REFINED QUANTUM INVARIANTS FOR THREE-MANIFOLDS WITH STRUCTURE

CHRISTIAN BLANCHET
Département de Mathématiques
2 rue de la Houssinière
BP 92208
F-44322 Nantes Cedex 3, France
E-mail blanchet@math.univ-nantes.fr

Introduction.

Following Witten's interpretation ([Wi]) of the Jones polynomial ([Jo]) in terms of *Topological Quantum Field Theory* , Reshetikhin and Turaev ([RT]) and then many others have constructed invariants of 3-manifolds now called *Quantum Invariants* (see [Tu2] for a detailed exposition, and [Vo] for a survey). The construction of Reshetikhin and Turaev involves representation theory of quantum groups. This point of view gives a deep insight into the algebraic questions related to the subject, however it is not immediately accessible for the beginner. Among these quantum invariants those called the $SU(2)$-invariants can be obtained easily from the skein theory associated with the Kauffman bracket ([Ka]). This was first observed by Lickorish ([Li1],[Li2],[Li3]) and then systematically studied in [BHMV1]. Section 1 deals with this skein method. Starting with a formal skein theory, we discuss the construction of 3-manifolds invariants, and give the simplest examples. We think that this could be helpful for the beginner and hope that the method will be applied to new examples.

Once one has constructed a lot of 3-manifold invariants, the question is to understand their meaning, and this is far from clear at the moment. Let us discuss the example of $\tau^{SU(2)}$ at $q = e^{\frac{i\pi}{8}}$ ([KM]) which corresponds to θ_8 in [BHMV1] and [Bl1]. This invariant decomposes as a sum, over all spin structures on the manifold, of spin invariants. Moreover the spin invariant is (a version of) the well known Rochlin invariant. This was first observed by Kirby and Melvin and generalized independently in [KM], [Tu1] and [Bl1]. This example shows that considering refined invariants can help in understanding their geometrical meaning. Section 2 is about cohomological refinements of quantum invariants. According to H. Murakami ([Mu]) $\tau_r^{SU(n)}$ admits such refinements, for conveniently chosen r. He states also a decomposition formula in which cohomology classes are replaced by some spin type structures. In section 3, a topological definition of these structures is

1991 *Mathematics Subject Classification*: Primary 57N10; Secondary 57N25.

given. In spite of its simplicity, this description seems to be new.

In this paper we only consider invariants of 3-manifolds. We will say in section 4 a few words about extending this to a whole Topological Quantum Field Theory. Following the methods developed in [BHMV3] together with N. Habegger, G. Masbaum and P. Vogel, we have, in a joint work with G. Masbaum ([BM]), constructed and studied this extension, for the spin refined invariants obtained from the Kauffman bracket. Understanding other refined theories is our challenge!

The content of this paper was exposed at the *Mini-semester in Knot Theory* in Warsaw (Summer 1995). We thank the organizers and the Stefan Banach Center for their invitation and hospitality.

1. Three-manifolds invariants derived from a skein theory

Various skein modules of 3-manifolds have been defined and studied (see [P], [HP]). In each case the modules have presentations in which generators are links, and relations are local (skein) relations between them. The definition below gives a general nonsense. We emphasize the functorial property. Here we consider embeddings of 3-manifolds. In a concrete theory generated by links, an embedding $M \to M'$ carries links in M to links in M'.

Let $\mathcal{S}$ be a functor from the category of compact oriented 3-manifolds with isotopy classes of oriented embeddings to the category of k-modules. We will say that $\mathcal{S}$ is a *skein theory* if $\mathcal{S}$ is monoidal and involutive (see [ML]). Here k is a commutative ring with unit, supposed to be equipped with an involutive automorphism $\lambda \mapsto \overline{\lambda}$).

Remark 1. The monoidal property says that, up to canonical isomorphisms, one has $\mathcal{S}(M_1 \amalg M_2) = \mathcal{S}(M_1) \otimes \mathcal{S}(M_2)$ and $\mathcal{S}(\emptyset) = k$. Using the embedding $\varepsilon : \emptyset \to M$, we get the vector $\mathcal{S}(\varepsilon)(1) \in \mathcal{S}(M)$. In a concrete skein theory, defined using links, this vector is represented by the empty link in M. We will denote it by $\emptyset$.

