Finding overlapping communities
in multiplex networks

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Abstract

We define an approach to identify overlapping communities in mul-
tiplex networks, extending the popular clique percolation method for
simple graphs. The extension requires to rethink the basic concepts on
which the clique percolation algorithm is based, including cliques and
clique adjacency, to allow the presence of multiple types of edges.

Keywords: multiplex network, multi-graph, community de-
tection, clustering, overlapping

1 Introduction

Community detection, also known as graph clustering, is one of the main
tasks to study complex systems represented as networks. A large number
of community detection methods has appeared in the literature [1, 2], with
early methods sharing two main features: they group the nodes of the net-
work into a set of disjoint clusters — also called partitioning clustering, and
they operate on simple graphs, that is, graphs with at most one edge between
each pair of nodes, no edges connecting a node to itself and no attributes
(with the possible exception of weights on the edges).

However, simple graphs and partitioning clusterings do not accurately
represent the complexity of several types of systems. For example, in so-
cial networks individuals communicate with different groups of people, like
friends, colleagues, and family, and this determines multiple types of relationships between nodes, including multiple types of ties between the same pairs of nodes. In addition, people may belong to more than one community at the same time.

To increase the expressiveness of models based on simple graphs, multilayer networks [3, 4] and heterogeneous information networks [5] have been introduced — among other models, allowing nodes and edges to have different types and to be described by multiple attributes. A specific type of multilayer system, called multiplex network, is characterized by nodes that can be connected through multiple types of edges and has been used for almost one century in the field of social network analysis [6, 7]. For what concerns community detection, people have developed several methods to find overlapping communities in simple graphs [8]. Clique based methods [9, 10, 11], fuzzy community detection algorithms [12, 13] and link partitioning methods [14, 15] are examples of overlapping clustering algorithms.

To the best of our knowledge, these two lines of research have not met yet: while we have methods for overlapping community detection on simple graphs [8], and we have partitioning community detection methods for multilayer networks [16], the problem of detecting overlapping communities in multilayer networks has not been directly addressed. Some approaches convert the multilayer network to a simple graph [17, 18, 19, 20], and then employ existing methods. However, this may result in information losses, because the clustering algorithm would not know whether a set of edges belongs to the same or to different layers, potentially leading to the discovery of communities scattered across a large number of layers and weak ties. In this paper we introduce an approach to identify overlapping communities for the special case of multiplex networks.

In the next section we present the basic definitions and concepts needed to understand our extended method. These include multiplex networks, partitioning and overlapping communities in multiplex networks, and the original clique percolation method for simple graphs. Section 3 extends the basic concepts on which the clique percolation algorithm is based, including cliques and clique adjacency, to allow the presence of multiple types of edges. In this section we highlight how well-understood concepts like cliques can be extended in different ways when multiple layers are considered. In Section 4 we go through the main algorithmic steps used to compute communities. We conclude this paper with a discussion of the complexity and implications of our approach, and a list of future works.
2 Preliminaries

2.1 Multilayer and multiplex networks

Multilayer networks are data structures where the same node can belong to multiple contexts called layers. An example with four nodes and two layers is shown in Figure 1a.

Definition 1 (Multilayer network) Given a set of nodes $\mathcal{N}$ and a set of layers $\mathcal{L}$, a multilayer network is defined as a quadruple $M = (\mathcal{N}, \mathcal{L}, V, E)$ where $(V, E)$ is a graph and $V \subseteq \mathcal{N} \times \mathcal{L}$.

In this paper we focus on a specific type of multilayer network called multiplex network, where edges can only exist between nodes in the same layer. We can represent multiplex networks separating the nodes into different layers, as in Figure 1b, or use an alternative representation as an edge-labeled multi-graph, as in Figure 1c, where different colors represent the different layers.

Definition 2 (Multiplex network) A multiplex network is a multilayer network where $((n_1, l_1), (n_2, l_2)) \in E$ implies that $l_1 = l_2$.

