 5

\[
e\frac{d^2 y}{dx^2} + xy' - y = 0, \quad x \in [-1, 1], \\
y(-1) = 1, \quad y(1) = \ln 2.
\]

One just sets:

\[
eqns := \epsilon \frac{\partial^2 y(x)}{\partial x^2} + x \frac{\partial y(x)}{\partial x} - y(x) = 0; \\
ini := y(-1) = 1, \quad y(1) = 2; \\
\text{redordproc}(\eqns, \{[-1, 1], [1, 2]\});
\]

The singular perturbation problem can be converted into:

\[
x \frac{d}{dx} p(x) = 0, \quad p(1) = 2,
\]
\[ q(x) + x \frac{d}{dx} q(x) = -q(x), \quad q(-1) = 3, \]
\[ \frac{d}{dx} r(x) = x, \quad r(-1) = 0. \]

Solving above equations, we can get:
\[ p(x) = 2x, \quad q(x) = \frac{3}{x^2}, \quad r(x) = \frac{1}{2}x^2 - \frac{1}{2}. \]

Then we can get the approximate analytic solution of the SPP is as follows:
\[ y(x) = 2x + \frac{3}{x^2} e^{\left((1/2)x^2-(1/2)/\varepsilon\right)}. \]

5. Conclusion and remarks

Mechanization of reduction of order method for solving singular perturbation problems has proposed in this study. The procedure redordproc give not only the approximate analytic solutions of SPP, but also the numerical solutions. The method is very simple and effective for most of singular perturbation problems.

Of course, in this paper, we give the solution of left-end boundary layer problems of singular perturbation problems. If we want to obtain the solution of right-end boundary layer problems, we must convert Eq. (2.1) into the following form:

\[ f(x)p'(x) + g(x)p(x) = h(x), \quad p(a) = \beta, \]
\[ \frac{d}{dx} [f(x)q(x)] = g(x)q(x), \quad q(b) = \beta - p(b), \]
\[ r'(x) = f(x), \quad r(b) = 0. \]

Solving above equations, we can get the solution of (2.1) as follows:
\[ y(x) = p(x) + q(x)e^{-r(x)/\varepsilon}. \]

In Maple, there are three methods to establish functions: arrow operator (\(\rightarrow\)), command unapply and program. But we find an interesting thing which we cannot use arrow operator in programming when we want to establish a function. The only way to do it is using the command unapply to convert an expression into function.

The results of the examples indicated that the procedure redordproc of reducing order approximation method had advantages of simple idea, excellent property for operation and powerful competence. This would be useful for solving differential equations.
References