Real Quantifier Elimination by Computation of Comprehensive Gröbner Systems

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Contents of My Talks

Motivation

Todai Robot Project

- The motivation of our work has its roots in "Todai Robot Project".

  "Todai Robot Project" is the ongoing research project of artificial intelligence.

- The purpose of "Todai Robot Project" is to develop software which automatically produces an answer sheet for an entrance examination of "Todai".

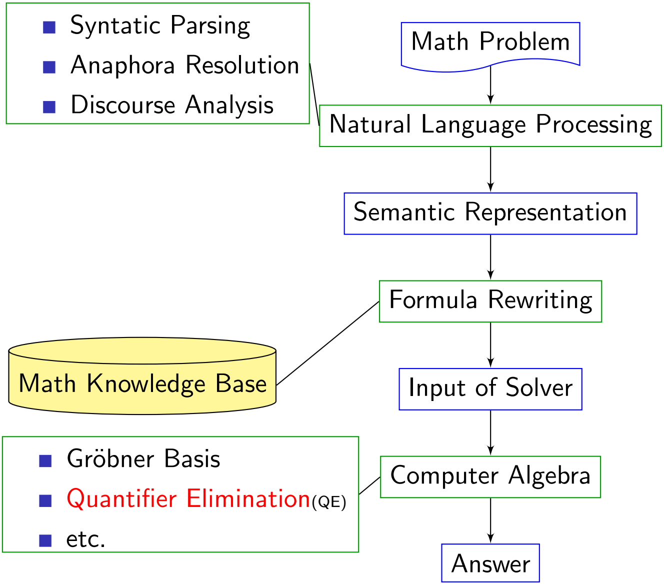
  University of Tokyo is known as "Todai" in Japan.

  University of Tokyo is the highest rank university in Japan.

  We have to obtain a sufficient score to pass by using our software.

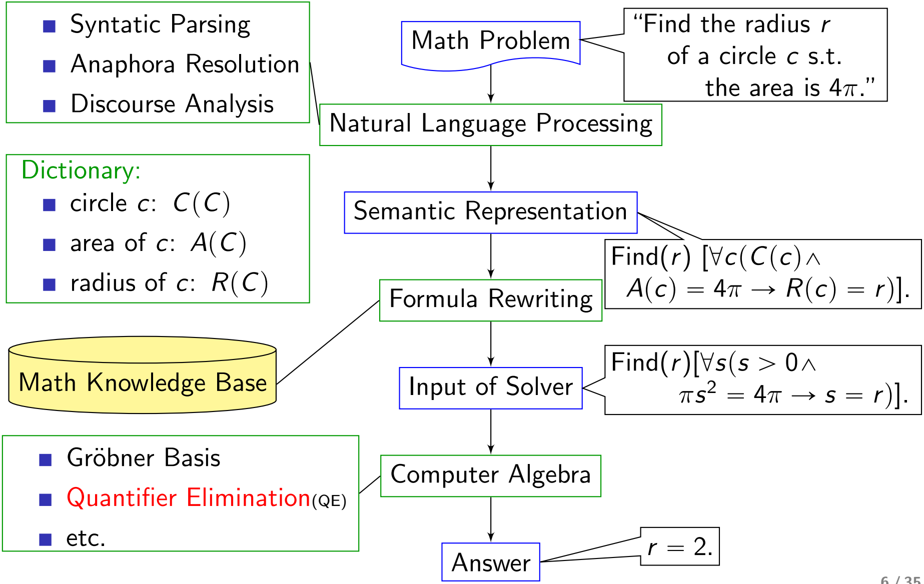
Todai Robot Project

- How does our software solve math problems?



Todai Robot Project

- How does our software solve math problems?

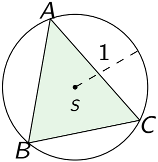


Quantified Formula with Many Equalities

- Our software often generates a quantified formula with many equalities.

Example $\triangle ABC$ is inscribed in a circle with the radius 1, $tan(\angle CAB) = m$ and $tan(\angle ABC) = n$. However $m, n \geqslant 3$. Let S be the area of $\triangle ABC$.

(1) Represent S on the terms of m, n.