Remark 2. Involutivity says that, up to a canonical isomorphism, the module $\mathcal{S}(-M)$ is equal to $\overline{\mathcal{S}(M)}$. If a fixed oriented diffeomorphism $g : M \xrightarrow{\sim} -M$ is given, then $\mathcal{S}(g)$ defines an anti-linear automorphism of $\mathcal{S}(M)$ (a linear isomorphism from $\mathcal{S}(M)$ to $\overline{\mathcal{S}(M)}$). This automorphism will be called the *mirror* and denoted $x \mapsto \overline{x}$. The map $(z, \alpha) \mapsto (\overline{z}, \alpha)$ gives such a g, in $\mathbf{D}^2 \times \mathbf{S}^1$ as well as in $\mathbf{S}^3 \subset \mathbf{C}^2$.

Remark 3. Using any oriented embedding $\mathbf{D}^3 \amalg \mathbf{D}^3 \to \mathbf{D}^3$ we get a product on the module associated with the 3-ball $\mathcal{S}(\mathbf{D}^3)$. We also get a product on the module $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$, by using a standard embedding $\mathbf{D}^2 \times \mathbf{S}^1 \amalg \mathbf{D}^2 \times \mathbf{S}^1 \to \mathbf{D}^2 \times \mathbf{S}^1$. This makes $\mathcal{S}(\mathbf{D}^3)$ and $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$ into commutative algebras with unit $1 = \emptyset$.

To an oriented embedding of a disjoint union of solid tori

$$g = \coprod_{i=1}^{m} g_i : \coprod_{i=1}^{m} \mathbf{D}^2_i \times \mathbf{S}^1_i \to M$$

is associated a multilinear map

$$\mathcal{S}(g) : \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)^{\otimes m} \to \mathcal{S}(M)$$

By the isotopy hypothesis, this map only depends on the framed link $L = (L_1, \ldots, L_m)$ underlying g.

We call a *bracket* any linear map $\langle\ldots\rangle : \mathcal{S}(\mathbf{S}^3) \to k$ involutive and such that the composition

$$\mathcal{S}(\mathbf{D}^3) \to \mathcal{S}(\mathbf{S}^3) \to k$$

is multiplicative. Here involutive means that the mirror image (see remark 2 above) is sent to the conjugate.

Notations. As already explained, a framed link $L = (L_1, \ldots, L_m)$ in the sphere gives a multilinear map $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)^{\otimes m} \to k$. The image of $x_1 \otimes \ldots \otimes x_m$ by this map is denoted by $\langle L_1(x_1), \ldots, L_m(x_m)\rangle$ or $\langle L(x_1, \ldots, x_m)\rangle$. Such an element is said to be obtained by *skein cabling*, or simply by *cabling*.
For $\epsilon \in \{-1, 0, 1\}$ we note U_ϵ the unknot with framing ϵ, and H_ϵ the Hopf link with linking number one and both components having framing ϵ.

A framed link L determines by surgery a 3-manifold which will be denoted by $S^3(L)$ (every compact oriented 3-manifold can be obtained in this way). As a consequence of Kirby's theorem ([Ki]), we have the following proposition.

Proposition 1.1. *If $\omega \in \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$ satisfies*

(K) $\forall x \in \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$ $\langle H_1(x, \omega)\rangle = \langle U_0(x)\rangle\langle U_1(\omega)\rangle$ and $\langle U_1(\omega)\rangle$ is invertible

then

$$\frac{\langle L(\omega, \ldots, \omega)\rangle}{\langle U_1(\omega)\rangle^{b_+}\langle U_{-1}(\omega)\rangle^{b_-}}$$

is an invariant of the surgered manifold $M = \mathbf{S}^3(L)$. Here b_+ (resp. b_-) is the number of positive (resp. negative) eigenvalues of the linking matrix B_L associated with L.

Remark. The vector ω is defined up to a multiplicative invertible factor λ and up to the kernel $\mathcal{N}_1$ of the bilinear form $\langle H_1(\ ,\ )\rangle$. The factor λ multiplies the invariant by $\lambda^{b_1(M)}$ where $b_1(M)$ is the first Betti number of M. Adding an element of $\mathcal{N}_1$ does not change the invariant; this is a corollary of the following lemma whose proof will be given forward.

Lemma 1.2. *If (K) has a solution, then for any framed link in $\mathbf{S}^3$ the multilinear form $\langle L(\ldots)\rangle$ is well defined on the quotient $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)/\mathcal{N}_1$.*

Remark. Understanding the kernel $\mathcal{N}_1$ is a key point in this construction. Multiplying by any x, in the algebra $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$, is a self-adjoint operator with respect to $\langle H_1(\ ,\ )\rangle$, thus $\mathcal{N}_1$ is an ideal. The first condition in (K) can be written

$$\forall x\ (x - \langle U_0(x)\rangle)\omega \in \mathcal{N}_1$$

Moreover it is sufficient above to consider x in a set of generators of $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$ as an algebra.