2.2 Communities in multilayer networks

Given a multiplex network $M = (\mathcal{N}, \mathcal{L}, V, E)$, we can group its nodes $\mathcal{N}$ into $q$ sets $C = \{C_1, \ldots, C_q\}$, where we allow different groups to overlap. For example, in Figure 1b we can group the nodes in two sets $C_1 = \{n_1, n_2, n_4\}$ and $C_2 = \{n_2, n_3, n_4\}$. Among the many possible ways of grouping nodes, we want to find one representing the (overlapping) community structure of the network.

Some methods, like [21], use a quality function to compare different ways of assigning the nodes to groups, for example assigning a higher quality to solutions where nodes that are connected together are included in the same
group. In other cases, the shape of the community structure is defined by the specific method used to discover it. For example, the clique percolation method (CPM) described in the next section provides a specific definition of overlapping community structure based on the concept of clique.

2.3 Clique percolation

The clique percolation method (CPM) was introduced by Palla et al. in 2005 [9]. For a given $k$, CPM builds up communities from $k$-cliques, that is, complete subgraphs in the network with $k$ nodes. Two $k$-cliques are said to be adjacent if they share $k−1$ nodes. A $k$-clique community is defined as a maximal union of $k$-cliques that can be reached from each other through a series of adjacent $k$-cliques. In general, if the number of links is increased above some critical point, a giant community would appear that covers a vast part of the system. Therefore, $k$ is chosen as the smallest value where no giant community appears. CPM allows overlapping communities in a natural way as a node can belong to multiple cliques.

Figure 2 shows a simplified example of how CPM works. Given a simple graph, first cliques are identified (in this example, we only have 3-cliques for simplicity), then adjacent cliques are grouped together to define two communities with one common node.

3 Multiplex clique percolation

Our extended CPM algorithm for multiplex networks (CPM$^m$) follows the same main steps of CPM. However, the concepts on which it is based must be extended to multiplex networks. In particular, we need to define:

1. What is a clique on multiple layers/edge types.
2. When two multiplex cliques can be considered adjacent.

3. How adjacent cliques should be grouped to build communities.

### 3.1 Cliques on multiple layers

While a clique on a simple graph is a well understood structure, defined as a set of nodes that are all connected to each other, the same concept can be extended in different ways for multiplex networks depending on how multiple edge types can contribute to the clique connectivity. For example, we may require that a clique on $m$ layers contains all the possible edges on all these layers, or we can allow different pairs of nodes to be connected by different edge types. In the following we provide three increasingly relaxed alternative definitions of multiplex clique. We notate $L_{ij}$ the set of edge types (layers) between nodes $i$ and $j$.

- **$k$-$m$-type(1) cliques**
  We define a $k - m - type(1)$ clique as a subgraph in the multilayer network with $k$ nodes that includes a combination of at least $m$ different $k$-cliques coming from $m$ different networks. In other words, a $k - m - type(1)$ clique is a subgraph with $k$ nodes like $C$ where

$$\left| \bigcap_{i,j \in C} L_{ij} \right| \geq m$$

- **$k$-$m$-type(2) cliques**
  A $k - m - type(2)$ clique is defined as a subgraph with $k$ nodes where each two vertices of the subgraph are connected on at least $m$ networks. Therefore, for a $k - m - type(2)$ clique like $C$ we have

$$\left| L_{ij} \right| \geq m$$

- **$k$-$m$-type(3) cliques**
  A $k - m - type(3)$ clique is defined as a subgraph with $k$ nodes where each two vertices of the subgraph are directly connected on at least one network and at least $m$ different edge types exist among all edges in the subgraph. Therefore, for a $k - m - type(3)$ clique like $C$ we have

$$\left| \bigcup_{i,j \in C} L_{ij} \right| \geq m$$

A $k$-$m$-type(1) clique is also a $k$-$m$-type(2) clique and a $k$-$m$-type(2) clique is also a $k$-$m$-type(3) clique. Similar to the case of cliques on simple graphs, we can have the concept of maximality of cliques in multilayer networks. For $a \in \{1, 2, 3\}$, an induced subgraph like $C$ is a maximal $k$-$m$-type(a) clique if: 1) $C$ is a $k$-$m$-type(a) clique, 2) There is no $m' > m$ such
that $C$ is also $k$-$m'$-type(a) clique, or in other words, $C$ is maximal on $m$, and 3) $C$ is not included in any $k'$-$m$-type(a) clique where $k' > k$, in other words, $C$ is maximal on $k$.

