ISOTOPY CONVERGENCE THEOREM

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Abstract

When approximating a space curve, it is natural to consider whether the knot type of the original curve is preserved in the approximant. This preservation is of strong contemporary interest in computer graphics and visualization. We establish a criterion to preserve knot type under approximation that relies upon convergence in both distance and total curvature.

Keywords: Knot; Ambient isotopy; Convergence; Total curvature; Visualization.

1 Introduction

Convergence for curve approximation is often in terms of distance, such as in Weierstrass approximation theorem [11]. But an approximation in terms of distance does not necessarily yield ambient isotopic equivalence. However, ambient isotopic equivalence is a fundamental concern in knot theory, and a theoretical foundation for curve approximation algorithms in computer graphics and visualization.

So a natural question is what criterion will guarantee ambient isotopic equivalence for curve approximation? The answer is that, besides convergence in distance, an additional hypothesis of total curvature will be sufficient, that is, convergence in both distance and total curvature.

2 Related Work

The Isotopy Convergence Theorem presented here is motivated by the question about topological integrity of geometric models in computer graphics and visualization. The publications [1, 2, 7, 9] are among the first that provided algorithms to ensure ambient isotopic approximations. The paper [6] provided existence criteria for a PL approximation of a rational spline, but did not include any specific algorithms.

Recent progress was made for the class of Bézier curves, by providing stopping criteria for subdivision algorithms to ensure ambient isotopic equivalence for Bézier curves of any degree n [4], extending the previous work of [9], that had been restricted to degree less than 4.

This work here extends to a much broader class of curves, piecewise C^2 curves, where there is no restriction on approximation algorithms. Because of its generality, this pure mathematical result is potentially applicable to both theoretical and practical areas.

3 Preliminaries

Use $\mathcal{C}$ to denote a compact, regular, C^2, simple, parametric, space curve. Let $\{C_i\}_1^\infty$ denote a sequence of piecewise C^2, parametric curves. Suppose all curves are parametrized on $[0,1]$, that is, $\mathcal{C} = \mathcal{C}(t)$ and $C_i = C_i(t)$ for $t \in [0,1]$. Denote the sub-curve of $\mathcal{C}$ corresponding to $[a,b] \subset [0,1]$ as $\mathcal{C}_{[a,b]}$, and similarly use $C_{i[a,b]}$ for C_i. Denote a total curvature as $T_\kappa(\cdot)$.

The definitions [8] of total curvatures of both PL curves and C^2 curves are standard. These can be naturally extended to define total curvatures of piecewise C^2 curves, for which the concept of exterior angles [8] is needed.

Definition 3.1 (Exterior angles of piecewise C^2 curves) *For a piecewise C^2 curve $\gamma(t)$, define the exterior angle at some t_i to be the angle between two vectors $\gamma'(t_i-)$ and $\gamma'(t_i+)$ where*

$$\gamma'(t_i-) = \lim_{h\to 0} \frac{\gamma(t_i) - \gamma(t_i - h)}{h},$$

and

$$\gamma'(t_i+) = \lim_{h\to 0} \frac{\gamma(t_i + h) - \gamma(t_i)}{h}.$$

Definition 3.2 (Total curvatures of piecewise C^2 curves) *Suppose that a piecewise C^2 curve $\phi(t)$ is not C^2 at finitely many parameters $t_1, \cdots, t_n$. Denote the sum of the total curvatures of all the C^2 sub-curves as $T_{\kappa 1}$, and the sum of exterior angles at $t_1, \cdots, t_n$ as $T_{\kappa 2}$. Then the total curvature of $\phi(t)$ is $T_{\kappa 1} + T_{\kappa 2}$.*

Definition 3.3 *We say that $\{C_i\}_1^\infty$ converges to $\mathcal{C}$ in distance if for any $\epsilon > 0$, there exits an integer N such that $\max_{t\in[0,1]} |C_i(t) - \mathcal{C}(t)| < \epsilon$ for all $i \geq N$.*

Definition 3.4 *We say that $\{C_i\}_1^\infty$ converges to $\mathcal{C}$ in total curvature if for any $\epsilon > 0$, there exits an integer N such that $|T_\kappa(C_i) - T_\kappa(\mathcal{C})| < \epsilon$ for all $i \geq N$. We designate this property as convergence in total curvature.*

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4 Isotopy Convergence

Convergence in distance provides a lower bound of the total curvatures of approximants.