About the proof of proposition 1.1. This proposition can be proved using the well known Kirby theorem [Ki] as refined in [FR] and [RT] (see [Tu2] ch. 2). There are two non standard points here.

First one must check that changing the orientation of one component of the link L does not modify the value of $\langle L_1(\omega), \ldots, L_m(\omega)\rangle$. This will follows from lemma 1.3 .

Second one must show that (K) implies that $\langle U_{-1}(\omega)\rangle$ is also invertible. This is a consequence of lemma 1.4 .

Lemma 1.3. *If ω is a solution of (K), then the skein element $\breve{\omega} = \mathcal{S}(j)(\omega)$, where j is the diffeomorphism $(z, \alpha) \mapsto (\overline{z}, \overline{\alpha})$ of $\mathbf{D}^2 \times \mathbf{S}^1$, is equal to ω in the quotient $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)/\mathcal{N}_1$.*

Lemma 1.4. *If ω is a solution of (K). Let $\overline{\omega} \in \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$ be the mirror image of ω. In the quotient $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)/\mathcal{N}_1$, one has $\overline{\omega} = \lambda\omega$ with $\lambda\overline{\lambda} = 1$.*

Recall (see remark 2 above) that $\overline{\omega}$ is defined using the diffeomorphism $(z, \alpha) \mapsto (\overline{z}, \alpha)$.

As a consequence, after multiplying by a square root of λ (extend the scalars if necessary), we get a solution of (K) equal to its mirror image. The corresponding invariant of 3-manifolds is then involutive (M and $-M$ have conjugate invariants).

Proof of lemma 1.3. Let us denote, for $x \in \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$, $\mathcal{S}(j)(x) = \breve{x}$. By isotopy, for any $x \in \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$,

$$\langle U_\epsilon(\breve{x})\rangle = \langle U_\epsilon(x)\rangle \quad \epsilon \in \{0, 1\}, \text{ and } \langle H_1(x, \breve{\omega})\rangle = \langle H_1(\breve{x}, \omega)\rangle$$

It follows that $\breve{\omega}$ is a solution of (K), equal to ω up to $\mathcal{N}_1$.

Notations. A right handed twist induces an automorphism of $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$ denoted by t. For $x \in \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$, let c_x be the operator defined by $c_x(y) = h(x \otimes y)$, where h is induced by a link in the torus $\mathbf{D}^2 \times \mathbf{S}^1$ whose first component is parallel to a meridian and whose second component is the core of the torus (standardly framed). Note that t and c_x commute.

The map $c = [x \mapsto c_x]$ is a representation of $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$ onto itself, dual to multiplication with respect to the bilinear form $\langle H_0(\ ,\ )\rangle$. Namely

$$\forall x, y, z \in \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1) \quad \langle H_0(c_x(y), z)\rangle = \langle H_0(y, xz)\rangle$$

By computing in two different ways the expression $\langle H_0(t(x)t(\breve{\omega}), t(\omega))\rangle$ we have.

Lemma 1.5. If ω is a solution of (K), then

$$\forall x \in \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1) \quad \langle H_0(x, \omega)\rangle = \langle H_0(t(x), \omega)\rangle$$

Proof of lemma 1.4. Using the lemma above, we have first

$$\begin{aligned} \langle H_0(x\omega, \omega)\rangle &= \langle H_0(t(x\omega), \omega)\rangle \\ &= \langle U_0(x)\rangle\langle U_{-1}(\omega)\rangle\langle U_1(\omega)\rangle \end{aligned}$$

We have also

$$\begin{aligned} \langle H_0(x\omega, \omega)\rangle &= \langle H_0(\omega, c_x(\omega))\rangle \\ &= \langle H_0(\omega, tc_x(\omega))\rangle \\ &= \langle H_0(x\omega, t(\omega))\rangle \\ &= \langle U_{-1}(x\omega)\rangle\langle U_1(\omega)\rangle \end{aligned}$$

Hence

$$\forall x \in \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1) \quad \langle U_{-1}(\overline{x}\omega)\rangle = \langle U_0(\overline{x})\rangle\langle U_{-1}(\omega)\rangle$$

Using the mirror automorphism we show that $\overline{\omega}$ is a solution of (K). Thus, modulo $\mathcal{N}_1$, $\overline{\omega} = \lambda\omega$. Taking the mirror once more, $\omega = \lambda\overline{\lambda}\omega$. Using invertibility of $\langle U_1(\omega)\rangle$, we have $\lambda\overline{\lambda} = 1$.

Note. The drawings corresponding to the computation above prove the refinement of Kirby's theorem ([RT]) saying that negative Fenn-Rourke moves can be deduced from positive and special negative ones.

