Statistical approximation for new positive linear operators of Lagrange type

M. Mursaleen, Faisal Khan and Asif Khan

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
mursaleenm@gmail.com; faisalamu2011@gmail.com; asifjnu07@gmail.com

Abstract

In this paper, we prove some approximation results in statistical sense and establish some direct theorems for the positive linear operators constructed by the means Lagrange type polynomials. We compute error estimation by using modulus of continuity with the help of Matlab and give its algorithm. Furthermore, we show graphically the convergence of our operators to various functions.


Keywords and phrases: Lagrange polynomial; $A$-statistical convergence; positive linear operators; modulus of continuity; error estimate; Peetre’s $K$-functional; Lipschitz class.

1 Introduction and preliminaries

Statistical convergence [16], and its variants, extensions and generalizations have been proved to be an active area of recent researches in summability theory, e.g. lacunary statistical convergence [17], $\lambda$–statistical convergence [28], $A$–statistical convergence [20], statistical $A$–summability [12], statistical summability $(C, 1)$ [23], statistical summability $(H, 1)$ [24], statistical summability $(\tilde{N}, p)$ [25] and statistical $\sigma$–summability [29] etc.. Following the work of Gadjiv and Orhan [18], these statistical summability methods have been used in establishing many approximation theorems (e.g. [9], [11], [22], [10], [26], [30], [12], [36], [5], [8], [27] and [33]). Recently the statistical approximation properties have also been investigated for several operators. For instance, in [3] Butzer and Hahn operators; in [4] and [32] $q$–analogue of Stancu-Beta operators; in [15] Bleimann, Butzer and Hahn operators; in [19] Baskakov-Kantorovich operators; in [34] Szász-Mirakjan operators; in [35] analogues of Bernstein-Kantorovich operators; and in [31] $q$–Lagrange polynomials were defined and their statistical approximation properties were investigated. Most recently, the statistical summability of Walsh-Fourier series has been discussed in [1]. In this paper, we construct a new family of operators with the help of Erkuş-Srivastava polynomials, establish some $A$-statistical approximation properties and direct theorems.
First we recall the following definitions:

Let $\mathbb{N}$ denote the set of all natural numbers. Let $K \subseteq \mathbb{N}$ and $K_n = \{ k \leq n : k \in K \}$. Then the \textit{natural density} of $K$ is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of the set $K_n$. A sequence $x = (x_k)$ of real numbers is said to be \textit{statistically convergent} to $L$ (cf. Fast [16]) provided that for every $\epsilon > 0$ the set $\{ k \in \mathbb{N} : |x_k - L| \geq \epsilon \}$ has natural density zero, i.e. for each $\epsilon > 0$,

$$
\lim_{n} \frac{1}{n} \{ k \leq n : |x_k - L| \geq \epsilon \} = 0.
$$

In this case, we write $st \lim_{n} x_n = L$. Note that every convergent sequence is statistically convergent but not conversely.

Let $A = (a_{nk}), n, k = 1, 2, 3...$ be an infinite matrix. For a given sequence $x = (x_k)$, the $A$-transform of $x$ is defined by $Ax = ((Ax)_n)$, where $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$, provided the series converges for each $n$. We say that $A$ is regular if $\lim_n (Ax)_n = L = \lim_n x$. Let $A$ be a regular matrix.

We say that a sequence $x = (x_k)$ is $A$-\textit{statistically convergent} to a number $L$ (cf. Kolk [20]) if for every $\epsilon > 0$,

$$
\lim_n \sum_{k : |x_k - L| \geq \epsilon} a_{nk} = 0.
$$

In this case, we denote this limit by $st_A \lim_{n} x_n = L$.

Note that for $A = C_1 := (c_{jn})$, the Cesàro matrix of order 1, $A$-statistical convergence reduces to the statistical convergence.

2 Construction of a new operator and its properties

The familiar (two-variable) polynomials $g_n^{(\alpha, \beta)}(x, y)$, which are generated by

$$
(1 - xz)^{-\alpha}(1 - yz)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha, \beta)}(x, y)z^n \quad (|z| < \min\{|x|^{-1}, |y|^{-1}\}),
$$

are known as the Lagrange polynomials which occur in certain problems in statistics [14]. Recently, Chain [6], introduced and systematically investigated the multivariable extension of the classical Lagrange polynomials $g_n^{(\alpha, \beta)}(x, y)$. These multivariable Lagrange polynomials, which are popularly known in the literature as the Chan-Chyan-Srivastava polynomials, are generated by (see [6]; see also [7])

$$
\prod_{j=1}^{r} (1 - x_jz)^{-\alpha_j} = \sum_{n=0}^{\infty} g_n^{(\alpha_{1, \ldots, \alpha_{r}})}(x_1, \ldots, x_r)z^n, \quad \alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, r);
$$

