Supply Chain Network Capacity Competition with Outsourcing: A Variational Equilibrium Framework

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Abstract This paper develops a supply chain network game theory framework with multiple manufacturers/producers, with multiple manufacturing plants, who own distribution centers and distribute their products, which are distinguished by brands, to demand markets, while maximizing profits and competing noncooperatively. The manufacturers also may avail themselves of external distribution centers for storing their products and freight service provision. The manufacturers have capacities associated with their supply chain network links and the external distribution centers also have capacitated storage and distribution capacities for their links, which are shared among the manufacturers and competed for. We utilize a special case of the Generalized Nash Equilibrium problem, known as a variational equilibrium, in order to formulate and solve the problem. A case study on apple farmers in Massachusetts is provided with various scenarios, including a supply chain disruption, to illustrate the modeling and methodological framework as well as the potential benefits of outsourcing in this sector.

Keywords: Generalized Nash Equilibrium, game theory, supply chains, capacity competition, outsourcing, variational inequalities, networks

1. Introduction

The logistics landscape, from warehousing to distribution, underpinning supply chains is dealing with increased competition and tightened capacity along with increasing consolidation (cf. Langley Jr. (2015)) with 29% of shippers in a recent survey noting that they have engaged with a larger number of third party logistics providers to get access to gain capacity. For example, according to Phillips (2015), with a strong dollar and consumption in the U.S. improving, more products are entering the U.S. and vacancy rates for industrial real estate, consisting primarily of warehouse properties, are at historic lows. In some parts of Southern California, where much of the country's containerized cargo lands from Asia, industrial real estate vacancy has stayed below 5% in 2015. In addition, shortages of warehouse workers as well as truckers are posing further challenges and adding to the competition. 73% of shippers noted that they increased their use of outsourced logistics services in 2015, as compared to a figure of 68% in the previous year, with 35% of shippers indicating that they are returning to insourcing many of their logistics activities, with this figure being higher than the 26% reported the year previously (cf. Langley Jr. (2015)).

The new competitive landscape has affected different industrial sectors, including healthcare, automotive, and food. For example, UPS now provides various healthcare companies with comprehensive third-party logistics services so as to improve the efficiency of their supply chains. UPS has recently built several healthcare logistics hubs in Asia and the Pacific in order to catch up with the rapid growth in the demand for pharmaceuticals in this part of the globe (cf. Pharmaceutical Commerce (2013)). In addition, UPS, Fedex, and DHL are also making significant investments in healthcare logistics in the U.S. given the aging population and also demands put on requirements for the handling and transport of pharmaceuticals, which can be perishable (Brennan and Golden (2015)). Another example consists of British Petroleum (BP), Chevron Corp., Atlas Supply, and GATX. In order to focus on their core competencies, BP and Chevron formed Atlas Supply for the supply and delivery of auto parts to their $6,500$ service stations. Atlas decided to outsource all the logistics to GATX, which took responsibility to run five distribution centers and maintain inventory at each service station (Andel (1995) and Simchi-Levi, Kaminsky, and Simchi-Levi (2000)). According to the World Economic Forum (2016), two competitors, Nestle and PepsiCo, are sharing warehouse capabilities, in the form of storage, packing operations, and the distribution of fresh and chilled food products destined for their retail customers in Belgium and Luxembourg.

In the consumer goods sector Inbound Logistics (2010) reported that manufacturers sometimes share truck and warehouse space where similar loads are destined for the same store or retailer warehouse. As a result, time and money may be saved. Shippers and carriers are able to better justify transportation moves and costs, end customers can better allocate labor for unloading full truckloads, and the energy, pollution, and congestion generated by transportation can all be significantly reduced. For example, Kimberly-Clark Corporation has been very innovative in sharing warehouses as well as freight service provision with multiple different companies, including Unilever and Kellogg, in several European countries (Cooke (2011)) with results of cost reduction and improvement in customer service.

Clearly, firms in supply chains have difficult decisions to make and must optimize within their own supply chain network capacities and also as they compete for shared capacities of third party logistics providers for both distribution center space as well as freight service provision to their demand markets. In this paper, hence, we develop a competitive supply chain network model consisting of multiple firms involved in the manufacture/production of a similar, substitutable, product distinguished by each firm's brand. The firms have available to them their manufacturing plants and distribution centers, and supply the same demand points, which can correspond to retailers. Each firm has a capacity associated with its supply chain network economic activities of production, transportation, storage, and distribution. In addition, the firms may avail themselves of external distribution enters, to which they can outsource any or all of the storage of their products and also the ultimate delivery to the demand markets. With the external distribution center storage links and freight service provision links there are also associated capacities and the firms compete for storage and freight service provision.

The model that we develop, because of the shared or "coupling" constraints, is a Generalized Nash Equilibrium (GNE) model. Although the Nash (1950, 1951) equilibrium concept of noncooperative behavior has stimulated a wide spectrum of supply chain network models, including supply chain network equilibrium models (cf. Nagurney, Dong, and Zhang (2002), Nagurney (2006), Qiang et al. (2013), Toyasaki, Daniele, and Wakolbinger (2014), and the references therein), as well as models in which supply chain competition among vertically integrated firms is captured (see, e.g., Masoumi, Yu, and Nagurney (2012), Yu and Nagurney (2013), Nagurney, Yu, and Floden (2013, 2015)), there has been only limited development of GNE models for supply chain networks. In Nash equilibrium problems, the strategies of players, that is, decision-makers in the noncooperative game, affect the utility functions of the others, but the feasible set of each player only depends on his/her strategies. In contrast, in a Generalized Nash Equilibrium game, the strategies of decision-makers and, hence, their feasible sets, also depend on the strategies played by the other decision-makers. Nash equilibrium problems can be formulated as variational inequality problems (Gabay and

Moulin (1980)) whereas Generalized Nash equilibrium problems are, typically, formulated as quasi-variational inequality problems. The state of the art of the theory, algorithms, and applications is more advanced for the former problems (cf. Nagurney (1999)) than for the latter (see, e.g., Fischer, Herrich, and Schonefeld (2014)). This may be a reason for the dearth of supply chain models formulated as GNE problems.

As noted in Nagurney, Alvarez Flores, and Soylu (2016), the Generalized Nash Equilibrium problem dates to Debreu (1952) and Arrow and Debreu (1954), although it was not termed as such. Rosen (1965) provided a formal definition of a normalized Nash equilibrium, provided qualitative properties, and proposed an algorithm. Bensoussan (1974) formulated the GNE problem as a quasi-variational inequality. For background on the GNE problem, we refer the interested reader to von Heusinger (2009) and the recent review by Fischer, Herrich, and Schonefeld (2014). For possible recent approaches to solving GNE problems based on global optimization see Aguiar e Oliveira Jr. and Petraglia (2016).

Nagurney, Alvarez Flores, and Soylu (2016) focused on post-disaster humanitarian relief and constructed an integrated network model in which disaster relief NGOs compete for financial funds from donors while also deriving utility from providing relief through their supply chains to multiple points of demand. The shared constraints consisted not of capacities on the links, as is the case for the model developed in this paper, but, rather, of lower and upper bounds for relief supplies at demand points in order to ensure that needs of the victims are met but not at the expense of material convergence and oversupply. The GNE model was of a structure that enabled its reformulation as an optimization problem, based on the elegant work of Li and Lin (2013), who also proposed an oligopoly model with capacities and differentiated products and then solved a duopoly problem with linear underlying demand price and cost functions. The GNE supply chain network model in this paper does not have a structure amenable to reformulation as an optimization problem as in Li and Lin (2013). Nevertheless, we make use of a *variational equilibrium* (cf. Facchinei and Kanzow (2010), Kulkarni and Shanbhag (2012)), which is a specific kind of GNE. The variational equilibrium allows for alternative variational inequality formulations of our supply chain network Generalized Nash Equilibrium model with capacity competition and outsourcing. What is notable about a variational equilibrium (see also Luna (2013)) is that the Lagrange multipliers associated with the shared or coupling constraints associated the the external distribution centers and subsequent freight service provision are the same for all players in the game. This also has a nice economic and equity interpretation.

Although there are few Generalized Nash Equilibrium models for supply chain networks, multiple GNE models have been constructed for the energy sector (see, e.g., Contreras, Klusch, and Krawczyk (2004), Krawczyk (2005), and the references therein). In addition, there is very interesting recent research in service provisioning in cloud systems using Generalized Nash Equilibrium models (see Ardagna, Panicucci, and Passcantando (2013) and Passacantando, Ardagna, and Savi (2016)). Jiang and Pang (2011) focused on network capacity competition in the airline industry using a Generalized Nash Equilibrium approach. Ang et al. (2013) proposed a novel supply chain model with multiple suppliers and a single manufacturer, which is a bilevel game in which suppliers' frequencies of delivery are captured. The authors formulated the problem as a GNE problem, provided qualitative properties, and considered the case that can be converted and solved as a variational inequality problem. Li and Nagurney (2017) developed a multitiered supply chain network game theory model with suppliers, manufacturers, and demand markets and also provided metrics for performance assessment of supply chains. They formulated the model, which includes capacities faced by suppliers and manufacturers, as a variational inequality problem. As in Ang et al. (2013), in this paper, we are concerned with the general mathematical structure of the problem, possible global optimal solutions, and the uniqueness of the solution. For an excellent edited volume on game theory and equilibria, which includes several chapters on supply chain networks, see Chinchuluun et al. (2008).