The graph in Figure 3(a) is a maximal 3-3-type(1) clique which is also a maximal 3-3-type(2) and a maximal 3-3-type(3) clique. The graph in Figure 3(b) is a maximal 3-2-type(1), a maximal 3-3-type(2) and a maximal 3-6-type(3) clique. The graph in Figure 3(c) is not any type(1) clique, but it is a maximal 3-1-type(2) and a maximal 3-5-type(3) clique. It is worth noting that the existing hierarchy in the definition of cliques is not preserved for maximal cliques; e.g. figure 1(b) is a maximal 3-2-type(1) clique but not a maximal 3-2-type(3) clique.

### 3.2 Adjacency

As a generalization of the definition of adjacency, we first suppose that the adjacent $k$-$m$-type(1) cliques should share $k - 1$ nodes as the only constraint on adjacency. Figure 4(a) shows a series of adjacent 3-3-type(1) cliques. As we see, adjacent cliques do not necessarily share any edge types on all pairs and they might share edge types only on their common pairs of nodes (e.g., cliques 1 and 2). It is worth noting that more diversity among the edge types in external connections of adjacent cliques results in denser internal connection. In addition, cliques which have a distance of only one clique, still share some edge types on some of their pairs of nodes (e.g., cliques 1 and 3 in Figure 4(a)), however, when the distance of cliques becomes greater than one, they might have completely different edge labels (cliques 1 and 4 in Figure 4(a)).

To gain more uniformity among edge types, we also consider a constraint respecting the edge labels for adjacency of cliques. Two $k$-$m$-type(1) cliques are said to be $m'$-adjacent if they share $k - 1$ nodes and they also share at least $m'$ edge types on all of their pairs of nodes. Figure 4(b) shows a
series of 3-adjacent 3-3-type(1) cliques. As we see, although this constraint enforce uniformity among edge labels in small neighborhoods, it cannot still guarantee the repetition of edge types for all cliques which are reachable from each other (e.g., cliques 1 and 2 in Figure 4(b)).

3.3 Communities
To enforce uniformity among edge types throughout the whole community, we need more constraints than constraints on cliques’ adjacency. We define a \((k - m - \text{type}(1))_{(m',m'')}\) community as the maximal union of \(m'\)-adjacent \(k\)-\(m\)-type(1) cliques where all cliques share at least \(m''\) edge types on all of their pairs of nodes. Therefore, a \((k - m - \text{type}(1))_{(m',m'')}\) community is a group of nodes where the nodes form \(k\)-clique communities on at least \(m''\) different layers.

4 Algorithm (sketch)
In this section we discuss the algorithmic steps needed to detect all communities in a multiplex network. We will focus on the strongest definition of community, that is, \((k - m - \text{type}(1))_{(m',m'')}\) communities, and without loss of generality we will assume that \(m = m' = m''\).

• Locating the cliques
  Our method is based on first locating all \(k - m - \text{type}(1)\) cliques. The example in the top left of Figure 5 shows all 3 – 3 – type(1) cliques in a multilayer network.
Clique-Clique Adjacency Matrix
In a simple graph, each clique can be included in exactly one community, therefore, communities can be identified from a clique-clique overlap matrix (see [9] for the details). However, this statement is not necessarily true for $k-m$-type(1) cliques and $(k-m-type(1))_{(m,m)}$ communities in multiplex networks. As an example, $C_8$ in the top left of Figure 5 is a (non-maximal) 3-3-type(1) clique and it has two 3-adjacent cliques, $C_7$ and $C_9$. However, $C_7$ and $C_9$ cannot be included in the same $(3-3-type(1))_{(3,3)}$ community as they do not share the required number of edge types. Therefore, $C_8$ can be included in at least two different communities, one including $C_7$ and the other including $C_9$. Because of the more complicated relations between cliques, we use a clique-clique adjacency matrix rather than the overlap matrix, represented as an adjacency graph in Figure 5. The corresponding adjacency matrix is a binary symmetric matrix with diagonal of zeros. For given $k$ and $m$, the elements of the matrix indicate whether pairs of cliques are $m$-adjacent or not.