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Clearly, the defined generating function (2.2) yields the explicit representation given by [2, p. 140, Eq. (6)]

\[
g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{k_1+\ldots+k_r=n} (\alpha_1)_{k_1} \ldots (\alpha_r)_{k_r} \frac{x_1^{k_1}}{k_1!} \ldots \frac{x_r^{k_r}}{k_r!},
\]

or, equivalently, by [10, p. 522, Eq. (17)]

\[
g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{n_r-1=0}^{n} \sum_{n_r-2=0}^{n} \ldots \sum_{n_1=0}^{n_2} \frac{(\alpha_1)_{n_1} (\alpha_2)_{n_2-n_1} \ldots (\alpha_r)_{n-n_r-1}}{n_1! (n_2-n_1)! \ldots (n-n_r-1)!} x_1^{n_1} x_2^{n_2-n_1} \ldots \]

\[\ldots x_r^{n-n_r-1}.
\]

On the other hand, Altin and Erkuş [2] presented a multivariable extension of the so-called Lagrange-Hermite polynomials generated by (see, [2])

\[
\prod_{j=1}^{r} \left(1 - x_j z^j\right)^{-\alpha_j} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) z^n,
\]

\[
\alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, r);
\]

\[
|z| < \min\{|x_1|^{-1},\ldots,|x_r|^{-1/r}\}.
\]

(2.5)

The case \(r = 2\) of the polynomials given by (2.5) corresponds to the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli et al. [4].

The multivariable polynomials

\[
U_{n,\ell_1,\ldots,\ell_r}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r),
\]

which are defined by the following generating function [6, p. 268, Eq. (3)]:

\[
\prod_{j=1}^{r} \left(1 - x_j z^j\right)^{-\alpha_j} = \sum_{n=0}^{\infty} U_{n,\ell_1,\ldots,\ell_r}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) z^n,
\]

\[
\alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, r); \quad \ell_j \in \mathbb{N}
\]

\[
(j = 1, 2, \ldots, r); \quad |z| < \min\{|x_1|^{-1/\ell_1},\ldots,|x_r|^{-1/\ell_r}\},
\]

(2.6)

are a unification (and generalization) of several known families of multivariable polynomials including (for example) Chan-Chyan-Srivastava polynomials

\[
g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)
\]

defined by (2.2) (see, for details, [13]), Obviously, the Chan-Chyan-Srivastava polynomials

\[
g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)
\]
follow as the special case of the polynomials due to Erkuş and Srivastava [13]

\[ U_{n,\ell_1,\ldots,\ell_r}^{\alpha_1,\ldots,\alpha_r}(x_1,\ldots,x_r), \]

when

\[ \ell_j = 1 \quad (j = 1,\ldots,r), \]

where (as well as in what follows)

\[ \mathbb{N} = \{1,2,3,\ldots\} \quad \text{and} \quad \mathbb{N}_0 = \{0,1,2,\ldots\} = \mathbb{N} \cup \{0\}. \]

Moreover, the Lagrange-Hermite polynomials

\[ h_n^{(\alpha_1, \ldots, \alpha_r)}(x_1, \ldots, x_r), \]

follows as a special case of the polynomials [13]

\[ U_{n,\ell_1,\ldots,\ell_r}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r), \]

when

\[ \ell_j = 1 \quad (j = 1,\ldots,r). \]

The generating function (2.6) yields the following explicit representation ([13], p. 268, Eq. (4)):

\[ U_{n,\ell_1,\ldots,\ell_r}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{\ell_1k_1+\ldots+\ell_rk_r=n} (\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r} \frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}, \quad (2.7) \]

which, in the special case when

\[ \ell_j = 1 \quad (j = 1,\ldots,r), \]

corresponds to (2.3).