This paper is organized as follows. In Section 2, we develop the supply chain network Generalized Nash Equilibrium model with capacity and outsourcing and also present several special cases. We define the variational equilibrium and then present several alternative variational inequality formulations. We also discuss some qualitative properties, in particular, we provide existence results. In Section 3, we present the algorithm, which yields, at each iteration, closed form expressions for the product path flows of the firms as well as the Lagrange multipliers associated with the firms' own supply chain networks and the Lagrange multipliers associated with the shared constraints, which are under control of the distribution centers and subsequent freight service providers that the firms can outsource to. In Section 4, we present a case study. We summarize our results and present our conclusions in Section 5.

2. The Supply Chain Network Generalized Nash Equilibrium Model with Capacity Competition and Outsourcing

We consider a finite number of I firms, with a typical firm denoted by i, who are involved in the production, storage, and distribution of a substitutable product and who compete noncooperatively in an oligopolistic manner. The products associated with the firms are differentiated by their brands. Each firm is represented as a network of its economic activities (cf. Figure 1). Each firm i; $i = 1, \ldots, I$, owns n^i_M manufacturing (production) facilities and n^i_D distribution centers. In addition, there are n_{OD} external outsourcers available for the warehousing and the distribution of the products. The I firms compete for and may share space in the n_{OD} external distribution centers, and the same holds for the subsequent freight service provision for distribution to the n_R demand markets. Here, for the sake of generality, we refer to the bottom-tiered nodes in Figure 1 as demand markets. Of course, they may correspond to retailers. Let $G = [N, L]$ denote the graph consisting of the set of nodes N and the set of links L in Figure 1. Each firm seeks to determine its optimal product quantities that maximize its profits by using Figure 1 as a schematic.

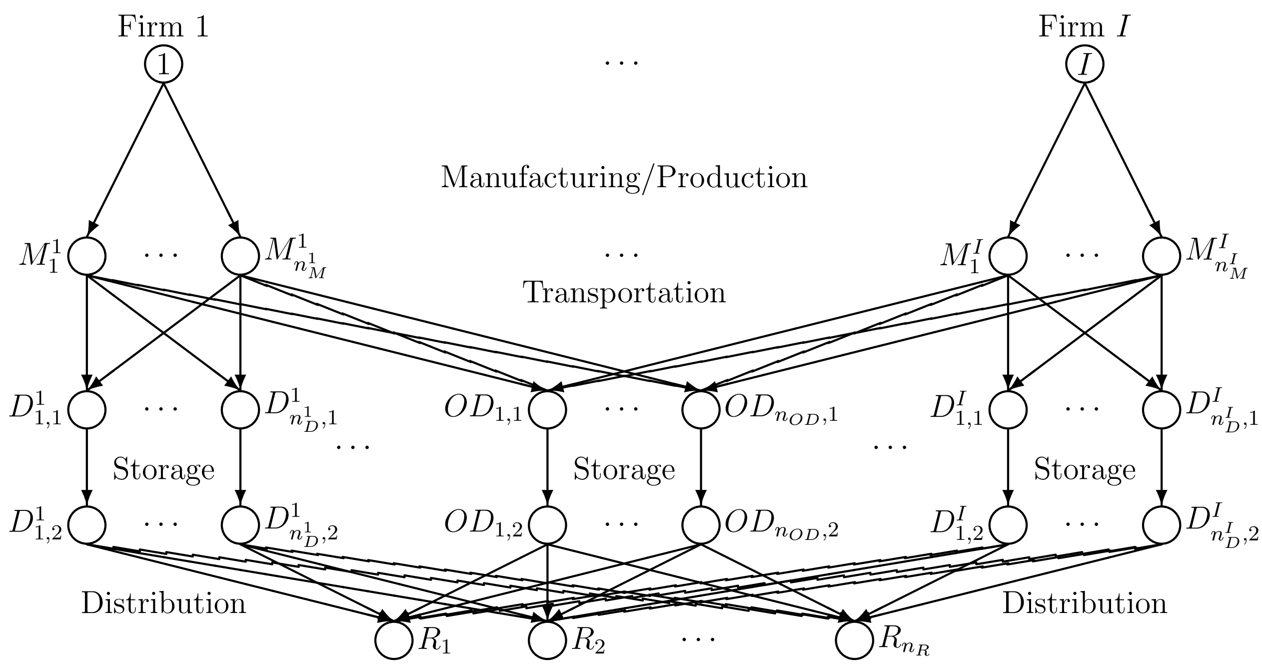

Figure 1: The Supply Chain Network Topology of the Oligopoly with Capacity Competition and Outsourcing

The production links from the top-tiered nodes i; $i = 1, \ldots, I$, representing firm i, in Figure 1 are connected to the production nodes of firm i, which are denoted, respectively, by: $M^i_1, \ldots, M^i_{n^i_M}$. The links from the production nodes, in turn, are connected to the distribution center nodes of each firm i; $i = 1, \ldots, I$, which are denoted by $D^i_{1,1}, \ldots, D^i_{n^i_D,1}$. These links correspond to the in-house transportation links between the production plants and the in-house distribution centers where the product is stored and then distributed to the demand markets. The links joining nodes $D^i_{1,1}, \ldots, D^i_{n^i_D,1}$ with nodes $D^i_{1,2}, \ldots, D^i_{n^i_D,2}$ correspond to the storage links. Finally, there are distribution links joining the nodes $D^i_{1,2}, \ldots, D^i_{n^i_D,2}$ for $i = 1, \ldots, I$ with the demand market nodes: $R_1, \ldots, R_{n_R}$.

In addition, each firm has the option to exploit the external distribution centers. The links joining the production nodes $M^i_1, \ldots, M^i_{n^i_M}$; $i = 1, \ldots, I$, with the external distribution centers $OD_{1,1}, \ldots, OD_{n_{OD},1}$ are transportation links. The links joining nodes $OD_{1,1}, \ldots, OD_{n_{OD},1}$ with nodes $OD_{1,2}, \ldots, OD_{n_{OD},2}$ correspond to the shared storage links that the firms compete for space at. The distribution links from the nodes $OD_{1,2}, \ldots, OD_{n_{OD},2}$ are connected to the demand market nodes: $R_1, \ldots, R_{n_R}$; these links may also be shared by the firms and they correspond to freight service provision. Competition also takes place here since there are capacities not only associated with the shared distribution centers but also with freight provision. Of course, the firms also have capacities associated with their own production, transportation, storage, and distribution links, which are not shared. We discuss the capacity constraints after we present the conservation of flow equations. We then discuss the underlying supply chain network link total cost functions and the demand price functions.

The additional notation for the model is given in Table 1.

The following conservation of flow equations must hold for each firm i: $i = 1, \ldots, I$:

$$\sum_{p \in P^i_k} x^i_p = d_{ik}, \quad \forall k, \tag{1}$$

that is, the demand for each firm's product at each demand market must be satisfied by the product flows from the firm to that demand market.

Moreover, the path flows must be nonnegative; that is, for each firm i; $i = 1, \ldots, I$:

$$x^i_p \geq 0, \quad \forall p \in P^i. \tag{2}$$

Furthermore, the expression that relates the link flows of each firm i; $i = 1, \ldots, I$, to the path flows is given by:

$$f^i_a = \sum_{p \in P} x^i_p \delta_{ap}, \quad \forall a \in L, \tag{3}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Hence, the flow of a firm's product on a link is equal to the sum of that product flows on paths that contain that link.

Table 1: Notation for the Supply Chain Model with Capacity Competition and Outsourcing

Notation	Definition
L^i	the links comprising the supply chain network of firm i; $i = 1, \ldots, I$, that it owns/controls, with a total of n_{L^i} elements. These links include firm i's links to its manufacturing nodes; the links from manufacturing nodes to its distribution centers, its storage links, and the links from its distribution centers to the demand markets as well as the links from its manufacturing nodes to the external distribution centers.
L^S	the links consisting of storage links associated with the external distribution centers and the links associated with freight service provision from the external distribution centers to the demand markets, with a total of n_{L^S} elements. These links can be shared by the I firms, if capacity allows.
L	the full set of links in the supply chain network economy with $L = \cup_{i=1}^{I} L^i \cup L^S$ with a total of n_L elements.
P_k^i	the set of paths in firm i's supply chain network terminating in demand market k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$.
P^i	the set of all n_{P^i} paths of firm i; $i = 1, \ldots, I$.
P	the set of all n_P paths in the supply chain network economy.
$x_p^i; p \in P_k^i$	the nonnegative path flow of firm i's product to demand market k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$. We group firm i's product path flows into the vector $x^i \in R_+^{n_{P^i}}$. We then group all the firms' product path flows into the vector $x \in R_+^{n_P}$.
f_a^i	the nonnegative flow of product i on link a, $\forall a \in L$; $i = 1, \ldots, I$. We group the link flows for each i into the vector $f^i \in R_+^{n_{L^i} + n_{L^S}}$. We then group the vectors f^i; $i = 1 \ldots, I$, into the vector $f \in R_+^{\sum_{i=1}^{I} n_{L^i} + I \times n_{L^S}}$.
d_{ik}	the demand for the product of firm i at demand market k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$. We group the $\{d_{ik}\}$ elements for firm i into the vector $d^i \in R_+^{n_R}$ and all the demands into the vector $d \in R_+^{I \times n_R}$.
u_a^i	the capacity on link $a \in L^i$; $i = 1, \ldots, I$.
u_a	the capacity on link $a \in L^S$.
$\hat{c}_a^i(f)$	the total operational cost associated with link a, $\forall a \in L$ and all firms i; $i = 1, \ldots, I$.
$\rho_{ik}(d)$	the demand price function for the product of firm i at demand market k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$.