Proof of lemma 1.2. Let $L = (L_1, \dots, L_m)$ be a framed link in $\mathbf{S}^3$, we have to show that $\langle L_1(x_1), \dots, L_m(x_m)\rangle$ is zero if x_1 is in $\mathcal{N}_1$. The proof is in three steps.

If $L_1 = U_1$, the definition of $\mathcal{N}_1$ gives the result.

Then using the properties of ω it is shown that $\mathcal{N}_1$ is fixed by the automorphism t of $\mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$ induced by a right handed twist. This gives the result if L_1 is the unknot with any framing.

In the general case, the component L_1 can be unknotted by changing some crossings. Inserting an ω around each changed crossing reduces the problem to the preceding case.

Example 1. Kauffman bracket skein theory. Given an invertible element A in a ring k (equipped with an involution sending A to A^{-1}), the skein module $\mathcal{K}(M)$ is the free k-module generated by isotopy classes of banded links (embedded copies of $\mathbf{S}^1 \times [0,1]$), quotiented by the usual Kauffman relations ([Ka]). The equation (K) has been discussed in [BHMV1]. The result is that A must be a root of unity whose order is an even integer $2p$. For each p there is, up to changing the ring and normalizing, a unique invariant θ_p. In the notation coming from Chern-Simons gauge theory, θ_p corresponds to the $SU(2)$-invariant for even p and to the $SO(3)$-invariant for odd p.

Example 2. Skein theory associated with linking. Let q be an invertible element in k (equipped with an involution sending q to q^{-1}). Define the skein module $\mathcal{L}(M)$ to be the free k-module generated by isotopy classes of framed links, quotiented by the local relations

$$L_+ = qL_0 \quad L_- = q^{-1}L_0 \quad L \amalg U_0 = L$$

Here L_+, L_- et L_0 are the same except in a ball $D^2 \times [0,1]$ where their projection on the disc $D^2 \times \{0\}$ have respectively, a positive crossing, a negative crossing and no crossing; U_0 is an unknot with the framing given by the disc it bounds. The algebra $\mathcal{L}(\mathbf{D}^2 \times \mathbf{S}^1)$ is isomorphic to $k[y, y^{-1}]$. The condition (K) has a solution only if q is a root of unity whose order is either an odd integer N, or is $2N$ with N even. In each case, if N is invertible in k, an invariant is produced, which is equal to the Z_N-invariant derived from linking matrices in [MOO].

Example 3. HOMFLY theory. Quantum $SU(n)$-invariants of 3-manifolds have been obtained by Turaev and Wenzl ([TW]) and studied by Kohno and Takata ([KT]). Recently, following a combinatorial approach of Morton ([Mo]), Yokota ([Yo]) gave a construction of this invariants based on HOMFLY skein theory. The case $n = 3$ was already given by Ohtsuki and Yamada ([OY]). The construction of Yokota enters easily in our description.

The $SU(n)$ specialized HOMFLY skein module of a 3-manifold M is defined to be the free module generated by framed links in M, quotiented by the relations

$$aL_+ - a^{-1}L_- = (a^n - a^{-n})L_0$$

$$L \amalg U_0 = \frac{a^{n^2} - a^{-n^2}}{a^n - a^{-n}} L$$

$$L^{+f} = a^{f(n^2-1)}L$$

In the first two relations, notations are standards; in the third one L^{+f} is obtained from L by adding the integer f to the framing. Here a is an invertible element in k (equipped with an involution sending a to a^{-1}).

If a is a primitive $(k+n)n$-th root of unity, then Lemma 3.2, and Proposition 4.3 in [Yo] show that (K) has a solution. The needed computation is related with combinatorics in the algebra of Young diagrams studied by Morton and Aiston ([MA]).

2. Cohomological refinements.

The homology of the surgered manifold $M = \mathbf{S}^3(L)$ can be described using a Mayer-Vietoris argument. We want to give a precise statement for the group $H^1(M;\mathbf{Z}/n)$. The group $H^1(\mathbf{S}^3 - L;\mathbf{Z}/n)$ is canonically isomorphic to $(\mathbf{Z}/n)^m$. The inclusion map induces a monomorphism $\phi_L : H^1(M;\mathbf{Z}/n) \to H^1(\mathbf{S}^3 - L;\mathbf{Z}/n) \simeq (\mathbf{Z}/n)^m$ whose image is the kernel of the linking matrix B_L, reduced modulo n. An elementary Kirby move between L and L' gives a diffeomorphism between $\mathbf{S}^3(L')$ and $\mathbf{S}^3(L)$ (defined up to isotopy). This diffeomorphism induces the isomorphism

$$\phi_{L,L'} : \mathrm{Ker}(B_L) \simeq H^1(\mathbf{S}^3(L);\mathbf{Z}/n) \to H^1(\mathbf{S}^3(L');\mathbf{Z}/n) \simeq \mathrm{Ker}(B_{L'})$$