Let

ϕ_1 be $x_0x_3 - x_0x_4 + x_1x_2 - x_1x_3 - x_2x_5 + x_4x_5 \geqslant 0$,

ϕ_2 be $(x_5 - x_0)x_4 - x_2 - (x_3 - x_2)x_1 - x_0 \geqslant 0$,

ϕ_3 be $(x_5 - x_0)((1/2)x_0 + (1/2)x_5 + x_7) + (x_3 - x_2)((1/2)x_2 + (1/2)x_3 - x_6) = 0$,

ϕ_4 be $(x_1 - x_5)((1/2)x_5 + (1/2)x_1 - x_7) + (x_4 - x_3)((1/2)x_3 + (1/2)x_4 - x_6) = 0$,

ϕ_5 be $((x_7 - x_0)^2 + (x_6 - x_2)^2)^{1/2} = 1$,

ϕ_6 be $|x_0x_3 - x_0x_4 + x_1x_2 - x_1x_3 - x_2x_5 + x_4x_5|/((x_1 - x_0)(x_5 - x_0) + (x_4 - x_2)(x_3 - x_2)) = m$,

ϕ_7 be $|x_0x_3 - x_0x_4 + x_1x_2 - x_1x_3 - x_2x_5 + x_4x_5|/((x_0 - x_5)(x_1 - x_5) + (x_2 - x_3)(x_4 - x_3)) = n$,

ϕ_8 be $m \geqslant 3 \wedge n \geqslant 3$,

ϕ_9 be $|x_5 - x_0x_4 - x_2 + x_3 - x_2x_1 - x_0|/2 = S$ and

ϕ be $\exists x_0 \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \exists x_7 (\bigwedge_{1 \leqslant i \leqslant 9} \phi_i)$.

- ϕ can be not solved within 1 hour by the existing QE software SyNRAC@Maple, RegularChains@Maple, Resolve@Mathematica, Reduce@Mathematica, QEPCAD and RedLog@Reduce.

Quantified Formula with Many Equalities

- We need to establish a practical implementation of QE
  for a quantified formula with many equalities.
- We improve the following work:

  1998: Weispfenning, V. :
  A New Approach to Quantifier Elimination for Real Algebra.

- We call for short "comprehensive Gröbner system" "CGS" and
  "real QE by computation of CGSs" "CGS-QE".

Real QE by Computation of CGSs

- CGS-QE is a special QE method for the input formula

$$\exists\bar{x}((\bigwedge_i f_i = 0) \wedge (\bigwedge_i p_i > 0) \wedge (\bigwedge_i q_i \neq 0)),$$

where $\bar{X} = X_1, \ldots, X_n$, $\bar{Y} = Y_1, \ldots, Y_m$, $f_i, p_i, q_i \in K[\bar{Y}, \bar{X}]$.

- CGS-QE uses "Real Root Counting Theorem (Pedersen)" and "CGS".
  - In Section **Real Root Counting**, we modify "Real Root Counting Theorem" for improving CGS-QE.
  - In Section **Comprehensive Gröbner System**, we show its definition.

Real Root Counting

Notations

- R denotes a real closed field,

  C its algebraic closed extension and

  K a computable subfield of R.

- Let $\bar{X}$ be variables $X_1, \ldots, X_n$.

- $T(\bar{X})$ denotes the set of all terms consisting of variables in $\bar{X}$.

- In this section, let I be a zero dimensional ideal in $K[\bar{X}]$.

- Let $V_R(I) = \{\bar{c} \in R^n | \forall f \in I \ f(\bar{c}) = 0\}, V_C(I) = \{\bar{c} \in C^n | \forall f \in I \ f(\bar{c}) = 0\}$.

Real Root Counting Theorem

- Let $v_1, \ldots, v_d$ be the basis of the residue class ring $A = K[\bar{X}]/I$.
- For $p \in A$ and each i, j $(1 \leqslant i, j \leqslant d)$, we consider the followings:
  - Let $Q_{p,i,j}$ be the trace of a linear map $A \to A$ by $f \mapsto pv_iv_ja$ for $a \in A$.
  - Let M_p^I be a symmetric matrix $(M_p^I)_{(i,j)} = Q_{p,i,j}$.
  - The signature of M_p^I is denoted $\sigma(M_p^I)$.

Pedersen $\sigma(M_p^I) = \#(\{\bar{c} \in V_R(I) | p(\bar{c}) > 0\}) - \#(\{\bar{c} \in V_R(I) | p(\bar{c}) < 0\})$.