The following relationship is established between the polynomials due to Erkuş and Srivastava [13] and the Chan-Chyan-Srivastava polynomials by applying the generating functions (2.2) and (2.6), in [21].

\[ \sum_{n=0}^{\infty} U_{n,\ell_1,\ldots,\ell_r}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)z^n = \prod_{i=1}^{r-1} (1-x_i z_{\ell_i})^{-\alpha_i} = \prod_{i=1}^{r} \prod_{j=1}^{\ell_i} (1 - \omega^{(i,j)} z)^{-\alpha_i} = \sum_{n=0}^{\infty} g_n^{(\alpha_1,\ldots,\alpha_r,\ldots,\alpha_r)}(\omega_{11},\ldots,\omega_{1\ell_1},\ldots,\omega_{r1},\ldots,\omega_{r\ell_r})z^n, \quad (2.8) \]

where it is tacitly assumed that the following set:

\[ \omega^{(i,j)} : 1 \leq i \leq r \quad \text{and} \quad 1 \leq j \leq \ell_i \quad (\ell_i \in \mathbb{N}; i = 1,\ldots,r), \]
Throughout this paper we assume that 

\[ 1 - \left( x_i, z \right)^{\ell_i} \quad (i = 1, \ldots, r), \]

exists such that

\[ (1 - x_i z^{\ell_i})^{-\alpha_i} = \prod_{j=1}^{\ell_i} \left\{ (1 - \omega^{(i,j)} z)^{-\alpha_j} \right\} \quad (i = 1, \ldots, r). \]

Thus, by the assertion (2.8), we obtain the desired relationship as follows:

\[ U_n^{(\alpha_1, \ldots, \alpha_r)}(x_1, \ldots, x_r) = g_n^{(\alpha_1, \ldots, \alpha_1, \ldots, \alpha_r, \ldots, \alpha_r)}(\omega^{(1,1)}, \ldots, \omega^{(1,\ell_1)}, \ldots, \omega^{(r,1)}, \ldots, \omega^{(r,\ell_r)}). \]

Now by using the Erkuş-Srivastava multivariable polynomials given by (2.2), we introduce the following family of positive linear operators on \( C[0,1] \):

\[
T_n^{\omega^{(1,1)}, \ldots, \omega^{(1,\ell_1)}, \ldots, \omega^{(r,1)}, \ldots, \omega^{(r,\ell_r)}}(f; x) = \prod_{i=1}^{r} \left\{ (1 - x_i z^{\ell_i}) \right\} n \sum_{m=0}^{\infty} g_n^{(\alpha_1, \ldots, \alpha_1, \ldots, \alpha_r, \ldots, \alpha_r)}(\omega^{(1,1)}, \ldots, \omega^{(1,\ell_1)}, \ldots, \omega^{(r,1)}, \ldots, \omega^{(r,\ell_r)})(\omega^{(1,1)}, \ldots, \omega^{(1,\ell_1)}, \ldots, \omega^{(r,1)}, \ldots, \omega^{(r,\ell_r)}) z^n
\]

\[
\times f \left( \frac{k_r}{n + k_r - 1} \right)
\]

\[
= \prod_{i=1}^{r} \prod_{j=1}^{\ell_i} \left\{ (1 - \omega^{(i,j)} z) \right\} n \sum_{m=0}^{\infty} \omega^{(1,1)}, \ldots, \omega^{(1,\ell_1)}, \ldots, \omega^{(r,1)}, \ldots, \omega^{(r,\ell_r)}) z^n
\]

\[
\times f \left( \frac{k_r}{n + k_r - 1} \right),
\]

where

\[ \alpha_j \in \mathbb{C} \ (j = 1, 2, \ldots, r); \ \ell_j \in \mathbb{N} \ (j = 1, 2, \ldots, r); \ |z| < \min\{|x_1|^{-1/\ell_1}, \ldots, |x_r|^{-1/\ell_r}\}. \]

Throughout this paper we assume that

\[ \omega^{(i,j)} = \left\{ \omega^{(i,j)} \right\}_{n \in \mathbb{N}} \quad , 1 \leq i \leq r \text{ and } 1 \leq j \leq \ell_i \quad (\ell_i \in \mathbb{N}; i = 1, \ldots, r), \]

are sequences of real numbers such that

\[ 0 < \omega^{(i,j)} < 1. \]

For convenience taking \( r = 1, \ \ell_1 = 2, \ \alpha_1 = \alpha_2 = n \) in (2.9), we have

\[
T_n^{\omega^{(1,1)}, \omega^{(1,2)}}(f; x) = (1 - \omega^{(1,1)} x) n \sum_{m=0}^{\infty} g_n^{(\alpha_1, \alpha_2)}(\omega^{(1,1)}, \omega^{(1,2)})(\omega^{(1,1)}, \omega^{(1,2)}) x^n f \left( \frac{k}{n + k - 1} \right)
\]

\[
= (1 - \omega^{(1,1)} x)^n (1 - \omega^{(1,2)} x)^n \sum_{m=0}^{\infty} \left\{ \sum_{k_1=m}^{\infty} f \left( \frac{k}{n + k - 1} \right) \frac{(\omega^{(1,1)} x)^{k_1}}{k_1!} (\omega^{(1,2)} x)^{k_1} \right\} x^n.
\]