The formula for the usual positive Fenn-Rourke move is

$$\phi_{L,L'}(c_1,\ldots,c_{m-1},0) = (c_1,\ldots,c_{m-1},c'_m)$$
$$\text{with } \; c'_m = -\sum_{i<m} b'_{im} c_i$$

Here L_m is a trivial component, with framing one, in a ball; the other components of L slide over this component to obtain the link L', b'_{im} is the corresponding coefficient of the matrix $B_{L'}$.

This can be used to construct invariants for pairs (M,σ), $\sigma \in H^1(M;\mathbf{Z}/n)$. Suppose that the skein module $\mathcal{S} = \mathcal{S}(\mathbf{D}^2 \times \mathbf{S}^1)$ is $\mathbf{Z}/n$-graded as an algebra

$$\mathcal{S} = \bigoplus_{\nu=0}^{n-1} \mathcal{S}_\nu$$

Suppose moreover that this grading is *compatible with cabling*. By this we mean that for any framed link $L = (L_1,\ldots,L_m)$ in $\mathbf{D}^2 \times \mathbf{S}^1$, and for any homogeneous elements $x_1,\ldots,x_m$ of respective degrees $d_1,\ldots,d_m$, the skein element $L(x_1,\ldots,x_m)$ is homogeneous of degree equal to $\sum \lambda_i d_i$, where λ_i is the algebraic intersection of L_i with a meridian disc. This implies that the twist t and the c_x are graded operators.

PROPOSITION 2.1. *If the vectors $\omega_\nu \in \mathcal{S}_\nu$, $\nu = 0,\ldots,n-1$, satisfy the condition*

$$\forall \nu \; \forall x_\nu \in \mathcal{S}_\nu \; \langle H_1(x_\nu,\omega_{-\nu})\rangle = \langle U_0(x_\nu)\rangle\langle U_1(\omega_0)\rangle \text{ and } \langle U_1(\omega_0)\rangle \text{ is invertible}$$

then, provided $(c_1,\ldots,c_m)$ lies in the kernel of B_L,

$$\frac{\langle L(\omega_{c_1},\ldots,\omega_{c_m})\rangle}{\langle U_1(\omega_0)\rangle^{b_+}\langle U_{-1}(\omega_0)\rangle^{b_-}}$$

is an invariant of the surgered manifold $M = \mathbf{S}^3(L)$ equipped with the cohomology class $\sigma = \phi_L^{-1}(c_1,\ldots,c_m) \in H^1(M;\mathbf{Z}/n)$.

The condition in the hypothesis above can be reduced to a unique equation if the grading satisfies the condition (WG) below (we will say that the algebra $\mathcal{S}$ is well graded).

(WG) For all ν there exists $y_\nu \in \mathcal{S}_\nu$ such that $\mathcal{S}_\nu = y_\nu \mathcal{S}_0$ and $\langle U_0(y_\nu)\rangle$ is invertible.

LEMMA 2.2. If the grading of $\mathcal{S}$ satisfies (WG), and $\omega_0 \in \mathcal{S}_0$ satisfies

$$\forall x_0 \in \mathcal{S}_0 \; \langle H_1(x_0,\omega_0)\rangle = \langle U_0(x_0)\rangle\langle U_1(\omega_0)\rangle \text{ and } \langle U_1(\omega_0)\rangle \text{ is invertible}$$

then the hypothesis of Proposition 2.1 is satisfied.

P r o o f. Take $\omega_\nu = \langle U_0(y_\nu)\rangle^{-1} y_\nu \omega_0$.

In the interesting known examples, the cohomological invariant, constructed with a given bracket, appears as a refinement of the one without structure in the following precise sense: the latter decomposes as a sum, over all cohomological classes, of the refined ones. The following theorem gives a sufficient condition for existence of a cohomological invariant satisfying such a decomposition property.