From Adjacency Matrix to Communities
As previously mentioned, each clique can be included in different com-
munities with different combinations of its adjacent cliques. Here our objective is using the adjacency matrix and the information regarding the edge labels simultaneously to find communities in the multilayer network. The adjacency matrix corresponds to the adjacency matrix of a graph where the nodes represent cliques. We refer to this graph as clique-adjacency graph. Two cliques can be included in at least one community if: 1) there exists a path between the corresponding nodes in the graph, and 2) for all nodes in the path the corresponding cliques share at least $m$ edge types on all of their pairs. We call the latter rule as cliques’ constraint. It should be noted that cliques with these specifications can be included in more than one community. It can be realized that each community corresponds to a maximal tree in the modeled graph where the cliques’ constraint holds for all nodes in the tree. So the problem is equivalent to recognizing all such maximal trees in the graph.

The graph in the top right of Figure 5 shows the clique-adjacency graph for the multilayer network in this figure for $k = m = 3$ where the set of common edge types between pairs of nodes in each clique has been associated to the corresponding nodes in the clique-adjacency graph. In this simple example, the graph does not contain any cycle. In the bottom right of the figure, we can see all maximal trees with mentioned constraint in this graph. As we see, $C_8$ can be included in two communities; one formed by the cliques $\{C_6, C_7, C_8\}$ and the other formed by $\{C_1, C_2, C_5, C_9, C_{10}\}$. No new clique can be added to these sets without violating the cliques’ constraint. The bottom left of the figure shows all $(3 - 3 - type(1))_{(3,3)}$ communities in this example.

5 Discussion and future work

In this paper we have extended the CPM method to be able to identify overlapping communities in multiplex networks. We have focused on the formal definition of the method, discussing how to extend existing concepts to the multiplex context. Our next step is to validate the method on real datasets, to study its empirical behavior.

However, from the formal extension of the method some interesting aspects already emerge:

- Multiple extensions can be defined based on how we want the different layers to contribute to the community structure. Our three extensions of the concept of clique are related to each other, representing a range of approaches from the more restrictive to the more flexible. Therefore, we can limit the choice of the most appropriate method to a few options, and through the experimental validation of the approach we aim
at clearly characterizing the expected kind of communities produced by each alternative.

• Clique adjacency must consider both the nodes and the edge types.

• Constraining clique adjacency to a maximum number of different edge types, so that different cliques contain some common edge types, is not sufficient to enforce uniformity at the community level, that is, inside the same group of adjacency-reachable cliques there can be cliques not sharing any edge type. To obtain more homogeneous communities we then need to define a limit to the heterogeneity we want to accept.

• The attempt to keep communities homogeneous results in a phenomenon not visible when single graphs and the original method are used. While in CPM the same node can belong to multiple communities, in CPM\textsuperscript{m} whole cliques can belong to different communities, as it is the case for C\textsubscript{8} in Figure 5. In CPM, the whole subgraph containing C\textsubscript{8} would result in a single big community, without being broken into two overlapping ones. This shows an example where not considering the information about the different edge types would result in the detection of larger communities scattered on several layers.

We are currently in the process of testing our algorithm on real datasets. Therefore, we will soon update this paper with evidence about the computational complexity of CPM\textsuperscript{m} and the quality or kind of clusters it can identify. From the formalization of the method, we can see that it is at least as complex as CPM in the worst case. Our hypothesis, that we are currently testing, is that CPM\textsuperscript{m} has a similar practical behavior, where the sparsity of the networks and their community structures make the practical execution time acceptable in real cases.

References