Corollary $\sigma(M_1^I) = \#(V_R(I))$.

Remark We can compute $\sigma(M_p^I)$
by computing the number of the sign cheanges
of the coefficients of the chracteristic polynomial of M_p^I.

In CGS-QE, by using the obvious equivalent relations

$$“p > 0 \Leftrightarrow \exists z\ z^2 p = 1”$$

and

$$“q \neq 0 \Leftrightarrow \exists w\ wq = 1”$$

we reduce “the degree of a charateristic polynomial”.

Real Root Counting Theorem

- Let $p_1, \ldots, p_s \in K[\bar{X}]$ and $\bar{Z}$ be new variables $Z_1, \ldots, Z_s$.
- Let $J = I + \langle Z_1^2 p_1 - 1, \ldots, Z_s^2 p_s - 1 \rangle$ be an ideal in $K[\bar{X}, \bar{Z}]$.

Corollary $\#(V_R(J)) = 2^s \#(\{\bar{c} \in V_R(I) | p_1(\bar{c}) > 0, \ldots, p_s(\bar{c}) > 0\})$.

- Let I' be the elimination ideal $J \cap K[\bar{X}]$.

Corollary $I' = \langle 1 \rangle \vee p_i$ is invertible in $K[\bar{X}]/I'$ for $i = 1, \ldots, s$.

- We assume that p_i has the inverse p_i' in $K[\bar{X}]/I'$ for $i = 1, \ldots, s$.

Corollary $J = I' + \langle Z_1^2 - p_1', \ldots, Z_s^2 - p_s' \rangle$.

Real Root Counting Theorem

- Let $J = I + \langle Z_1^2 - p_1, \ldots, Z_s^2 - p_s \rangle$ with $p_1, \ldots, p_s \in K[\bar{X}]$.

- Let $B_I = \{t_1, \ldots, t_k\} \subset T(\bar{X})$ be a basis of $K[\bar{X}]/I$ and
  $B_J = \{t_1 Z_1^{e_1} Z_2^{e_2} \cdots Z_s^{e_s}, \ldots, t_k Z_1^{e_1} Z_2^{e_2} \cdots Z_s^{e_s} | (e_1, e_2, \ldots, e_s) \in \{0,1\}^s\}$.

Then B_J forms a basis of $K[\bar{X}, \bar{Z}]/J$.

- For $g \in K[\bar{X}]$, we consider the followings:
  - M_g^J denote a symmetric matrix such as
    the matrix of **Pedersen** for J, g
    and χ_g^J its characteristic polynomial.
  - We consider also M_g^I and χ_g^I simiraly as M_g^J and χ_g^J.

Theorem $\chi_g^J(2^s X) = c \prod_{(e_1, e_2, \ldots, e_s) \in \{0,1\}^s} \chi^I_{g p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}}(X)$ (a non-zero constant c).

($\because$ See the proceedings)

Charateristic Polynomial

Example We consider $I = \langle (x_1^2 - x_2^2)(x_1 + 2x_2 - 1), (3x_1 + x_2 - 1)^2 \rangle$, $p_1 = x_1 - x_2$ and $p_2 = x_1 + x_2$. Let $J = I + \langle z_1^2 p_1 - 1, z_2^2 p_2 - 1 \rangle$, $I' = J \cap \mathbb{Q}[x_1, x_2]$ and $\succ$ be a term order such that $z_1 \succ z_2 \succ x_1 \succ x_2$.

- $\{25x_2^2 - 20x_2 + 4, x_1 + 2x_2 - 1, 9z_2^2 - 25x_2 - 5, z_1^2 - 75x_2 + 35\}$ is a Gröbner basis of J w.r.t. $\succ$.
- $I' = \langle 25x_2^2 - 20x_2 + 4, x_1 + 2x_2 - 1 \rangle$.
- Let $p_1' = 15Y - 7, p_2' = 5Y + 1$.
- $\chi^{I}_{p_1^2 p_2^2}(X)\chi^{I}_{p_1 p_2^2}(X)\chi^{I}_{p_1^2 p_2}(X)\chi^{I}_{p_1 p_2}(X)$ has a degree 24, whereas $\chi^{I'}_{1}(X)\chi^{I'}_{p_1'}(X)\chi^{I'}_{p_2'}(X)\chi^{I'}_{p_1' p_2'}(X)$ has a degree 8.
- The original CGS-QE computes $\chi^{I}_{p_1^2 p_2^2}(X)\chi^{I}_{p_1 p_2^2}(X)\chi^{I}_{p_1^2 p_2}(X)\chi^{I}_{p_1 p_2}(X)$.