(2.10)
Lemma 2.1. For each $x \in [0, 1]$ and $n \in \mathbb{N}$,
\[ T_{n}^{\omega^{(1,1)}, \omega^{(1,2)}} (f_{0}; x) = 1 \quad (f_{0}(x) = 1). \]

Lemma 2.2. For each $x \in [0, 1]$ and $n \in \mathbb{N}$,
\[ T_{n}^{\omega^{(1,1)}, \omega^{(1,2)}} (f_{1}; x) = 1 \quad (f_{1}(x) = x). \]

Proof. Let each $x \in [0, 1]$ be fixed. Then from (2.10), we get
\[
T_{n}^{\omega^{(1,1)}, \omega^{(1,2)}} (f_{1}; x) = \left( 1 - \omega^{(1,1)} x \right)^{n} \left( 1 - \omega^{(1,2)} x \right) \sum_{m=0}^{\infty} \left( \sum_{k_{1}=m}^{\infty} \left( \frac{k}{n + k - 1} \right) \frac{\omega_{n}^{(1,1)}}{k_{1}!} (n)_{k_{1}} x^{m} \right) x^{m}
\]
\[
= \left( 1 - \omega^{(1,1)} x \right)^{n} \left( 1 - \omega^{(1,2)} x \right) \sum_{m=0}^{\infty} \left( \sum_{k_{1}=m}^{\infty} \frac{\omega_{n}^{(1,1)}}{(k_{1} - 1)!} (n)_{k_{1} - 1} \right) x^{m}
\]
\[
= \omega_{n}^{(1,1)} x \left( 1 - \omega^{(1,1)} x \right)^{n} \left( 1 - \omega^{(1,2)} x \right) \sum_{m=0}^{\infty} \left( \sum_{k_{1}=m}^{\infty} \frac{\omega_{n}^{(1,1)}}{(k_{1} - 1)!} (n)_{k_{1} - 1} \right) x^{m-1}
\]
\[= \omega_{n}^{(1,1)} x, \quad 0 < \omega_{n}^{(1,1)} < 1, \ \omega_{n}^{(1,1)} \to 1. \]

Lemma 2.3. For each $x \in [0, 1]$ and $n \in \mathbb{N}$,
\[ T_{n}^{\omega^{(1,1)}, \omega^{(1,2)}} (f_{2}; x) \leq x^{2} \left( \omega_{n}^{(1,1)} \right)^{2} + x \frac{\omega_{n}^{(1,1)}}{n} \quad (f_{2}(x) = x^{2}). \]

Proof. Let each $x \in [0, 1]$ be fixed. Then from (2.10), we get
\[
T_{n}^{\omega^{(1,1)}, \omega^{(1,2)}} (f_{2}; x) = \left( 1 - \omega^{(1,1)} x \right)^{n} \left( 1 - \omega^{(1,2)} x \right) \sum_{m=0}^{\infty} \left( \sum_{k_{1}=m}^{\infty} \left( \frac{k}{n + k - 1} \right) \frac{\omega_{n}^{(1,1)}}{k_{1}!} (n)_{k_{1}} x^{m} \right) x^{m}
\]
\[
= \left( 1 - \omega^{(1,1)} x \right)^{n} \left( 1 - \omega^{(1,2)} x \right) \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \left( \frac{k}{n + k - 1} \right) \left( \frac{\omega_{n}^{(1,1)}}{(k_{1} - 1)!} (n)_{k_{1} - 1} \right) x^{m}
\]
\[
= \left( 1 - \omega^{(1,1)} x \right)^{n} \left( 1 - \omega^{(1,2)} x \right) \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \left( \frac{(k - 1) + 1}{n + k - 1} \right) \left( \frac{\omega_{n}^{(1,1)}}{(k_{1} - 1)!} (n)_{k_{1} - 1} \right) x^{m}
\]
\[
= \left( 1 - \omega^{(1,1)} x \right)^{n} \left( 1 - \omega^{(1,2)} x \right) \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \left( \frac{(k - 1)}{n + k - 1} \right) \left( \frac{\omega_{n}^{(1,1)}}{(k_{1} - 1)!} (n)_{k_{1} - 1} \right) x^{m}
\]
Remark 2.2. Let

\[ T_n^{\omega(1,1),\omega(1,2)}(t - x; x) = x\omega_n^{(1,1)} - x = x(\omega_n^{(1,1)} - 1). \]

Remark 2.1.