In addition, the link flows must satisfy the following capacity constraints. For links corresponding to the individual firm networks L^i; $i = 1, \ldots, I$, we must have that:

$$f_a^i \leq u_a^i, \quad \forall a \in L^i. \tag{4}$$

In other words, the flow on each link associated with a firm's network cannot exceed the capacity of that link.

Also, in the case of the links corresponding to the outsourced storage and distribution, the following capacity constraints must be satisfied:

$$\sum_{i=1}^{I} f_a^i \leq u_a, \quad \forall a \in L^S. \tag{5}$$

Hence, as noted earlier, the links comprising L^S can be shared among the firms. Since the products are substitutable, we can expect them to be of the same size and, therefore, constraints (5) are appropriate.

According to Table 1, the demand price function ρ_{ik}; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$, depends not only on the firm's demand for its product but also, in general, on the demands for the other firms' products. Hence, we also capture competition on the demand side. In view of (1), we may reexpress the demand price function, $\rho_{ik}(d)$, as:

$$\hat{\rho}_{ik} = \hat{\rho}_{ik}(x) \equiv \rho_{ik}(d), \quad \forall i, \forall k. \tag{6}$$

Also, according to Table 1, the total operational cost on link a for product i, $\hat{c}_a^i$, can depend on the flow of the product on that link as well as on the flows of other products on that link and on other links. The generality of the total operational link cost functions captures competition for resources on the individual firms' networks as well as on the shared component of the supply chain network.

We assume that the link total operational cost functions and the demand price functions are all continuously differentiable.

The utility/profit of firm i, U^i; $i = 1, \ldots, I$, is the difference between its revenue and its total costs:

$$U_i = \sum_{k=1}^{n_R} \rho_{ik}(d) d_{ik} - \sum_{a \in L^i \cup L^S} \hat{c}_a^i(f), \tag{7}$$

and the function is assumed to be concave.

Let X_i denote the vector of strategy variables associated with firm i; $i = 1, \ldots, I$, where X_i is the vector of path flows associated with firm i, that is,

$$X_i \equiv \{\{x_p\} | p \in P^i\} \in R_+^{n_{P^i}}. \tag{8}$$

X is then the vector of all firms' strategies, that is, $X \equiv \{\{X_i\}|i = 1, \ldots, I\}$.

Through the use of the conservation of flow equations (1) and (3), and the form of the total operational link cost functions and the demand price functions, we can define $\hat{U}_i(X) \equiv U_i$; $i = 1 \ldots, I$. We group the profits of all the firms into an I-dimensional vector $\hat{U}$, where

$$\hat{U} = \hat{U}(X). \tag{9}$$

Also, observe that, in view of the conservation of flow equations (3), we may rewrite the individual firms' capacity constraints (4) in terms of path flows as:

$$\sum_{p \in P} x_p^i \delta_{ap} \leq u_a^i, \quad \forall a \in L^i, \forall i. \tag{10}$$

Similarly, we may rewrite the shared capacity constraints (5) in terms of path flows such that:

$$\sum_{i=1}^{I} \sum_{p \in P} x_p^i \delta_{ap} \leq u_a, \quad \forall a \in L^S. \tag{11}$$

We now define the each firm i's individual feasible set K_i for $i = 1, \ldots, I$, as:

$$K_i \equiv \{x_p^i \geq 0, \forall p \in P^i \,\text{and}\, (10) \,\text{holds}\}. \tag{12}$$

In addition, we define the feasible set consisting of the shared constraints, $\mathcal{S}$, as:

$$\mathcal{S} \equiv \{x|(11) \,\text{holds}\}. \tag{13}$$

In the competitive oligopolistic market framework, each firm selects its product path flows in a noncooperative manner, seeking to maximize its own profit, until an equilibrium is achieved, according to the definition below.

Definition 1: Supply Chain Network Generalized Nash Equilibrium with Capacity Competition and Outsourcing

A path flow pattern $X^* \in K = \prod_{i=1}^{I} K_i$, $X^* \in \mathcal{S}$, constitutes a supply chain network Generalized Nash Equilibrium if for each firm i; $i = 1, \ldots, I$:

$$\hat{U}_i(X_i^*, \hat{X}_i^*) \geq \hat{U}_i(X_i, \hat{X}_i^*), \quad \forall X_i \in K_i, \forall X \in \mathcal{S}, \tag{14}$$

where $\hat{X}_i^* \equiv (X_1^*, \ldots, X_{i-1}^*, X_{i+1}^*, \ldots, X_I^*)$.

Hence, an equilibrium is established if no firm can unilaterally improve its profit by changing its product flows in the supply chain network, given the product flow decisions of the other firms, and subject to the capacity constraints, both individual and shared/coupling ones. We remark that both K and $\mathcal{S}$ are convex sets.

If there are no coupling, that is, shared, constraints in the above model, then X and X^* in Definition 1 need only lie in the set K, and, under the assumption of concavity of the utility functions and that they are continuously differentiable, we know that (cf. Gabay and Moulin (1980) and Nagurney (1999)) the solution to what would then be a Nash equilibrium problem (see Nash (1950, 1951)) would coincide with the solution of the following variational inequality problem: determine $X^* \in K$, such that

$$-\sum_{i=1}^{I} \langle \nabla_{X_i} \hat{U}_i(X^*), X_i - X_i^* \rangle \geq 0, \quad \forall X \in K, \tag{15}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the corresponding Euclidean space and $\nabla_{X_i} \hat{U}_i(X)$ denotes the gradient of $\hat{U}_i(X)$ with respect to X_i.

In our game theory supply chain network model, however, the strategies of the "players," which are the firms, affect not only the values of the others' objective functions, which are the profit functions, but also the strategies of the firms affect the other firms' strategies because of the shared constraints. These are sometimes also referred to as "coupling" constraints. Hence, although Nash equilibrium problems can be formulated as variational inequality problems, Generalized Nash Equilibrium problems can no longer directly be formulated as variational inequality problems, but, instead, are formulated as quasi-variational inequalities (see, e.g., Facchinei and Kanzow (2010)). However, it is well-known (cf. Luna (2013) and the references therein) that quasi-variational inequality problems are much harder to solve.

A refinement of the Generalized Nash Equilibrium (GNE) is what is known as a *variational equilibrium* and it is a specific type of GNE (see Kulkarni and Shabhang (2012)). In particular, in a GNE defined by a variational equilibrium, the Lagrange multipliers associated with the coupling constraints are all the same. This, in a sense, has a fairness interpretation and is reasonable from an economic standpoint. Specifically, we have the following definition:

Definition 2: Variational Equilibrium

A strategy vector X^* is said to be a variational equilibrium of the above Generalized Nash

Equilibrium game if $X^* \in K, X^* \in \mathcal{S}$ is a solution of the variational inequality:

$$-\sum_{i=1}^{I} \langle \nabla_{X_i} \hat{U}_i(X^*), X_i - X_i^* \rangle \geq 0, \quad \forall X \in K, \forall X \in \mathcal{S}. \tag{16}$$

By utilizing a variational equilibrium, we can take advantage of the well-developed theory of variational inequalities, including algorithms (cf. Nagurney (1999) and the references therein), which are in a more advanced state of development and application than algorithms for quasi-variational inequality problems.

We now expand the terms in variational inequality (16).