Charateristic Polynomial

- We compute the saturation ideal I' of I w.r.t. polynomials $p_1, \ldots, p_s$.
- The dimension of $K[\bar{x}]/I'$ is smaller than it of $K[\bar{x}]/I$.
- We can reduce the degree of our charateristic polynomial.
- By using a primary decomposition of I,

  we can certainly remove the unnecessary portion from I.
- For parametric polynomial ideals,

  this computation or even factorization of a polynomial

  becomes a significantly heavy computation.
- Using the relation $q \neq 0 \Leftrightarrow \exists W \ Wq = 1$,

  we further can reduce the degree of a charateristic polynomial.

Comprehensive Gröbner System

Notations

- Let $\bar{X}$ be main variables $X_1, \ldots, X_n$.
- Let $\bar{Y}$ be parameters $Y_1, \ldots, Y_m$.
- Given a term order, $LM(f)$, $LT(f)$, $LC(f)$ denotes the leading monomial, the leading term, the leading coefficient of a polynomial f, respectively.

Algebraic Partition

Algebraic Partition Let S be a subset of an affine space C^n for some natural number n. A finite set $\{\mathcal{S}_1, \ldots, \mathcal{S}_t\}$ of non-empty subsets of S is called an algebraic partition of S if it satisfies the properties **1**, **2**, **3**:

1 $\cup_{i=1}^{t} \mathcal{S}_i = S$.

2 $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ if $i \neq j$.

3 For each i, $\mathcal{S}_i = V_C(I_1) \backslash V_C(I_2)$ for some ideals I_1, I_2 of $K[\bar{Y}]$.

- Each $\mathcal{S}_i$ is called a segment.
- We identify each $\mathcal{S}_i$ with its defining formula.

Comprehensive Gröbner System

- Let S be a subset of C^n and $\succ$ be a term order on $T(\bar{X})$.

CGS For finite $F \subset K[\bar{Y}, \bar{X}]$, a finite set $\mathcal{G} = \{(\mathcal{S}_1, G_1), \ldots, (\mathcal{S}_s, G_s)\}$ satisfying the properties **1**, **2**, **3**, **4** is called a CGS of F over S with parameters $\bar{Y}$ w.r.t. $\succ$:

1 Each G_i is a finite subset of $K[\bar{Y}, \bar{X}]$.

2 $\{\mathcal{S}_1, \ldots, \mathcal{S}_s\}$ is an algebraic partition of S.

3 For each $\bar{c} \in \mathcal{S}_i$, $G_i(\bar{c}, \bar{X}) = \{g(\bar{c}, \bar{X}) | g(\bar{Y}, \bar{X}) \in G_i\}$ is a Gröbner basis of the ideal $\langle \{f(\bar{c}, \bar{X}) | f(\bar{Y}, \bar{X}) \in F\} \rangle$ in $C[\bar{X}]$ w.r.t. $\succ$.

4 For each $\bar{c} \in \mathcal{S}_i$, $LC(g)(\bar{c}) \neq 0$ for any element g of G_i.

Main Algorithm

Notations 1

- In this section, we assume that ϕ forms of a formula
  $$(\bigwedge_{1\leqslant i\leqslant r} f_i = 0) \wedge (\bigwedge_{1\leqslant i\leqslant s} p_i > 0) \wedge (\bigwedge_{1\leqslant i\leqslant t} q_i \neq 0)$$
  ,where $\bar{X} = X_1, \ldots, X_n$, $\bar{Y} = Y_1, \ldots, Y_m$, $f_i, p_i, q_i \in \mathbb{Q}[\bar{Y}, \bar{X}]$, $f_i, p_i, q_i \notin \mathbb{Q}[\bar{Y}]$.

- **Free**$(\psi, \bar{X})$ and **NonFree**$(\psi, \bar{X})$ denote
  the free part and non-free part of ψ w.r.t. the variables $\bar{X}$.