\[ T_n^{\omega(1,1),\omega(1,2)}(t - x; x) = x\omega_n^{(1,1)} - x = x(\omega_n^{(1,1)} - 1). \]

Remark 2.2. Let \( x \in [0, 1] \), we have

\[ T_n^{\omega(1,1),\omega(1,2)}((t - x)^2; x) \leq x^2(u_n^{(2)})^2 + \frac{xu_n^{(2)}}{n} - 2x^2u_n^{(2)} + x^2 \]

Proof. Since \( T_n^{\omega(1,1),\omega(1,2)} \) is linear, we get

\[ T_n^{\omega(1,1),\omega(1,2)}((t - x)^2; x) = T_n^{\omega(1,1),\omega(1,2)}(t^2; x) - 2xT_n^{\omega(1,1),\omega(1,2)}(t; x) + x^2T_n^{\omega(1,1),\omega(1,2)}(1; x) \]

\[ \leq x^2(\omega(1,1))^2 + \frac{x\omega_n^{(1,1)}}{n} - 2x^2\omega_n^{(1,1)} + x^2 \]
3 A– statistical approximation

Let $C[a,b]$ be the linear space of all real valued continuous functions $f$ on $[a,b]$ and let $T$ be a linear operator which maps $C[a,b]$ into itself. We say that $T$ is positive if for every non-negative $f \in C[a,b]$, we have $T(f,x) \geq 0$ for all $x \in [a,b]$. We know that $C[a,b]$ is a Banach space with norm

$$\|f\|_{C[a,b]} := \sup_{x \in [a,b]} |f(x)|, \; f \in C[a,b].$$

For typographical convenience, we will write $\|\cdot\|$ in place of $\|\cdot\|_{C[a,b]}$ if no confusion arises.

**Theorem 3.1.** Let $A = (a_{jn})$ be a non-negative regular summability matrix. Then

$$st_A - \lim_n \omega^{(1,1)}_n = 1,$$

if and only if, for all $f \in C[0,1]$,

$$st_A - \lim_n \|T^{(1,1)}_n \omega^{(1,2)} (f) - f\| = 0.$$  \hspace{1cm} (3.2)

**Proof.** Suppose that (3.2) holds for all $f \in C[0,1]$. Then we have

$$st_A - \lim_n \|T^{(1,1)}_n \omega^{(1,2)} (f_1) - f_1\| = 0,$$

since $f_1 \in C[0,1]$. By Lemma 2.2, we have

$$\|T^{(1,1)}_n \omega^{(1,2)} (f_1) - f_1\| = 1 - \omega^{(1,1)}_n.$$  \hspace{1cm} (3.4)

By (3.3) and (3.4), we immediately get

$$st_A - \lim_n \omega^{(1,1)}_n = 1.$$  \hspace{1cm} (3.5)

Conversely, suppose that (3.1) holds. Then from Lemma 2.1, we have $\lim_n \|T^{(1,1)}_n \omega^{(1,2)} (f_0) - f_0\| = 0$. Hence

$$st_A - \lim_n \|T^{(1,1)}_n \omega^{(1,2)} (f_0) - f_0\| = 0, \; (f_0(x) = 1).$$  \hspace{1cm} (3.5)

Also from Lemma 2.2, it follows that

$$\|T^{(1,1)}_n \omega^{(1,2)} (f_1) - f_1\| = 1 - \omega^{(1,1)}_n.$$  \hspace{1cm} (3.6)

Therefore by using (3.1), we get

$$st_A - \lim_n \|T^{(1,1)}_n \omega^{(1,2)} (f_1) - f_1\| = 0, \; (f_1(x) := x).$$  \hspace{1cm} (3.6)
Now we claim that
\[ st\lim_n \|T_n^{\omega^{(1,1)},\omega^{(1,2)}}(f_2) - f_2\| = 0, \quad (f_2(x) := x^2). \] (3.7)

By Lemma 2.3, we have
\[ \|T_n^{\omega^{(1,1)},\omega^{(1,2)}}(f_2) - f_2\| \leq 2(1 - \omega_n^{(1,1)}) + \frac{\omega_n^{(1,1)}}{n}. \] (3.8)

Now, for a given \( \epsilon > 0 \), we define the following sets:
\[ D := \{ n : \|T_n^{\omega^{(1,1)},\omega^{(1,2)}}(f_2) - f_2\| \geq \epsilon \}, \]
\[ D_1 := \{ n : 1 - \omega_n^{(1,1)} \geq \frac{\epsilon}{4} \}, \]
\[ D_2 := \{ n : \frac{\omega_n^{(1,1)}}{n} \geq \frac{\epsilon}{2} \}. \]