THEOREM 2.3. *Suppose the grading of $\mathcal{S}$ satisfies (WG), and $\omega_0 \in \mathcal{S}_0$ is such that*

$$\text{(KC)} \quad \begin{cases} \forall x_0 \in \mathcal{S}_0 \ \langle H_1(x_0,\omega_0)\rangle = \langle U_0(x_0)\rangle\langle U_1(\omega_0)\rangle \text{ and } \langle U_1(\omega_0)\rangle \text{ is invertible} \\ \forall \nu \neq 0 \ \forall x_\nu \in \mathcal{S}_\nu \ \langle H_1(x_\nu,\omega_0)\rangle = 0 \end{cases}$$

then there exists $\omega_\nu \in \mathcal{S}_\nu$, $\nu = 0,\dots,n-1$ such that the formula

$$\tau(M,\sigma) = \frac{\langle L(\omega_{c_1},\dots,\omega_{c_m})\rangle}{\langle U_1(\omega_0)\rangle^{b_+}\langle U_{-1}(\omega_0)\rangle^{b_-}}$$

is an invariant of the surgered manifold $M = \mathbf{S}^3(L)$ equipped with the cohomology class $\sigma = \phi_L^{-1}(c_1,\dots,c_n) \in H^1(M;\mathbf{Z}/n)$.
Moreover, if $\omega = \omega_0 + \ldots + \omega_{n-1}$ then

$$\tau(M) = \frac{\langle L(\omega,\dots,\omega)\rangle}{\langle U_1(\omega)\rangle^{b_+}\langle U_{-1}(\omega)\rangle^{b_-}}$$

is an invariant of the surgered manifold $M = \mathbf{S}^3(L)$ which satisfies the decomposition property

$$\forall M \quad \tau(M) = \sum_{\sigma \in H^1(M;\mathbf{Z}/n)} \tau(M,\sigma)$$

The decomposition formula is a consequence of the lemma below, which can be shown as for lemma 1.2.

LEMMA 2.4. *In the hypothesis of theorem 2.3, $\langle L(\omega_{c_1},\dots,\omega_{c_m})\rangle = 0$ if $(c_1,\dots,c_m)$ is not in the kernel of B_L mod n.*

E x a m p l e 1. In the Kauffman bracket skein theory, the algebra $\mathcal{K} = \mathcal{K}(\mathbf{D}^2 \times \mathbf{S}^1)$ is $\mathbf{Z}/2$-graded. The equation (KC) has a solution only if A is a root of unity whose order is congruent to 8 modulo 16. The corresponding decomposition theorem has been stated in [Bl1]. Note that, although $\mathcal{K} \simeq k[z]$, the degree does not give a usable $\mathbf{Z}/n$-grading on the skein algebra $\mathcal{K}$, for $n > 2$. This is because the Kauffman skein relation in the solid torus is not homogeneous with respect to this degree.

E x a m p l e 2. S k e i n t h e o r y $\mathcal{L}$ a s s o c i a t e d w i t h l i n k i n g. Using Gauss sum computations, we can show that if $N = 2^{2l+1}$, the invariant Z_N of section 1, example 2, admits a $\mathbf{Z}/2^l$ cohomological refinement.

E x a m p l e 3. In [Mu], H. Murakami gives cohomological refinements of the quantum $SU(n)$-invariants. This refinements as well as the corresponding decomposition formula can be obtained using HOMFLY skein theory.

3. Spin type structures.

In [Mu], H. Murakami states also a decomposition formula in which cohomology classes are replaced by some spin type structures (see remark 2.7 in his paper). He observes that for $n = 2$ these are spin structures, and the corresponding refinements were studied in [KM] and [Bl1]. For $n > 2$, he only gives a combinatorial description of the structures, and asks for a topological interpretation. We are going to give a topological definition for these structures. From the combinatorial description in the case of 3-manifolds we will then obtain a version of the results of the previous section for 3-manifolds equipped with these spin type structures. More about these structures will be found in [Bl2].

Suppose n is an even integer. Then there exists, up to homotopy, a unique non trivial map $g : BSO \to K(\mathbf{Z}/n, 2)$. Define the fibration

$$\pi_n : BSpin(\mathbf{Z}/n) \to BSO$$

to be the pull-back, using g, of the path fibration over $K(\mathbf{Z}/n, 2)$. For $n = 2$ this construction is well known, and $BSpin(\mathbf{Z}/2) = BSpin$ is a classifying space for the universal covering $Spin$ of the group SO. The space $BSpin(\mathbf{Z}/n)$ is a classifying space for the non trivial central extension of the Lie group SO by $\mathbf{Z}/n$. This extension will be denoted by $Spin(\mathbf{Z}/n)$ in [Bl2] whence the notation $BSpin(\mathbf{Z}/n)$. Remark that $Spin(\mathbf{Z}/n)$ is a sub-group of $Spin^c$.

Now we can use the fibration π_n to define structures (see [St]). Let $\gamma_{Spin(\mathbf{Z}/n)} = \pi_n^*(\gamma_{SO})$ be the pull-back of the canonical vector bundle over BSO.