Specifically, by definition, we have that

$$-\nabla_{X_i} \hat{U}_i(X) = \left[-\frac{\partial \hat{U}_i}{\partial x_p^i}; p \in P_k^i; k = 1, \ldots, n_R \right]. \tag{17}$$

We also know that, in view of (1) and (7), for paths $p \in P_k^i$:

$$-\frac{\partial \hat{U}_i}{\partial x_p^i} = -\frac{\partial (\sum_{l=1}^{n_R} \rho_{il}(d) \sum_{q \in P_l^i} x_q^i - \sum_{b \in L^i \cup L^S} \hat{c}_b^i(f))}{\partial x_p^i}. \tag{18}$$

Making use of (1) and (3) and the expressions:

$$\frac{\partial \hat{C}_p^i(x)}{\partial x_p^i} \equiv \sum_{a \in L^i \cup L^S} \sum_{b \in L^i \cup L^S} \frac{\partial \hat{c}_b^i(f)}{\partial f_a^i} \delta_{ap}, \tag{19a}$$

$$\frac{\partial \hat{\rho}_{il}(x)}{\partial x_p^i} \equiv \frac{\partial \rho_{il}(d)}{\partial d_{ik}}. \tag{19b}$$

we obtain:

$$-\frac{\partial \hat{U}_i}{\partial x_p^i} = \left[\frac{\partial \hat{C}_p^i(x)}{\partial x_p^i} - \hat{\rho}_{ik}(x) - \sum_{l=1}^{n_R} \frac{\partial \hat{\rho}_{il}(x)}{\partial x_p^i} \sum_{q \in P_l^i} x_q^i \right]. \tag{20}$$

In view of (20), it is clear that variational inequality (16) is equivalent to the variational inequality that determines the vector of equilibrium path flows $x^* \in K$, $x^* \in \mathcal{S}$ such that:

$$\sum_{i=1}^{I} \sum_{k=1}^{n_R} \sum_{p \in P_k^i} \left[\frac{\partial \hat{C}_p^i(x^*)}{\partial x_p^i} - \hat{\rho}_{ik}(x^*) - \sum_{l=1}^{n_R} \frac{\partial \hat{\rho}_{il}(x^*)}{\partial x_p^i} \sum_{q \in P_l^i} x_q^{i*} \right] \times [x_p^i - x_p^{i*}] \geq 0, \quad \forall x \in K, \forall x \in \mathcal{S}. \tag{21}$$

Variational inequality (16) can also be expressed in terms of link flows as follows: determine the vector of equilibrium link flows and the vector of demands $(f^*, d^*) \in \mathcal{K}^0$, such that:

$$\sum_{i=1}^{I} \sum_{a \in L^i \cup L^S} \left[\sum_{b \in L^i \cup L^S} \frac{\partial \hat{c}_b^i(f^*)}{\partial f_a^i} \right] \times [f_a^i - f_a^{i*}]$$

$$+ \sum_{i=1}^{I} \sum_{k=1}^{n_R} \left[-\rho_{ik}(d^*) - \sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^*)}{\partial d_{ik}} d_{il}^* \right] \times [d_{ik} - d_{ik}^*] \geq 0, \quad \forall (f, d) \in \mathcal{K}^0 \qquad (22)$$

where $\mathcal{K}^0 \equiv \{(f, d) | \exists x \geq 0, \, (1), \, (3), \, (4), \, \text{and} \, (5) \, \text{hold}\}$.

Existence of a solution to variational inequality (21) and to variational inequality (22) is guaranteed since each of the feasible sets is closed and bounded. Indeed, since all the links in the supply chain network in Figure 1 have capacities imposed on them, the path flows as well as the link flows are bounded. Also, uniqueness of an equilibrium link flow and demand pattern solving variational inequality (22) is guaranteed under conditions of strict monotonicity on the function that enters the variational inequality (cf. Nagurney (1999)).

We now present alternative variational inequalities to the one in (16) by utilizing the expanded form (21). The alternative variational inequality in path flows, which includes Lagrange multipliers, is defined over the nonnegative orthant, and we will utilize it for computational purposes, since the algorithmic scheme that we propose for its solution, the Euler method, will yield closed form expressions at each iteration for the variables, both the path flows and the Lagrange multipliers.

Let λ_a; $a \in L^i$; $\forall i$ and η_a; $a \in L^S$ denote the Lagrange multipliers associated with constraints (10) and (11), respectively.

Theorem 1: Alternative Variational Inequality Formulations of the Variational Equilibrium in Path Flows and in Link Flows

The variational equilibrium (16) is equivalent to the variational inequality: determine the vector of equilibrium path flows, and the vector of optimal Lagrange multipliers, $(x^*, \lambda^*, \eta^*) \in \mathcal{K}$*, such that:*

$$\sum_{i=1}^{I} \sum_{k=1}^{n_R} \sum_{p \in P_k^i} \left[\frac{\partial \hat{C}_p^i(x^*)}{\partial x_p^i} + \sum_{a \in L^i} \lambda_a^* \delta_{ap} + \sum_{a \in L^S} \eta_a^* \delta_{ap} - \hat{\rho}_{ik}(x^*) - \sum_{l=1}^{n_R} \frac{\partial \hat{\rho}_{il}(x^*)}{\partial x_p^i} \sum_{q \in P_l^i} x_q^{i*} \right]$$

$$\times [x_p^i - x_p^{i*}] + \sum_{i=1}^{I} \sum_{a \in L^i} \left[u_a^i - \sum_{p \in P} x_p^{i*} \delta_{ap} \right] \times [\lambda_a - \lambda_a^*]$$

$$+\sum_{a\in L^S}\left[u_a-\sum_{i=1}^{I}\sum_{p\in P}x_p^{i*}\delta_{ap}\right]\times[\eta_a-\eta_a^*]\geq 0,\qquad (x,\lambda,\eta)\in\mathcal{K},\tag{23}$$

where $\mathcal{K}\equiv\{(x,\lambda,\eta)|x\in R_+^{n_P},\ \lambda\in R_+^{\sum_{i=1}^{I}n_{L^i}},\ \eta\in R_+^{n_{L^S}}\}$.

The variational inequality (23), in turn, can be rewritten in terms of link flows as: determine the vector of equilibrium link flows, the vector of demands, and the vector of optimal Lagrange multipliers, $(f^*,d^*,\lambda^*,\eta^*)\in\mathcal{K}^1$*, such that:*

$$\sum_{i=1}^{I}\sum_{a\in L^i}\left[\sum_{b\in L^i}\frac{\partial\hat{c}_b^i(f^*)}{\partial f_a^i}+\lambda_a^*\right]\times[f_a^i-f_a^{i*}]+\sum_{i=1}^{I}\sum_{a\in L^S}\left[\sum_{b\in L^S}\frac{\partial\hat{c}_b^i(f^*)}{\partial f_a^i}+\eta_a^*\right]\times[f_a^i-f_a^{i*}]$$

$$+\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[-\rho_{ik}(d^*)-\sum_{l=1}^{n_R}\frac{\partial\rho_{il}(d^*)}{\partial d_{ik}}d_{il}^*\right]\times[d_{ik}-d_{ik}^*]$$

$$+\sum_{i=1}^{I}\sum_{a\in L^i}[u_a^i-f_a^{i*}]\times[\lambda_a-\lambda_a^*]+\sum_{a\in L^S}\left[u_a-\sum_{i=1}^{I}f_a^{i*}\right]\times[\eta_a-\eta_a^*]\geq 0,\qquad (f,d,\lambda,\eta)\in\mathcal{K}^1,\tag{24}$$

where $\mathcal{K}^1\equiv\{(f,d,\lambda,\eta)|\exists x\geq 0,\ (1)$ *and* (3) *hold, and* $\lambda\geq 0,\ \eta\geq 0\}$.

Proof: Variational inequality (23) follows from the Karush Kuhn Tucker conditions (see also Lemma 1.2 in Yashtini and Malek (2007)). Variational inequality (24) then follows from variational inequality (23) by making use of the conservation of flow equations. □

It is interesting that the supply chain network oligopoly model with capacity competition and outsourcing contains, as a special case, the supply chain network problem without capacity competition for shared distribution centers and freight service providers, with the supply chain network topology depicted in Figure 2. A spectrum of supply chain network models, in which there are no coupling constraints, with similar topologies to the one in Figure 2, have been formulated and studied in the literature (see, e.g., Nagurney (2010)) with applications including fashion (Nagurney, Yu, and Floden (2015)) and pharmaceuticals with the use of generalized networks to capture product perishability (Masoumi, Yu, and Nagurney (2012)) as well as sustainability (Nagurney, Yu, and Floden (2013)).

Of course, another special case of our model arises when the manufacturers/producers don't own any distribution centers and must outsource storage as well as freight service provision with the underlying supply chain network topology then corresponding to the one given in Figure 3.

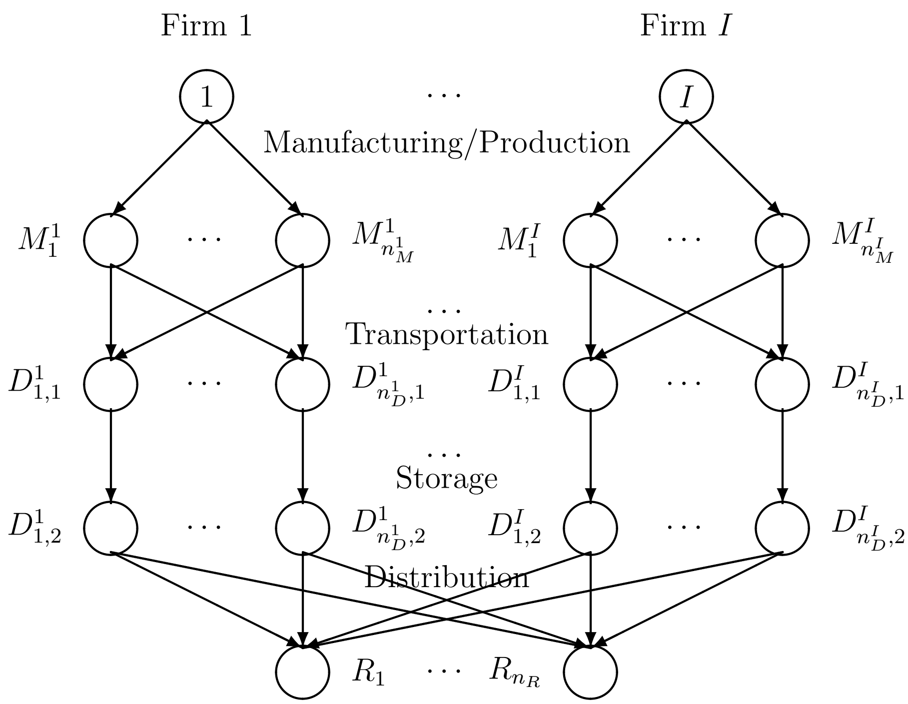

Figure 2: The Supply Chain Network Topology of the Oligopoly with No Shared Distribution Centers and Freight Service Providers