From (3.8), it is easy to see that \( D \subseteq D_1 \cup D_2 \). Then, for each \( j \in \mathbb{N} \), we get
\[ \sum_{n \in D} a_{jn} \leq \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn}. \] (3.9)

Using (3.3), we get
\[ stA - \lim_n (1 - \omega_n^{(1,1)}) = 0, \]
and
\[ stA - \lim_n \frac{\omega_n^{(1,1)}}{n} = 0. \]

Now using the above facts and taking the limit as \( j \to \infty \) in (3.9), we conclude that
\[ \lim_j \sum_{n \in D} a_{jn} = 0, \]
which gives (3.7). Now, combining (3.5)–(3.7), and using the statistical version of Korovkin approximation theorem (see Gadjiv and Orhan [[18], Theorem 1]), we get the desired result.

This completes the proof of the theorem.

In a similar manner, we can extend Theorem 3.1 to the \((i,j)\)-dimensional case for the operators \( T_n^{\omega^{(1,1)},\ldots,\omega^{(1,i_1)},\ldots,\omega^{(r,1)},\ldots,\omega^{(r,r)}}(f; x) \) given by (2.9) as follows.

**Theorem 3.2.** Let \( A = (a_{jn}) \) be a non-negative regular summability matrix. Then
\[ stA - \lim_n \omega_n^{(i,j)} = 1, \]
if and only if, for all \( f \in C[0,1] \)
\[ stA - \lim_n \|T_n^{\omega^{(1,1)},\ldots,\omega^{(1,i_1)},\ldots,\omega^{(r,1)},\ldots,\omega^{(r,r)}}(f) - f\| = 0. \]
Remark 3.1. If, in Theorem 2.2, we replace \( A = (a_{jn}) \) by the identity matrix, we immediately get the following theorem which is a classical case of Theorem 2.2.

**Theorem 3.3.** \( \lim_{n} \omega_n^{(i,j)} = 1 \), if and only if, for all \( f \in C[0,1] \), the sequence \( T_n^{\omega(1,1),\ldots,\omega(1,t_1),\ldots,\omega(r,1),\ldots,\omega(r,t_r)}(f) \) is uniformly convergent to \( f \) on \([0,1]\).

Finally, we display an example which satisfies all hypotheses of Theorem 3.2, but not of Theorem 3.3. Therefore, this indicates that our \( A \)-statistical approximation in Theorem 3.2 is stronger than its classical case.

Take \( A = C_1 := (c_{jn}) \), the Cesàro matrix of order 1 and

\[
\omega^{(i,j)} := (\omega_n^{(i,j)})_{n \in \mathbb{N}} \quad (j = 1, \ldots, r - 1)
\]

be sequences of real numbers defined by

\[
\omega_n^{(i,j)} := \begin{cases} 
\frac{1}{3} & \text{if } n = m^2, \ (m \in \mathbb{N}); \\
1 - \frac{1}{n+i} & \text{otherwise}.
\end{cases}
\]

We then observe that

\( 0 < \omega_n^{(i,j)} < 1 \quad (n \in \mathbb{N}) \)

and also that

\( st_A - \lim_n \omega_n^{(i,j)} = 1 \)

Therefore, by Theorem 3.2, we have that for all \( f \in C[0,1] \)

\[
st_A - \lim_n \|T_n^{\omega(1,1),\ldots,\omega(1,t_1),\ldots,\omega(r,1),\ldots,\omega(r,t_r)}(f) - f\| = 0.
\]

However, since the sequence \( \omega_n^{(i,j)} \) defined by (3.10) is non-convergent, Theorem 3.3 does not hold in this case.

### 4 Direct Theorems

By \( C_B[0,1] \), we denote the space of all real valued continuous bounded functions \( f \) on the interval \([0,1]\), the norm \( \|\cdot\| \) on the space \( C_B[0,1] \) is given by

\[
\|f\| = \sup_{0 \leq x \leq 1} |f(x)|
\]

The Peetre’s \( K \)-functional is defined by

\[
K_2(f, \delta) = \inf\{|\|f - g\| + \delta\|g''\| : g \in W^2\}.
\]
where
\[ W^2 = \{ g \in C_B[0, 1] : g', g'' \in C_B[0, 1] \}. \]