Definition. A $\mathbf{Z}/n$ *spin type structure* (or $Spin(\mathbf{Z}/n)$*-structure*, or *spin structure with mod n coefficients*) on a manifold M is an homotopy class of fiber maps from the stable tangent bundle τ_M to $\gamma_{Spin(\mathbf{Z}/n)}$.

If non empty the set of these structures, denoted $Spin(M;\mathbf{Z}/n)$, is affinely isomorphic to $H^1(M;\mathbf{Z}/n)$, by obstruction theory. Moreover the obstruction for existence is a class $w_2(M;\mathbf{Z}/n) \in H^2(M;\mathbf{Z}/n)$, which is the image of the Stiefel-Whitney class $w_2(M)$ by the homomorphism induced by the inclusion of coefficients $\mathbf{Z}/2 \hookrightarrow \mathbf{Z}/n$.

The Stiefel-Whitney class $w_2(M)$ is zero for every compact oriented 3-manifold, hence $\mathbf{Z}/n$ spin type structures exist on a 3-manifold $M = \mathbf{S}^3(L)$. The following theorem gives a combinatorial description of these structures. Recall that M is the boundary of a 4-manifold W_L called the trace of the surgery. To each $\sigma \in Spin(M;\mathbf{Z}/n)$ is associated a relative obstruction $w_2(\sigma;\mathbf{Z}/n)$ in $H^2(W_L, M;\mathbf{Z}/n)$. The group $H^2(W_L, M;\mathbf{Z}/n)$ is free of rank $m = \sharp L$. Taking the coordinates of the relative obstruction we get a map $\psi_L : Spin(M;\mathbf{Z}/n) \to (\mathbf{Z}/n)^m$.

Theorem 3.1. *The map ψ_L is injective, and its image is the set of those $(c_1, \ldots, c_m)$ which are solutions of the following ($\mathbf{Z}/n$-characteristic equation)*

$$B_L \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \frac{n}{2} \begin{pmatrix} b_{11} \\ \vdots \\ b_{mm} \end{pmatrix} \pmod n$$

Here the b_{ii} are the diagonal values of the linking matrix B_L.

Proof. First we compute the absolute obstruction $w_2(W_L;\mathbf{Z}/n) = \xi_*(w_2(W_L))$, where ξ_* is induced by the morphism of coefficients $\xi : \mathbf{Z}/2 \hookrightarrow \mathbf{Z}/n$. If x is an integral 2-cycle and $[x]_a$ denotes its homology class modulo an integer a, $w_2(W_L) \in H^2(W_L;\mathbf{Z}/2)$ is determined by the equation

$$\forall x \ < w_2(W_L), [x]_2 >= x.x \pmod 2$$

Hence $w_2(W_L;\mathbf{Z}/n)\in H^2(W_L;\mathbf{Z}/n)$ is determined by

$$\forall x \ < w_2(W_L;\mathbf{Z}/n),[x]_n >= \xi(x.x)=\frac{n}{2}x.x \pmod n$$

Now by functoriality, the relative obstruction lives in the inverse image of the absolute one under the map induced by inclusion $H^2(W_L,M;\mathbf{Z}/n)\to H^2(W_L;\mathbf{Z}/n)$. Using the affine structure over $H^1(M;\mathbf{Z}/n)$, we obtain an affine bijection between $Spin(M;\mathbf{Z}/n)$ and this inverse image. Whence the lemma by writing the equation above using coordinates.

As in section 2 there is a formula for the bijection $\psi_{L,L'}$ corresponding to a Kirby move. Using the $\mathbf{Z}/n$-characteristic equation we see that the coefficient for an unknotted component with framing ± 1 is $n/2$. For the usual positive Fenn-Rourke move, the formula is

$$\psi_{L,L'}(c_1,\ldots,c_{m-1},n/2)=(c_1,\ldots,c_{m-1},c'_m)$$
$$\text{with}\ \ c'_m=n/2-\sum_i b'_{im}c_i$$

PROPOSITION 3.2. *If the vectors $\omega_\nu\in\mathcal{S}_\nu$, $\nu=0,\ldots,n-1$, satisfy the condition*

$$\forall\nu\ \forall x_\nu\in\mathcal{S}_\nu\ \langle H_1(x_\nu,\omega_{n/2-\nu})\rangle=\langle U_0(x_\nu)\rangle\langle U_1(\omega_{n/2})\rangle \text{ and } \langle U_1(\omega_{n/2})\rangle \text{ is invertible}$$

then, provided $(c_1,\ldots,c_m)$ satisfies the $\mathbf{Z}/n$-characteristic equation,

$$\frac{\langle L(\omega_{c_1},\ldots,\omega_{c_m})\rangle}{\langle U_1(\omega_{n/2})\rangle^{b_+}\langle U_{-1}(\omega_{n/2})\rangle^{b_-}}$$

is an invariant of the surgered manifold $M=\mathbf{S}^3(L)$ equipped with the $\mathbf{Z}/n$ spin type structure $\sigma=\psi_L^{-1}(c_1,\ldots,c_m)$.