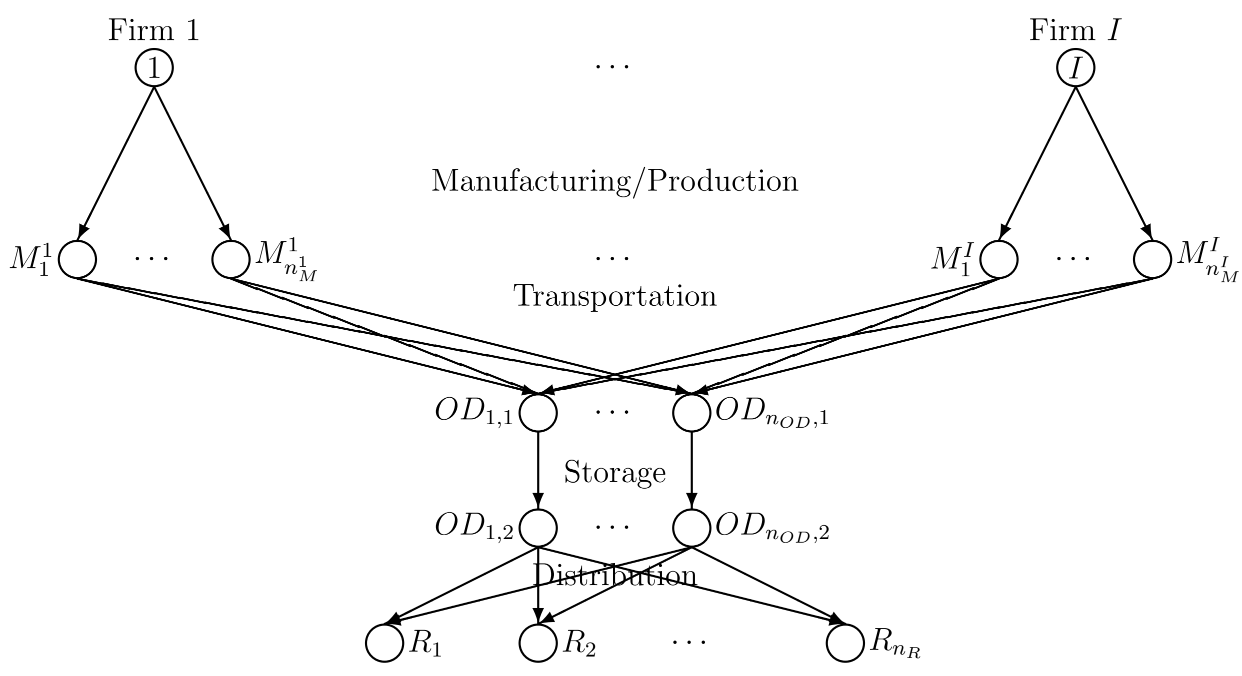

Figure 3: The Supply Chain Network Topology of the Oligopoly with Capacity Competition and Outsourcing of Distribution and Freight Service Provision and No Ownership of Distribution Centers and Freight Service Provision by the Firms

3. The Algorithm

The Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993) is presented in this Section. Specifically, at an iteration τ of the Euler method (see also Nagurney and Zhang (1996)) one computes:

$$X^{\tau+1} = P_{\mathcal{K}}(X^\tau - a_\tau F(X^\tau)), \tag{25}$$

where $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ and F is the utility function that enters the variational inequality problem (16).

As shown in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996), for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$. Specific conditions for convergence of this scheme as well as various applications to the solutions of network oligopolies can be found in Nagurney and Zhang (1996), Nagurney, Dupuis, and Zhang (1994), Nagurney (2010), Nagurney and Yu (2012), and Masoumi, Yu, and Nagurney (2012).

3.1 Explicit Formulae for the Euler Method Applied to the Alernative Variational Inequality Formulation

The elegance of this procedure for the computation of solutions to the supply chain network with capacity competition and outsourcing in Section 2 can be seen in the following explicit formulae. The closed form expressions for the path flows at iteration $\tau+1$ are as follows. For each path $p \in P^i_k$, $\forall i, k$, compute:

$$x_p^{i^{\tau+1}} = \max\{0, x_p^{i^\tau} + a_\tau(\hat{\rho}_{ik}(x^\tau) + \sum_{l=1}^{n_R} \frac{\partial \hat{\rho}_{il}(x^\tau)}{\partial x_p^i} \sum_{q \in P_l^i} x_q^{i^\tau} - \frac{\partial \hat{C}_p^i(x^\tau)}{\partial x_p^i} - \sum_{a \in L^i} \lambda_a^\tau \delta_{ap} - \sum_{a \in L^S} \eta_a^\tau \delta_{ap})\},$$

$$\forall p \in P^i_k;\ i = 1, \ldots, I;\ k = 1, \ldots, n_R. \tag{26}$$

The Lagrange multipliers for the individual firms' link $a \in L^i; i = 1, \ldots, I$, can be computed as:

$$\lambda_a^{\tau+1} = \max\{0, \lambda_a^\tau + a_\tau(\sum_{p \in P} x_p^{i^\tau} \delta_{ap} - u_a^i)\},\ \forall a \in L^i; i = 1, \ldots, I. \tag{27}$$

The computation process for the Lagrange multipliers for the shared link $a \in L^S$, can be given as:

$$\eta_a^{\tau+1} = \max\{0, \eta_a^\tau + a_\tau(\sum_{i=1}^{I} \sum_{p \in P} x_p^{i^\tau} \delta_{ap} - u_a)\},\ \forall a \in L^S. \tag{28}$$

The number of strategic variables x_p, as well as the number of the paths, in the supply chain network, grow linearly in terms of the number of nodes in the supply chain network.

Therefore, even a supply chain network with hundreds of demand markets is still tractable within our proposed modeling and computational framework.

4. Case Study

In this section we present a case study in order to illustrate the modeling framework and its relevance to applications. The case study consists of four examples inspired by a food supply chain application in which the food is fresh produce, specifically, apples. The case study is based on our experiences with apple growers in western Massachusetts. We consider two farmers that grow the apples, which, because of their quality, are represented by brands. Each farmer has two areas in which he grows his apples and each farm supplies its produce to two major retailers in the form of supermarkets in western Massachusetts. In the examples we vary the supply chain network topologies and describe additional details below. The top-most links in the supply chain network topologies for the four examples (see Figures 4 through 7) correspond to production links and these links also include the harvesting, processing, and packaging costs. The unit of the flows in these supply chain network examples is bushel(s) of apples.

The Euler method described in the preceding section was implemented in FORTRAN and a Linux system at the University of Massachusetts Amherst was used for the implementation and the computation of solutions below. The convergence tolerance $\epsilon = 10^{-6}$; that is, the Euler method was deemed to have converged if the absolute value of the difference of the successive computed iterates of the variables differed by no more than this ϵ. We initialized the Euler method by setting the demands for each firm's brand at each demand market to 100 and distributing the demand among the path flows equally for each set of farm/demand market pairs. The Lagrange multipliers were all initialized to 0.00. The sequence $\{a_\tau\} = .1\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$.

The link definitions for all the supply chain network examples, along with the total operational cost functions, are reported in Table 2. The link capacities and the demand price functions are given subsequently.

The supply chain network with the full set of nodes and links is in Example 3, Figure 6. The other examples in the case study have a subset of nodes and links to illustrate different scenarios.

The cost functions are constructed according to the information gathered from Berkett (1994) and CISA (2016). It is assumed that Farm 1 has 200 acres and Farm 2 has 100 acres of land. Therefore, the labor and machinery costs of Farm 1 are expected to be higher

Table 2: Definition of Links and Associated Total Operational Cost Functions for the Numerical Examples