- For an element $(\mathcal{S}, G)$ of a CGS $\mathcal{G}$
  w.r.t. a term order $\succ$ with main variables $\bar{X}$,
  MaxIndVar$(\bar{X}, G, \succ)$ denotes some maximal independent set
  among $\bar{X}$ w.r.t. an ideal $\langle G(\bar{c})\rangle$ for $\bar{c} \in \mathcal{S}$.

Notations 2

- Let M be a real symmetric matrix, $\chi(X)$ be its characteristic polynomial.
  - We assume that $\chi(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0$.
  - We assume that $\chi(-X) = (-1)^d X^d + b_{d-1}X^{d-1} + \cdots + b_0$.

  Remark $b_i = a_i$ if i is even, $b_i = -a_i$ if i is odd.

  - S_+ denotes $\#$(sign changes in $(1, a_{d-1}, \ldots, a_0)$)
  - S_- denotes $\#$(sign changes in $(-1)^d, b_{d-1}, \ldots, b_0$).

  Remark $S_+ = \#(\{c \in \mathbb{R} | c > 0 \wedge \chi(c) = 0\})$,
  $S_- = \#(\{c \in \mathbb{R} | c < 0 \wedge \chi_(c) = 0\})$.

Notations 3

- Let I be a zero dimensional ideal in a polynomial ring over $\mathbb{Q}$.
  Using the same notations as in **Real Roots Counting Theorem**,
  let S_+ and S_- be defined from M_1^I as in **Notations 2**.

Remark $\#(V_{\mathbb{R}}(I)) = \sigma(M_1^I) > 0 \Leftrightarrow S_+ \nVdash S_-$.

- We can write $S_+ \nVdash S_-$ as a quantifier free first order formula.
  We denote such a formula by $I_d(a_0, \ldots, a_{d-1})$.

Main Algorithm

Algorithm MainQE

Input: a basic quantified formula $\exists \bar{X}\phi$ $\{\phi \equiv (\bigwedge_{1\leqslant i\leqslant r} f_i = 0) \wedge (\bigwedge_{1\leqslant i\leqslant s} p_i > 0) \wedge (\bigwedge_{1\leqslant i\leqslant t} q_i \neq 0).\}$;

Output: an equivalent quantifier free formula ψ;

{In **MainQE**, we consider the dimension of a ideal generated by polynomials consiting of equalities.}

$\bar{Z} = Z_1, \ldots, Z_s, \bar{W} = W_1 \ldots, W_t \leftarrow$ new variables;
$\succ \leftarrow$ a term order of $T(\bar{X}, \bar{Z}, \bar{W})$ such that $\bar{Z}, \bar{W} \gg \bar{X}$;
$F \leftarrow \{f_1, \ldots, f_r, Z_1^2 p_1 - 1, \ldots, Z_s^2 p_s - 1, W_1 q_1 - 1, \ldots, W_t q_t - 1\}$;
$\mathcal{G} \leftarrow$ a CGS of F w.r.t. $\succ$ with parameters $\bar{Y}$; $\quad \psi \leftarrow$ *false*;
for $(\mathcal{S}, G) \in \mathcal{G}$ **do**
    if $G(\bar{c}, \bar{X}, \bar{Z}, \bar{W})$ is $\{0\}$ for $\bar{c} \in \mathcal{S}$ **then**
        $\psi \leftarrow \psi \vee S$; $\{(G \cap \mathbb{R}[\bar{X}])(\bar{c}, \bar{X}) = \{0\}.\}$
    else if $\langle G(\bar{c}, \bar{X}, \bar{Z}, \bar{W})\rangle$ is zero dimensional for $\bar{c} \in \mathcal{S}$ **then**
        $\psi \leftarrow \psi \vee$ **ZeroDimQE**$(\mathcal{S}, G, \succ)$; $\{(G \cap \mathbb{R}[\bar{X}])(\bar{c}, \bar{X})$ is also zero dimensional.$\}$
    else
        $\psi \leftarrow \psi \vee$ **NonZeroDimQE**$(\phi, \mathcal{S}, G, \succ)$; $\{(G \cap \mathbb{R}[\bar{X}])(\bar{c}, \bar{X})$ is not also zero dimensional.$\}$
    end if
end for
return ψ;