THEOREM 3.3. *Suppose the grading of $\mathcal{S}$ satisfies (WG), and $\omega_{n/2}\in\mathcal{S}_{n/2}$ satisfies*

$$\text{(KS)}\ \begin{cases}\forall x_0\in\mathcal{S}_0\ \langle H_1(x_0,\omega_{n/2})\rangle=\langle U_0(x_0)\rangle\langle U_1(\omega_{n/2})\rangle \text{ and } \langle U_1(\omega_{n/2})\rangle \text{ is invertible}\\ \forall\nu\neq 0\ \forall x_\nu\in\mathcal{S}_\nu\ \langle H_1(x_\nu,\omega_{n/2})\rangle=0\end{cases}$$

then there exists $\omega_\nu\in\mathcal{S}_\nu$, $\nu=0,\ldots,n-1$ such that the formula

$$\tau(M,\sigma)=\frac{\langle L(\omega_{c_1},\ldots,\omega_{c_m})\rangle}{\langle U_1(\omega_{n/2})\rangle^{b_+}\langle U_{-1}(\omega_{n/2})\rangle^{b_-}}$$

is an invariant of the surgered manifold $M=\mathbf{S}^3(L)$ equipped with the $\mathbf{Z}/n$ spin type structure $\sigma=\psi_L^{-1}(c_1,\ldots,c_m)$.
Moreover, if $\omega=\omega_0+\ldots+\omega_{n-1}$ then

$$\tau(M)=\frac{\langle L(\omega,\ldots,\omega)\rangle}{\langle U_1(\omega)\rangle^{b_+}\langle U_{-1}(\omega)\rangle^{b_-}}$$

is an invariant of the surgered manifold $M=\mathbf{S}^3(L)$ which satisfies the decomposition property

$$\forall M\quad \tau(M)=\sum_{\sigma\in Spin(M;\mathbf{Z}/n)}\tau(M,\sigma)$$

Example 1. The invariant coming from the Kauffman bracket skein theory admits a spin refinement (with a decomposition theorem), if A is a root of unity whose order is congruent to 0 modulo 16 ([Bl1]).

Example 2. If $N=2^{2l}$, the invariant Z_N of the skein theory $\mathcal{L}$ associated with linking admits a $\mathbf{Z}/2^l$ spin refinement.

Example 3. For n even the $SU(n)$ specialized HOMFLY theory admits $\mathbf{Z}/n$ spin type refinements ([Mu]).

4. Towards TQFTs.

A universal method for extending an invariant of closed 3-manifolds to a whole Topological Quantum Field Theory is described in [BHMV3] (see [At], [Tu2] or [BHMV3] for axioms of TQFT). Roughly speaking we could say that the problem is to have a pasting formula to compute the invariant out of pieces. If a manifold is cut by a separating surface Σ, each piece gives a vector in a module associated to the surface, and the invariant is obtained using a hermitian form on this module. Such a formula is clear if one proceeds as follows. Take the free module generated by the manifolds with boundary this surface, and divide by the kernel of the natural pairing defined using the invariant of closed manifolds. This has to be computable; precisely we want each module to have finite rank. The problem is reduced a lot if some natural surgery axioms are satisfied. First one needs an extension of the invariant to closed manifolds with (colored) links. (In our language a colored link is a skein element obtained by cabling some framed link.) Second, in most cases, the required index 2 surgery formula, is obtained only after resolving what is called *the framing anomaly*; this can be done by using p_1-structures. In [BHMV3], following the lines above, TQFT's are constructed from the Kauffman bracket. It is shown that the modules associated to surfaces (with p_1-structure) are free, of finite ranks given by the Verlinde formula.

How does this extend to the refined invariants? In joint work with Gregor Masbaum ([BM]) this question is solved for the spin invariants derived from the Kauffman bracket. A finiteness theorem and dimension formulae are given, and it is shown that the multiplicativity axiom holds in a $\mathbf{Z}/2$-graded sense. The relation between the spin theories and the *unspun* ones is described by a *transfer map*. This map identifies the 'unspun' module of a surface Σ with the invariant part, under a natural $\widetilde{H}_0(\Sigma, \mathbf{Z}/2)$-action, of the modules obtained in the spin theory (more precisely, the zero-graded part of the spin theory). As already noted, understanding other refined theories is our challenge!

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