Link a	From Node	To Node	$\hat{c}_a^1(f_a^1)$	$\hat{c}_a^2(f_a^2)$
1	1	M_1^1	$0.03(f_1^1)^2+3f_1^1$	–
2	1	M_2^1	$0.02(f_2^1)^2+2f_2^1$	–
3	M_1^1	$D_{1,1}^1$	$0.01(f_3^1)^2+4f_3^1$	–
4	M_2^1	$D_{1,1}^1$	$0.025(f_4^1)^2+3f_4^1$	–
5	$D_{1,1}^1$	$D_{1,2}^1$	$0.035(f_5^1)^2+5f_5^1$	–
6	$D_{1,2}^1$	R_1	$0.02(f_6^1)^2+2f_6^1$	–
7	$D_{1,2}^1$	R_2	$0.03(f_7^1)^2+5f_7^1$	–
8	2	M_1^2	–	$0.01(f_8^2)^2+6f_8^2$
9	2	M_2^2	–	$0.01(f_9^2)^2+6f_9^2$
10	M_1^2	$D_{1,1}^2$	–	$0.02(f_{10}^2)^2+4f_{10}^2$
11	M_2^2	$D_{1,1}^2$	–	$0.02(f_{11}^2)^2+4f_{11}^2$
12	$D_{1,1}^2$	$D_{1,2}^2$	–	$0.03(f_{12}^2)^2+5f_{12}^2$
13	$D_{1,2}^2$	R_1	–	$0.02(f_{13}^2)^2+8f_{13}^2$
14	$D_{1,2}^2$	R_2	–	$0.035(f_{14}^2)^2+5f_{14}^2$
15	M_1^1	$OD_{1,1}$	$0.01(f_{15}^1)^2+6f_{15}^1$	–
16	M_2^1	$OD_{1,1}$	$0.02(f_{16}^1)^2+5f_{16}^1$	–
17	M_1^2	$OD_{1,1}$	–	$0.02(f_{17}^2)^2+5f_{17}^2$
18	M_2^2	$OD_{1,1}$	–	$0.02(f_{18}^2)^2+6f_{18}^2$
19	$OD_{1,1}$	$OD_{1,2}$	$0.01(f_{19}^1)^2+f_{19}^1$	$0.01(f_{19}^2)^2+f_{19}^2$
20	$OD_{1,2}$	R_1	$0.012(f_{20}^1)^2+2f_{20}^1$	$0.012(f_{20}^2)^2+2f_{20}^2$
21	$OD_{1,2}$	R_2	$0.01(f_{21}^1)^2+f_{21}^1$	$0.01(f_{21}^2)^2+f_{21}^2$

(see cost functions for links 1 and 2) than they are for Farm 2 (refer to total link costs for links 8 and 9). The second production facility of Farm 1, M_2^1, is assumed to be smaller in land size than its first production facility, M_1^1. Therefore, the total cost function on link 2 is smaller than the total cost function on link 1. On the other hand, Farm 2 has identical production facilities, M_1^2 and M_2^2, which results in the total cost functions on links 8 and 9 being the same. Both of the farms have controlled atmospheric storage and similar costs of storage. Furthermore, Farm 1 owns more vehicles, machinery, and employees to transport and distribute the processed apples to the storage units and to the retailers. This means that the transportation and distribution costs of Farm 1 are lower than Farm 2's. The external distribution center has the lowest storage cost due to its size of storage and its business capability. Also, the cost of distributing the apples from the external distribution center (which, in effect, can serve as a wholesaler) to the supermarkets is relatively low, due to its location, market power, and the size of its freight fleet. Observe from Table 2 that the external distribution center charges both farmers the same price, in effect, for storage and distribution, as reflected in the total costs, since the two supermarkets are in proximity to one another. Indeed, these functions depend on the volume of each of the farmers' apples that the external distribution center handles in terms of storage and distribution to the supermarkets. Additionally, the time horizon for the case study examples or the supply chain activities is taken as 3-4 weeks, which corresponds to the total harvest time of apples.

Also, the link capacities, in bushels of apples, are as follows:

For Farm 1:

$$u^1_1 = 3000, \quad u^1_2 = 1000, \quad u^1_3 = 2000, \quad u^1_4 = 1000, \quad u^1_5 = 10000,$$

$$u^1_6 = 500, \quad u^1_7 = 300, \quad u^1_{15} = 2000, \quad u^1_{16} = 500.$$

For Farm 2:

$$u^2_8 = 1500, \quad u^2_9 = 500, \quad u^2_{10} = 1000, \quad u^2_{11} = 500, \quad u^2_{12} = 5000,$$

$$u^2_{13} = 400, \quad u^2_{14} = 200, \quad u^2_{17} = 1500, \quad u^2_{18} = 400.$$

For the External Distribution Center and Freight Service Provider:

$$u_{19} = 10000, \quad u_{20} = 1000, \quad u_{21} = 1000.$$

The capacities on the links associated with the farms are constructed based on size of land, the available manpower, machinery, and vehicles. In general, since Farm 1 is larger in size, in terms of the number of employees and machinery than Farm 2, the capacities on its links are larger. However, the storage and transportation capacities of the external distribution center or the wholesaler, as expected, are as high or higher than those associated with the individual farms.

The demand price functions for the apples from Farm 1 and Farm 2 are as follows:

Farm 1:

$$\rho_{11}(d) = -0.002d_{11} - 0.001d_{21} + 90,$$

$$\rho_{12}(d) = -0.003d_{12} - 0.001d_{22} + 100,$$

Farm 2:

$$\rho_{21}(d) = -0.002d_{21} - 0.001d_{11} + 80,$$

$$\rho_{22}(d) = -0.0025d_{22} - 0.001d_{12} + 100.$$

Consumers ate the second supermarket are willing to pay a higher price for each brand of apples.

Example 1: Only Farmers' Storage Facilities Are Available

In Example 1, each farmer has a storage facility / distribution center for his apples. The supply chain network topology is depicted in Figure 4. In this example there are no available external distribution centers. This example serves as a baseline.

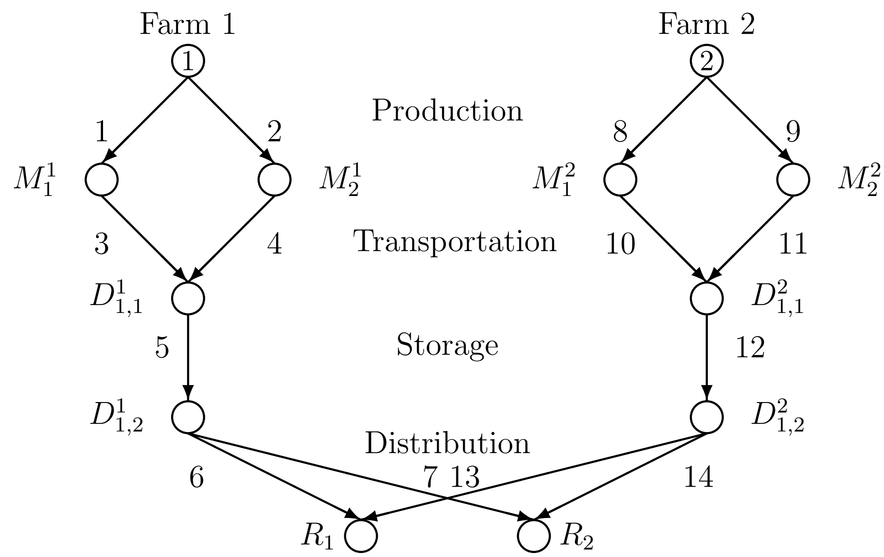

Figure 4: Example 1 Supply Chain Network Topology

The computed equilibrium link flow and Lagrange multiplier patterns are given in Table 3. Note that, because of the supply chain network topology in Figure 4, the only vector of equilibrium Lagrange multipliers is λ^* since there are no external distribution centers.

Observe from Table 3 that link 14 is at its capacity and, hence, the associated Lagrange multiplier is positive.

Also, for completeness, we report the computed equilibrium path flows.

The eight paths are defined as follows:

$$p_1 = (1, 3, 5, 6), \quad p_2 = (2, 4, 5, 6), \quad p_3 = (1, 3, 5, 7), \quad p_4 = (2, 4, 5, 7),$$

$$p_5 = (8, 10, 12, 13), \quad p_6 = (9, 11, 12, 13), \quad p_7 = (8, 10, 12, 14), \quad p_8 = (9, 11, 12, 14)$$

Table 3: Equilibrium Link Flow and Lagrange Multiplier Pattern for Example 1

Link a	f_a^{1*}	f_a^{2*}	λ_a^*
1	291.33	–	0.00
2	281.54	–	0.00
3	291.33	–	0.00
4	281.54	–	0.00
5	572.86	–	0.00
6	279.24	–	0.00
7	293.62	–	0.00
8	–	244.48	0.00
9	–	244.48	0.00
10	–	244.48	0.00
11	–	244.48	0.00
12	–	488.96	0.00
13	–	288.96	0.00
14	–	200.00	19.68

and the equilibrium product path flows are:

$$x_{p_1}^{1*} = 142.07, \quad x_{p_2}^{1*} = 137.17, \quad x_{p_3}^{1*} = 149.26, \quad x_{p_4}^{1*} = 144.37,$$

$$x_{p_5}^{2*} = 144.48, \quad x_{p_6}^{2*} = 144.48, \quad x_{p_7}^{2*} = 100.00, \quad x_{p_8}^{2*} = 100.00.$$

The incurred equilibrium prices at the demand markets are: $\rho_{11} = 89.15$, $\rho_{12} = 98.92$, $\rho_{21} = 79.14$, and $\rho_{22} = 98.71$. These prices are reasonable, since, typically, a bushel of apples in western Massachusetts commands a price of approximately \$80.

The incurred equilibrium demands, in turn, are as follows: $d_{11}^* = 279.24$, $d_{12}^* = 293.62$, $d_{21}^* = 288.96$, and $d_{22}^* = 200.00$. The profit of Farm 1 is 23,008.39 and that of Farm 2 is 18,135.58. Farm 1 attains a higher profit than Farm 2, since it sells its apples at a higher price at demand market, R_1. In addition, Farm 1 sells more than Farm 2 at the second demand market, R_2, even though the price of their apples are the same. This result is due, in part, to the capacity constraint of Farm 2 which also causes low profits for Farm 2.