Main Algorithm

Algorithm ZeroDimQE

Input: a component $(\mathcal{S}, G)$ of a CGS w.r.t. a term order $\succ$ of $T(\bar{X}, \bar{Z}, \bar{W})$ produced in **MainQE** s.t. $\langle G(\bar{c}, \bar{X}, \bar{Z}, \bar{W})\rangle$ is zero dimensional for $\bar{c} \in \mathcal{S}$;

Output: a quantifier free formula ψ s.t. $\mathcal{S} \wedge \exists \bar{X}\phi \Leftrightarrow \psi$;

{In **ZeroDimQE**, we use Real Root Counting Theorem.}

if $\langle G(\bar{c}, \bar{X}, \bar{Y}, \bar{Z})\rangle$ is $\langle 1\rangle$ for $\bar{c} \in \mathcal{S}$ **then**

  return *false*;

else

  $I \leftarrow \langle f_1', \ldots, f_{r'}'\rangle$;

  {G has a form $\{f_i, u_j Z_j^2 - p_j', v_k W_k - q_k' | 1 \leqslant i \leqslant r', 1 \leqslant j \leqslant s, 1 \leqslant k \leqslant t\}$ for $f_i', p_j', q_k' \in \mathbb{Q}[\bar{Y}, \bar{X}]$, $u_i, v_i \in \mathbb{Q}[\bar{Y}]$. Consider $\bar{Y}$ as parameters in the following.}

  $\chi(X) \leftarrow \prod_{(e_1, e_2, \ldots, e_s) \in \{0,1\}^s} \chi^I_{h_1^{e_1} h_2^{e_2} \cdots h_s^{e_s}}(X)$ with $h_i = p_i'/u_i$ for $i = 1, \ldots, s$;

  {For the construction of symmetric matrices, we need to use rational functions $\mathbb{Q}(\bar{Y})$. Let $\chi(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0$ for $a_{d-1}, \ldots, a_0 \in \mathbb{Q}(\bar{Y})$.}

  return $\mathcal{S} \wedge I_d(a_0, \ldots, a_{d-1})$;

  {Note also that we can easily transform the formula $I_d(a_0, \ldots, a_{d-1})$ into a formula using only polynomials. Reducing the degree, we can get a more simplified QE formula.}

end if

Main Algorithm

Algorithm NonZeroDimQE

Input: a basic quantified formula $\exists \bar{X} \phi$ and a component $(\mathcal{S}, G)$ of a CGS w.r.t. a term order $\succ$ of $T(\bar{X}, \bar{Z}, \bar{W})$ produced in **MainQE**;

Output: a quantifier free formula ψ s.t. $\mathcal{S} \wedge \exists \bar{X} \phi \Leftrightarrow \psi$;

{In **NonZeroDimQE**, we use recursive computations.}

$\bar{U} \leftarrow$ **MaxIndVar**$(\bar{X}, G, \succ)$; $\bar{X}' \leftarrow \bar{X} \backslash \bar{U}$; {We consider $\bar{X}'$ as new quantified variables.}

if $\bar{X}' = \emptyset$ **then**

    return **OtherQE**$(\mathcal{S} \wedge \exists \bar{X} \phi)$;

    {The equalitional constraints are all vanish. Then we does not use CGS-QE. We use the other QE algorithm}

else

    $\phi_1 \leftarrow$ **Free**$(\phi, \bar{X}')$; $\phi_2 \leftarrow$ **NonFree**$(\phi, \bar{X}')$; $\varphi \leftarrow \phi_1 \wedge$ **MainQE**$(\exists \bar{X}' \phi_2)$;

    {Let $\varphi_1 \vee \cdots \vee \varphi_l$ be a disjunctive normal form of φ.}

    for $1 \leqslant i \leqslant l$ **do**

        $\varphi_i^1 \leftarrow$ **Free**$(\varphi_i, \bar{U})$; $\varphi_i^2 \leftarrow$ **NonFree**$(\varphi_i, \bar{U})$; $\psi_i \leftarrow \varphi_i^1 \wedge$ **MainQE**$(\exists \bar{U} \varphi_i^2)$;

    end for

    $\psi \leftarrow \mathcal{S} \wedge (\psi_1 \vee \cdots \vee \psi_l)$;

    return ψ; {As long as the equalitional constraints with quantifiers exists, CGS-QE do not use CAD, etc.}

end if

Computation Data

Computation Data

- We implemeted our CGS-QE algorithm using "SyNRAC" on "Maple".
- We draw a comparison between the followings and our package(**O**).