By [10] there exists a positive constant \( c > 0 \) s.t.
\[ K_2(f, \delta) \leq cw_2(f, \delta^{1/2}), \delta > 0, \]
where the second order modulus of smoothness is
\[ w_2(f, \sqrt{\delta}) = \sup_{0 \leq h \leq \sqrt{\delta}} \sup_{0 \leq x \leq 1} |f(x + 2h) - 2f(x + h) + f(x)|. \]
Also for \( f \in C_B[0, 1] \), the usual modulus of continuity is given by
\[ w(f, \delta) = \sup_{0 \leq h \leq \delta} \sup_{0 \leq x \leq 1} |f(x + h) - f(x)|. \]

**Theorem 4.1.** Let \( f \in C_B[0, 1] \) and \( 0 \leq \omega_n^{(i,j)} < 1 \), then for all \( x \in [0, 1] \) and \( n \in \mathbb{N} \), there exists an absolute constant \( C > 0 \) s.t.
\[ |T_n^{\omega^{(1,1)}, \omega^{(1,2)}}(f, x) - f(x)| \leq Cw_2(f, \delta_n(x)) \]
where
\[ \delta_n^2(x) = x^2[(\omega_n^{(1,1)})^2 - 2\omega_n^{(1,1)} + 1] + \frac{x\omega_n^{(1,1)}}{n} \]

**Proof.** Let \( g \in W^2 \). From Taylor’s expansion
\[ g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - x)g''(u)du, \quad t \in [0, 1]. \]
and from Lemmas (2.1), (2.2) and (2.3), we get
\[ T_n^{\omega^{(1,1)}, \omega^{(1,2)}}(g, x) = g(x) + T_n^{\omega^{(1,1)}, \omega^{(1,2)}}(\int_x^t (t - x)g''(u)du, x) \]

hence
\[ |T_n^{\omega^{(1,1)}, \omega^{(1,2)}}(g, x) - g(x)| \leq |T_n^{\omega^{(1,1)}, \omega^{(1,2)}}(\int_x^t (t - x)g''(u)du, x)| \]
\[ \leq |I_n^{(1),u^{(2)}}((t - x)^2, x)| \|g''\| \]

using Remark 2.1, we obtain
\[ |T_n^{\omega^{(1,1)}, \omega^{(1,2)}}(g, x) - g(x)| \leq \left[ x^2(\omega_n^{(1,1)})^2 + \frac{x\omega_n^{(1,1)}}{n} - 2x^2\omega_n^{(1,1)} + x^2 \right] \|g''\| \]
On the other hand, by definition of $T^{(1,1),\omega^{(1,2)}}_n(f, x)$, we have
\[ |T^{(1,1),\omega^{(1,2)}}_n(f; x)| \leq \|f\| \]

Next
\[ |T^{(1,1),\omega^{(1,2)}}_n(f, x) - f(x)| \leq |T^{(1,1),\omega^{(1,2)}}_n((f-g); x) - (f-g)(x)| + |T^{(1,1),\omega^{(1,2)}}_n(g, x) - g(x)| \]
\[ \leq \|f-g\| + [x^2(\omega_n^{(1,1)})^2 + \frac{x\omega_n^{(1,1)}}{n} - 2x^2\omega_n^{(1,1)} + x^2]\|g''\|. \]

Hence taking infimum on the right hand side over all $g \in W^2$, we get
\[ |T^{(1,1),\omega^{(1,2)}}_n(f, x) - f(x)| \leq CK_2(f, \delta_n(x)) \]

In the view of the property of K-functional for every $0 < \omega_n^{(i,j)} < 1$, we get
\[ |T^{(1,1),\omega^{(1,2)}}_n(f, x) - f(x)| \leq Cw_2(f, \delta_n(x)) \]

This completes the proof of theorem.

**Theorem 4.2.** Let $f \in C_B[0, 1]$ be such that $f', f'' \in C_B[0, 1]$ and $0 < \omega_n^{(i,j)} < 1$, $j = 1, 2, 3, ...n$ such that $\omega_n^{(i,j)} \to 1$ as $n \to \infty$, then the following equality holds
\[ \lim_{n \to \infty} n \left( T^{(1,1),\omega^{(1,2)}}_n(f, x) - f(x) \right) = \frac{x}{2} f''(x) \]
uniformly on $[0, 1]$.