Example 2: An External Distribution Center Is Made Available But Only Farmer 2 Is Considering It

In Example 2, an external distribution center has become available but only the second farmer is interested in considering it. The supply chain network topology is given in Figure 5.

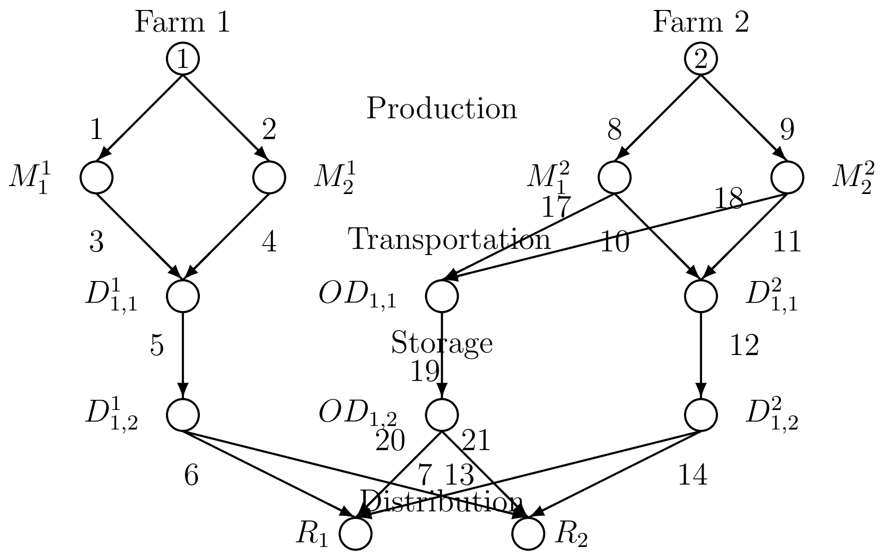

Figure 5: Example 2 Supply Chain Network Topology

Please refer to Table 2 for the complete data for Example 2.

The computed equilibrium link flow pattern and the Lagrange multiplier patterns are reported in Table 4. Recall that the Lagrange multipliers λ correspond to the firms' links that they have complete control over, whereas the Lagrange multipliers η correspond to the Lagrange multipliers associated with the links of the external distribution center, that is, links 19, 20, and 21 in Figure 5.

In addition to the previous eight paths defined for Example 1, there are now four additional paths as follows:

$$p_9 = (8, 17, 19, 20), \quad p_{10} = (9, 18, 19, 20), \quad p_{11} = (8, 17, 19, 21), \quad p_{12} = (9, 18, 19, 21).$$

The computed equilibrium product path flows for Example 2 are:

$$x_{p_1}^{1*} = 144.37, \quad x_{p_2}^{1*} = 139.35, \quad x_{p_3}^{1*} = 145.76, \quad x_{p_4}^{1*} = 140.77, \quad x_{p_5}^{2*} = 119.98, \quad x_{p_6}^{2*} = 39.94,$$

$$x_{p_7}^{2*} = 139.93, \quad x_{p_8}^{2*} = 60.07, \quad x_{p_9}^{2*} = 134.86, \quad x_{p_{10}}^{2*} = 40.23, \quad x_{p_{11}}^{2*} = 452.95, \quad x_{p_{12}}^{2*} = 359.76.$$

The incurred equilibrium prices at the demand markets are: $\rho_{11} = 89.10$, $\rho_{12} = 98.13$, $\rho_{21} = 79.05$, and $\rho_{22} = 94.65$. The incurred equilibrium demands, in turn, are as follows: $d_{11}^* = 283.73$, $d_{12}^* = 286.53$, $d_{21}^* = 335.01$, and $d_{22}^* = 1012.71$. The profit of Farm 1 is 22,760.00 and that of Farm 2 is 57,363.86. Observe that, in this example, Farm 2's profit dramatically increases from its value in Example 1; in fact, it more than triples. Since Farm

Table 4: Equilibrium Link Flow and Lagrange Multiplier Patterns for Example 2

Link a	f_a^{1*}	f_a^{2*}	λ_a^*	η_a^*
1	290.13	–	0.00	–
2	280.12	–	0.00	–
3	290.13	–	0.00	–
4	280.12	–	0.00	–
5	570.25	–	0.00	–
6	283.73	–	0.00	–
7	286.53	–	0.00	–
8	–	847.72	0.00	–
9	–	500.00	13.40	–
10	–	259.91	0.00	–
11	–	100.01	0.00	–
12	–	359.92	0.00	–
13	–	159.92	0.00	–
14	–	200.00	6.58	–
17	–	587.81	0.00	–
18	–	400.00	0.03	–
19	–	987.81	–	0.00
20	–	175.09	–	0.00
21	–	812.71	–	0.00

2 uses the external distribution center and its large storage facilities, the capacity limitations experienced in Example 1 for Farm 2 are eliminated. Now Farm 2 sells more of its apples at both demand markets and attains a higher profit. This result can also be seen from the fact that the optimal Lagrange multipliers on links 9, 14, and 18 are positive. Recall that link 14 corresponds to distribution from the second farmer's own distribution center to the second supermarket, whereas link 18 corresponds to the farm's transportation to the external distribution center. This means that Farm 2 transports its apples to the external distribution center and to the second supermarket at full capacity. Furthermore, a decrease in the price of Farm 2's apples at the second demand market, R_2, can be explained by the demand increase in this market.

Farm 2, by availing itself of the new service provided by the external distribution center, clearly gains economically, whereas Farm 1 experiences a decrease in profits as compared to its profit in Example 1.

Example 3: Both Farmers Are Considering the External Distribution Center

In Example 3, both farmers now are considering the option of the external distribution center and also still have available their own distribution centers, that is, storage facilities. The supply chain network topology for this example is depicted in Figure 6.

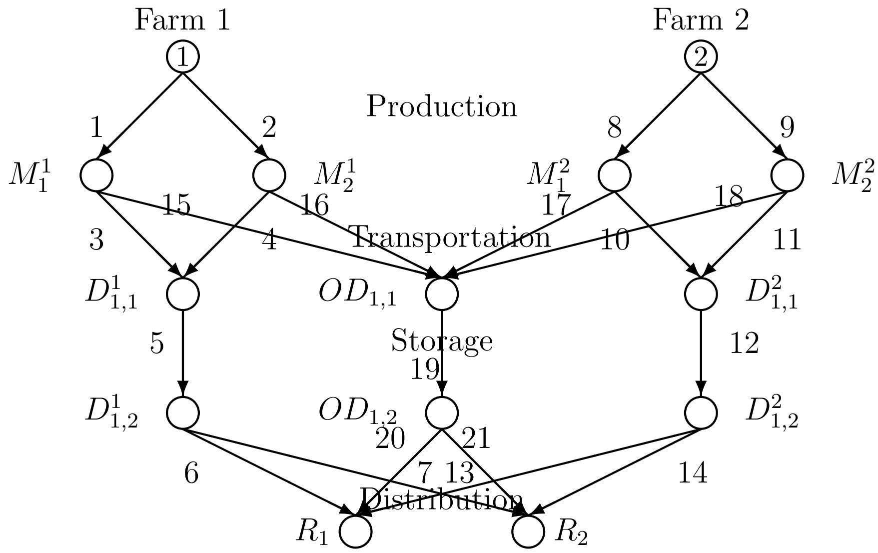

Figure 6: Example 3 Supply Chain Network Topology

The computed equilibrium link flow and Lagrange multiplier patterns are reported in Table 5.

The additional four paths to those in Example 2 for Example 3 are:

$$p_{13} = (1, 15, 19, 20), \quad p_{14} = (2, 16, 19, 20), \quad p_{15} = (1, 15, 19, 21), \quad p_{16} = (2, 16, 19, 21).$$

The computed equilibrium path flows are:

$$x_{p_1}^{1*} = 62.86, \quad x_{p_2}^{1*} = 114.71, \quad x_{p_3}^{1*} = 79.47, \quad x_{p_4}^{1*} = 130.33, \quad x_{p_5}^{2*} = 115.78, \quad x_{p_6}^{2*} = 52.36,$$

$$x_{p_7}^{2*} = 131.66, \quad x_{p_8}^{2*} = 68.34, \quad x_{p_9}^{2*} = 211.69, \quad x_{p_{10}}^{2*} = 135.28, \quad x_{p_{11}}^{2*} = 318.89, \quad x_{p_{12}}^{2*} = 244.01.$$

$$x_{p_{13}}^{1*} = 238.13, \quad x_{p_{14}}^{1*} = 242.8, \quad x_{p_{15}}^{1*} = 216.12, \quad x_{p_{16}}^{1*} = 220.88.$$

The incurred equilibrium prices at the demand markets are: $\rho_{11} = 88.17$, $\rho_{12} = 97.30$, $\rho_{21} = 78.31$, and $\rho_{22} = 95.54$. The incurred equilibrium demands, in turn, are as follows: $d_{11}^* = 658.56$, $d_{12}^* = 646.89$, $d_{21}^* = 515.12$, and $d_{22}^* = 762.91$. The profit of Farm 1 is 56,673.31 and that of Farm 2 is 42,412.05.