Theorem 1 *If $\{C_i\}_1^\infty$ converges to $\mathcal{C}$ in distance, then for $\forall \epsilon > 0$, there exits an integer N such that $T_\kappa(C_i) > T_\kappa(\mathcal{C}) - \epsilon$ for all $i \geq N$.*

Theorem 2 (Isotopy Convergence) *If $\{C_i\}_1^\infty$ converges to $\mathcal{C}$ in both distance and total curvature, then there exists an N such that C_i is ambient isotopic to $\mathcal{C}$ for all $i \geq N$.*

Let $\Im$ be a set of *pairwise disjoint* piecewise C^2 curves, (which is a link for closed curves), satisfying the same hypotheses as $\mathcal{C}$. Let $\Im_i$ be a set of piecewise C^2 parametric curves. The corollary bellow follows easily.

Corollary 3 [1] *If the sequence $\{\Im_i\}_1^\infty$ converges to $\Im$ in both distance and total curvature, then there exists an integer N such that $\Im_i$ is ambient isotopic to $\Im$ for all $i \geq N$.*

4.1 A representative example of offset curves

Offset curves are defined as locus of the points which are at constant distant along the normal from the generator curves [5]. They are widely used in various applications, and the related approximation problems were frequently studied [5]. It is well-known [10, p. 553] that offsets of spline curves need not be splines. Here we show a representative example as a catalyst to ambient isotopic approximations of offset curves.

Let $\mathcal{C}(t)$ be a compact, regular, C^2, simple, space curve parametrized in $[a, b]$, whose curvature κ never equals 1. Then define an offset curve by

$$\Omega(t) = \mathcal{C}(t) + N(t),$$

where $N(t)$ is the normal vector at t, for $t \in [a, b]$.

For example, let $\mathcal{C}(t) = (2\cos t, 2\sin t, t)$ for $t \in [0, 2\pi]$ be a helix, then it is an easy exercise for the reader to verify that the above assumptions of $\mathcal{C}$ are satisfied, with $\kappa = \frac{2}{5}$. Furthermore, it is straightforward to obtain the offset curve $\Omega(t) = (\cos t, \sin t, t)$, which is not a spline.

We first show that $\Omega(t)$ is regular. Let $s(t) = \int_a^t |\mathcal{C}'(t)| dt$ be the arc-length of $\mathcal{C}$. Then by Frenet-Serret formulas [3] we have

$$\Omega'(t) = \mathcal{C}'(t) + N'(t) = (1-\kappa)\frac{ds}{dt}T + \tau\frac{ds}{dt}B,$$

where T and B are the unit tangent vector and binormal vector respectively. Since $T \perp B$, if $(1-\kappa)\frac{ds}{dt} \neq 0$ then $\Omega'(t) \neq 0$. But $(1-\kappa)\frac{ds}{dt} \neq 0$ because $\kappa \neq 1$ and $\mathcal{C}(t)$ is regular by the assumption. Thus $\Omega(t)$ is regular.

Now we define a sequence $\{\Omega_i(t)\}_{i=1}^\infty$ to approximate $\Omega(t)$ by setting

$$\Omega_i(t) = \mathcal{C}(t) + \frac{i-1}{i}N(t).$$

It is obvious that $\{\Omega_i(t)\}_{i=1}^\infty$ converges in distance to $\Omega(t)$. For the convergence in total curvature, note that $\lim_{i\to\infty} \Omega_i'(t) = \Omega'(t)$ and $\lim_{i\to\infty} \Omega_i''(t) = \Omega''(t)$. It follows that

$$\lim_{i\to\infty} \Omega_i'(t) \times \Omega_i''(t) = \Omega'(t) \times \Omega''(t).$$

Since $|\Omega'(t)| \neq 0$ due to the regularity of $\Omega(t)$. Therefore

$$\lim_{i\to\infty} \frac{\Omega_i'(t) \times \Omega_i''(t)}{|\Omega_i'(t)|^3} = \frac{\Omega'(t) \times \Omega''(t)}{|\Omega'(t)|^3}.$$

This implies that, at each t, the curvature in the sequence converges to the curvature of $\Omega(t)$. Then the convergence in total curvature follows.

By the Isotopy Convergence Theorem (Theorem 2), we conclude that there exists a positive integer N such that $\Omega_i(t)$ is ambient isotopic to $\Omega(t)$ whenever $i > N$.

5 Conclusion

We derived the Isotopy Convergence Theorem as motivated by applications for knot theory, computer graphics, visualization and simulations. Future research directions may include using the Isotopy Convergence Theorem in knot classification and discovering applications in the area of computational topology.

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