**Proof.** By the Taylor’s formula we may write
\[ f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + r(t, x)(t - x)^2 \quad (4.1) \]
where $r(t, x)$ is the remainder term and $\lim_{t \to x} r(t, x) = 0$. Applying $T^{(1,1),\omega^{(1,2)}}_n(f; x)$ to (4.1) we obtain
\[ n(T^{(1,1),\omega^{(1,2)}}_n(f, x) - f(x)) = nT^{(1,1),\omega^{(1,2)}}_n(t - x; x) f'(x) \]
\[ + nT^{(1,1),\omega^{(1,2)}}_n((t - x)^2; x) \frac{f''(x)}{2} + nT^{(1,1),\omega^{(1,2)}}_n(r(t, x)(t - x)^2; x) \]

By the Cauchy-Schwartz inequality, we have
\[ T^{(1,1),\omega^{(1,2)}}_n(r(t, x)(t - x)^2; x) \leq \sqrt{T^{(1,1),\omega^{(1,2)}}_n(r^2(t, x)^2; x)} \sqrt{T^{(1,1),\omega^{(1,2)}}_n((t - x)^4; x)} \]
observe that $r^2(x, x) = 0$ and $r^2(x) \in C[0, 1]$. Then it follows from Theorem 4.1, that
\[ \lim_{n \to \infty} T^{(1,1),\omega^{(1,2)}}_n(r^2(t, x); x) = r^2(x, x) = 0 \quad (4.3) \]
uniformly with respect to $x \in [0, 1]$.

Now from (4.2), (4.3) and Remark 2.2, we get

$$\lim_{n \to \infty} n T_n^{\omega(1,1),\omega(1,2)} (r(t,x)(t-x)^2;x) = 0.$$  

Finally using Remark 2.1, we get the following

$$\lim_{n \to \infty} n (T_n^{\omega(1,1),\omega(1,2)}( f(x) - f(x)) = \lim_{n \to \infty} n \left( f'(x) T_n^{\omega(1,1),\omega(1,2)} ((t-x);x) \right) + \frac{1}{2} f''(x) T_n^{\omega(1,1),\omega(1,2)} ((t-x)^2;x) \right) \right)$$

$$+ \frac{1}{2} f''(x) T_n^{\omega(1,1),\omega(1,2)} ((t-x)^2;x)$$

$$+ T_n^{\omega(1,1),\omega(1,2)} (r(t,x)(t-x)^2;x)$$

$$= \frac{x}{2} f''(x).$$

5 Examples

In this section, we compute error estimation by using modulus of continuity with the help of Matlab and give its algorithm. We also show graphically the convergence of our operators to various functions.

Example 1. Let us take $f(x) = 1 + \sin(-6x^2)$. We compute error estimation by using modulus of continuity for operators (2.10) to the function $f(x) = 1 + \sin(-6x^2)$ shown in the following Table with the help of Matlab and its algorithm presented after the Table.

Error estimation table :

<table>
<thead>
<tr>
<th>n</th>
<th>max error bound at x=0.2</th>
<th>max error bound at x=0.5</th>
<th>max error bound at x=0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.0184</td>
<td>8.6785</td>
<td>19.8303</td>
</tr>
<tr>
<td>$10^4$</td>
<td>$2.448 \times 10^{-4}$</td>
<td>0.0026</td>
<td>0.0072</td>
</tr>
<tr>
<td>$10^6$</td>
<td>$4.318 \times 10^{-5}$</td>
<td>$8.938 \times 10^{-4}$</td>
<td>0.0032</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>$4.114 \times 10^{-5}$</td>
<td>$8.765 \times 10^{-4}$</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

Error estimation algorithm :

```matlab
syms x; w = 0.99;
n = [2, 10^4, 10^6, 10^{10}];
dat = zeros(4, 4);
for i = 1 : 4
```
\[ m = n(i); \]

\[ \text{erreestimate} = \text{inline} \left[ \text{char} \left\{ \left( x^2 w^2 + \frac{x w}{n(i)} - 2x^2 w + x^2 \right) \times \text{abs(diff}(1 + \sin(-6x^2), 2)) \right\} \right]; \]

\[ \text{dat}(i, 1) = n(i); \]
\[ \text{dat}(i, 2) = \text{erreestimate}.2; \]
\[ \text{dat}(i, 3) = \text{erreestimate}.5; \]
\[ \text{dat}(i, 4) = \text{erreestimate}.8; \]

\textit{end}

For \( n=2 \), and for different values of \( m = 5, 10, 140 \), the convergence of operators (2.10) to function

\[ f(x) = 1 + \sin(-6x^2) \]

is illustrated in figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Example 2.}
\end{figure}

\textbf{Example 2.} Similarly, for \( n=2 \), and for different values of \( m = 2, 30, 90 \), the convergence of operators (2.10) to function
is illustrated in figure 2.

\[ f(x) = (x - \frac{1}{3})(x - \frac{1}{2})(x - \frac{3}{4}) \]

![Graph showing function for different n and m values](image)

Figure 2:

**References**