Table 5: Equilibrium Link Flow and Lagrange Multiplier Patterns for Example 3

Link a	f_a^{1*}	f_a^{2*}	λ_a^*	η_a^*
1	596.67	–	0.00	–
2	708.78	–	0.00	–
3	142.33	–	0.00	–
4	245.04	–	0.00	–
5	387.37	–	0.00	–
6	177.57	–	0.00	–
7	209.80	–	0.00	–
8	–	778.03	0.00	–
9	–	500.00	10.63	–
10	–	247.45	0.00	
11	–	120.70	0.00	
12	–	368.15	0.00	
13	–	168.14	0.00	
14	–	200.00	10.20	–
15	454.34	–	0.00	–
16	463.74	–	0.00	–
17	–	530.58	0.00	–
18	–	379.29	0.00	–
19	918.08	909.88	–	0.00
20	480.99	346.97	–	0.00
21	437.10	562.91	–	12.41

With increased services and competition for them, Farm 2 experiences a drop in profits, as compared to those obtained in Example 2. Farm 1, on the other hand, more than doubles its profits by taking advantage of the services provided by the external distribution center.

Interestingly, the capacity on link 21 is reached, with an associated positive Lagrange multiplier associated with that shared distribution link from the external distribution center.

Also, again, the capacity on link 9 associated with Farm 2's second production facility is also attained. Clearly, Farm 2 should try to purchase more land in proximity to that facility since it is constraining its apple production capabilities and demand for its brand of apples.

Example 4: A Supply Chain Disruption Has Damaged Farmers' Distribution Centers

Example 4 considers a supply chain disruption. Both farmers have their storage facilities made unavailable due to a natural disaster, such as flooding. However, their produce has not been affected and still is being picked. Apples have to be kept at an appropriate temperature under refrigeration for quality retention. The external distribution center is, nevertheless, available to them. The supply chain network topology for this example is given in Figure 7.

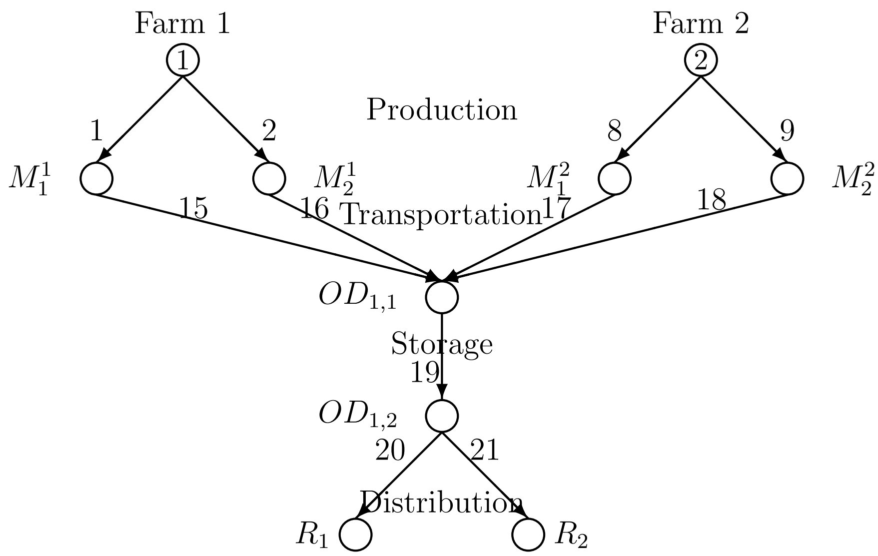

Figure 7: Example 4 Supply Chain Network Topology

The computed equilibrium link flow and Lagrange multiplier patterns are reported in Table 6.

Table 6: Equilibrium Link Flow and Lagrange Multiplier Patterns for Example 4

Link a	From Node	To Node	f_a^{1*}	f_a^{2*}	λ_a^*	η_a^*
1	1	M_1^1	513.45	–	0.00	–
2	1	M_2^1	500.00	–	0.00	–
8	2	M_1^2	–	574.26	0.00	–
9	2	M_2^2	–	400.00	0.00	–
15	M_1^1	$OD_{1,1}$	513.45	–	0.00	–
16	M_2^1	$OD_{1,1}$	500.00	–	3.07	–
17	M_1^2	$OD_{1,1}$	–	574.26	0.00	–
18	M_2^2	$OD_{1,1}$	–	400.00	9.38	–
19	$OD_{1,1}$	$OD_{1,2}$	1013.45	974.26	–	0.00
20	$OD_{1,2}$	R_1	576.09	411.62	–	0.00
21	$OD_{1,2}$	R_2	437.36	562.64	–	15.75

The computed equilibrium path flow for Example 4 is:

$$x_{p_{13}}^{1*} = 291.20, \quad x_{p_{14}}^{1*} = 284.90, \quad x_{p_{15}}^{1*} = 225.25, \quad x_{p_{15}}^{1*} = 222.25, \quad x_{p_{16}}^{1*} = 215.10,$$

$$x_{p_9}^{2*} = 249.52, \quad x_{p_{10}}^{2*} = 162.10, \quad x_{p_{11}}^{2*} = 324.75, \quad x_{p_{12}}^{2*} = 237.90.$$

The incurred equilibrium prices at the demand markets are: $\rho_{11} = 88.44$, $\rho_{12} = 98.13$, $\rho_{21} = 78.60$, and $\rho_{22} = 96.75$. The incurred equilibrium demands, in turn, are as follows: $d_{11}^* = 576.09$, $d_{12}^* = 437.36$, $d_{21}^* = 411.62$, and $d_{22}^* = 562.64$. The profit of Farm 1 is 46,427.75 and that of Farm 2 is 29,237.16. Following the supply chain disruption, which damages both farms' storage facilities, in that the capacities, are, in effect equal to zero on the corresponding links, both farmers suffer a decrease in profits.

As Table 6 reveals that link 16, associated with Farm 1, and link 18, associated with Farm 2, both of which are transportation links, are at their capacities and the associated Lagrange multipliers are positive. Also, the common, that is, shared, link 21, is also at its capacity with the Lagrange multiplier associated with this link being positive.

In Table 7, the profits of each farm for every example in the case study are summarized.

Table 7: Farm Profits for the Examples

Example	Farm 1 Profit	Farm 2 Profit
1	23,008.39	18,135.58
2	22,760.00	57,363.86
3	56,673.31	42,412.05
4	46,427.75	29,237.16

Clearly, both farms benefit, in terms of profits, by utilizing the external distribution center as revealed by the profit increases from Example 1 to Example 3. Furthermore, if, given a choice of operating their own distribution centers or using an external distribution center exclusively, a comparison of farm profits for Example 1, vis a vis the profits for Example 2, reveals that the latter is preferable.

The above results support what is happening increasingly in practice. First of all, Berkett (1994) states that farmers can harvest 600 bushels of apples per acre with a profit of approximately \$2200. In our case study, we assume that Farm 1 has 200 acres and Farm 2 has 100 acres of land. If the information gathered from Berkett (1994) is taken as an upper bound on profit and the path flows, then we can claim that our results on profit and flows are feasible and consistent with reality. Furthermore, according to a survey conducted by North Carolina State University (see Dunning (2014)), farms mostly use wholesalers as their primary marketing channel. Selling to a wholesaler can be more advantageous, especially for the small and middle-sized farms. For example, wholesalers can, typically, transport, store, and distribute higher volumes of fresh produce than the farmers can (Miles and Brown (2005)). Furthermore, since wholesalers maintain the relationships with the retailers, farms do not have to deal with the end customer directly; therefore, they can focus on the production side.

5. Summary and Conclusions

In this paper, we developed a supply chain network framework using game theory in which multiple manufacturers/producers have their own production facilities, distribution centers, and also distribute their products, which are distinguished by the firms' brands, to multiple demand points, which can correspond to retailers. In addition, they can outsource their product storage to external distribution centers who also provide freight service provision to the demand points. The firms have capacities associated with their supply chain network links, consisting of production, transportation, storage, and distribution, and compete also for the external distribution centers' services, which are also capacitated, but those links are shared, that is, common to the interested manufacturers. We assume noncooperative behavior with the manufacturers seeking to maximize their individual profits.

Due to the shared constraints, the governing equilibrium can no longer be directly formulated as a variational inequality problem. However, we utilize the concept of variational equilibrium, which is a special case of a Generalized Nash Equilibrium, with nice economic interpretations in that the Lagrange multipliers associated with the shared constraints are the same for the manufacturers. This problem is then analyzed qualitatively with existence guaranteed. Moreover, we propose an effective computational scheme, which resolves the problem into closed form expressions, at each iteration, for the product path flows and the Lagrange multipliers, until convergence is achieved.

We then illustrate the novel supply chain game theory framework with a case study consisting of producers that are apple farmers. Our case study reveals the benefits of external distribution centers and freight service provision in this sector.

The contributions in this paper add to the literature on supply chain network competition in terms of modeling advances with appropriate methodological foundations for a level of greater realism and generality for practice.

As for future research, the inclusion of stochastic elements would be worthwhile as well as the incorporation and investigation of competition for shared manufacturing plant resources, as occurs often in the high tech sector in the case of outsourcing of production. Finally, the exploration of alternative computational procedures based on global optimization techniques may also hold promise, along with the introduction of economies of scale in the Generalized Nash Equilibrium supply chain network model.

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