  SN SyNRAC@Maple: It is implemented CAD-QE, VS-QE.

  Reg RegularChains@Maple: It is implemented CAD-QE by regular chains.

  Res Resolve@Mathematica: It is implemented CAD-QE, VS-QE.

  Red Reduce@Mathematica: It is implemented CAD-QE, VS-QE.

  QC QEPCAD: It is implemented CAD-QE.

  hqe rlhqe@RedLog@Reduce: It is implemented CGS-QE.

  rqe rlqe@RedLog@Reduce: It is implemented CAD-QE, VS-QE.

- All the computations were done by the computer environment with an Intel CORE i7 CPU 2.40 GHz with 64 GB memory OS Ubuntu14.04.
- We show a part of our computation data.

Computation Data

1 $\exists c_2 \exists s_2 \exists c_1 \exists s_1 (r - c_1 + l(s_1 s_2 - c_1 c_2) = 0 \wedge z - s_1 - l(s_1 c_2 + s_2 c_1) = 0 \wedge s_1^2 + c_1^2 - 1 = 0 \wedge s_2^2 + c_2^2 - 1 = 0)$

2 $\exists x \exists y \exists z (xy + axz + yz - 1 = 0 \wedge xyz + xz + xy = a \wedge xz + yz - az - x - y - 1 = 0 \wedge axy = byz \wedge ayz = bzx)$

3 $\exists x \exists y \exists z (xy + axz + yz - 1 = 0 \wedge xyz + xz + xy = b \wedge xz + yz - az - x - y - 1 = 0)$

4 $\exists x_0 \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \exists x_7$

$$\sqrt{(x_7 - x_0)^2 + (x_6 - x_2)^2} = 1 \wedge \frac{|x_0x_3 - x_0x_4 + x_1x_2 - x_1x_3 - x_2x_5 + x_4x_5|}{(x_1 - x_0)(x_5 - x_0) + (x_4 - x_2)(x_3 - x_2)} = m \wedge \frac{|x_0x_3 - x_0x_4 + x_1x_2 - x_1x_3 - x_2x_5 + x_4x_5|}{(x_0 - x_5)(x_1 - x_5) + (x_2 - x_3)(x_4 - x_3)} = n \wedge$$

$$x_0x_3 - x_0x_4 + x_1x_2 - x_1x_3 - x_2x_5 + x_4x_5 \geqslant 0 \wedge (x_5 - x_0)x_4 - x_2 - (x_3 - x_2)x_1 - x_0 \geqslant 0 \wedge$$

$$(x_5 - x_0)(\tfrac{1}{2}x_0 + \tfrac{1}{2}x_5 + x_7) + (x_3 - x_2)(\tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 - x_6) = 0 \wedge$$

$$(x_1 - x_5)(\tfrac{1}{2}x_5 + \tfrac{1}{2}x_1 - x_7) + (x_4 - x_3)(\tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 - x_6) = 0 \wedge$$

$$m \geqslant 3 \wedge n \geqslant 3 \wedge \frac{|x_5 - x_0x_4 - x_2 + x_3 - x_2x_1 - x_0|}{2} = S$$

- The above **4** is the **Example** of Section **Motivation**.

Computation Data

Computing time is written in second.

'0' means that the computation time is within 1 second,

'×' means that the computation does not terminate within 1 hour,

'm' means memory exhaust and

'e' means the computation was crashed with some error.

	O	SN	Reg	Res	Red	QC	hqe	rqe
1	1	1	29	0	×	×	0	×
2	0	e	×	250	×	×	×	×
3	10	×	×	×	×	m	×	e
4	791	×	×	×	×	e	×	×

Conclusion

Conclusion

- Today, I talked the followoings:
  - CGS-QE

    CGS-QE use the followings:
    - Real Root Counting

      (Improving CGS-QE, we modify "Real Root Counting")
    - CGS
  - Computation Data
- Our future work is the simplification of outputs.
  - CGS-QE may return the complicated outputs.

Thank you for your attention!!