Topological Interpretation of Barbero-Immirzi Parameter

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Abstract

We set up a canonical Hamiltonian formulation for a theory of gravity based on a Lagrangian density made up of the Hilbert-Palatini term and, instead of the Holst term, the Nieh-Yan topological density. The resulting set of constraints in the time gauge are shown to lead to a theory in terms of a real $SU(2)$ connection which is exactly the same as that of Barbero and Immirzi with the coefficient of the Nieh-Yan term identified as the inverse of Barbero-Immirzi parameter. This provides a topological interpretation for this parameter. Matter coupling can then be introduced in the usual manner, *without* changing the universal topological Nieh-Yan term.

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I. INTRODUCTION

Hilbert-Palatini Lagrangian for pure gravity is written in terms of the connection fields ω_μ^{IJ} and tetrad e_μ^I as independent field variables. Its Holst generalisation is given in terms of the Lagrangian density [1]:

$$\mathcal{L} = \frac{1}{2}e\Sigma_{IJ}^{\mu\nu}R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2}e\Sigma_{IJ}^{\mu\nu}\tilde{R}_{\mu\nu}{}^{IJ}(\omega) \tag{1}$$

where,

$$\Sigma_{IJ}^{\mu\nu} := \frac{1}{2}(e_I^\mu e_J^\nu - e_J^\mu e_I^\nu)\,,\ R_{\mu\nu}{}^{IJ}(\omega) := \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK}\omega_{\nu]K}{}^J\,,\ \tilde{R}_{\mu\nu}{}^{IJ}(\omega) := \frac{1}{2}\epsilon^{IJKL}R_{\mu\nu KL}(\omega).$$

The second term is the Holst term with η^{-1} as the Barbero-Immirzi parameter [2, 3]. For $\eta = -i$, this Lagrangian density leads to the canonical formulation in terms of the self-dual Ashtekar connection which is a *complex* $SU(2)$ connection [4]. For real η, we have a Hamiltonian formulation in terms of a *real* $SU(2)$ connection, which coincides with the Barbero formulation for $\eta = 1$ [2, 5].

Inclusion of Holst term does not change the classical equation of motion of the Hilbert-Palatini action; there is no dependence on η in the equations of motion. In fact, when the connection equation $\omega_\mu{}^{IJ} = \omega_\mu{}^{IJ}(e)$ is used, Holst term is identically zero.

Adding matter in the generalised Lagrangian density (1) needs special care. In particular when spin $\frac{1}{2}$ fermions are included through minimal coupling, the classical equations of motion acquire a dependence on η [6]. However it is possible to modify the Holst term in such a way that the equations of motion remain unchanged. Such modification for spin $\frac{1}{2}$ fermionic matter and also those in the $N = 1, 2$ and 4 supergravities have been obtained [7, 8]. *When the connection equation of motion is used,* the modified Holst terms in each of these cases, become total divergences involving Nieh-Yan invariant density and divergence of axial current densities involving the fermion fields. The modified Holst term used in these formulations *changes* with the matter content of the theory.

It has been suggested that the Barbero-Immirzi parameter should have a topological interpretation in the same manner as the θ parameter of QCD [9]. For this to be the case, η should be the coefficient of a term in the Lagrangian density which is a *topological density.* Since such a term would be a total derivative for *all* field configurations, the classical equations of motion would remain unaltered. Such a term would be universal in the sense

that it would not change when any matter coupling to gravity is introduced. The Holst term in (1) or any of its modifications mentioned above do not have such a property.

In the four dimensional gravity, there are three possible topological densities, namely Pontryagin, Euler and Nieh-Yan. The first two are quadratic in the curvature tensor. The Nieh-Yan density contains a term linear in $R_{\mu\nu}{}^{IJ}(\omega)$ and an $R-$independent term. This is shown below to be associated with Barbero-Immirzi parameter.

The Nieh-Yan density is given by [11]:

$$I_{NY} = \epsilon^{\mu\nu\alpha\beta}\left[D_\mu(\omega)e_\nu^I \ D_\alpha(\omega)e_{I\beta} - \frac{1}{2}\Sigma_{\mu\nu}^{IJ} \ R_{\alpha\beta IJ}(\omega)\right] \ , \ \ D_\mu(\omega)e_\nu^I := \partial_\mu e_\nu^I + \omega_\mu{}^I{}_J e_\nu^J \ . \ (2)$$

This is a topological density, that is, it is a total divergence:

$$I_{NY} = \partial_\mu J_{NY}^\mu(e,\omega) \ , \ \ J_{NY}^\mu(e,\omega) := \epsilon^{\mu\nu\alpha\beta}e_\nu^I D_\alpha(\omega)e_{I\beta} \ . \tag{3}$$

Note that unlike the Pontryagin and Euler densities, the Nieh-Yan density *vanishes identically for a torsion free connection.*

The classical equations of motion from the Lagrangian density containing the Hilbert-Palatini term as well as the Nieh-Yan density:

$$\mathcal{L} = \frac{1}{2}e\Sigma_{IJ}^{\mu\nu}R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2}I_{NY} \tag{4}$$

are the same as those from the Hilbert-Palatini Lagrangian alone. We shall demonstrate that the canonical Hamiltonian formulation based on this new Lagrangian density also leads to a theory of real $SU(2)$ connections, exactly the same as that emerging from the theory with original Holst term. This in turn, for $\eta = 1$, is the Barbero formulation. Inclusion of matter now does not need any further modification and equations of motion continue to be independent of η for all couplings. This also allows a direct interpretation of the η parameter as a topological parameter in a manner analogous to the θ-parameter in QCD.

In a quantum framework, it is also possible to arrive at the canonical formulation based on the Lagrangian density (4) starting from the Hilbert-Palatini canonical formulation by rescaling the wave functional by $\exp\{\frac{i\eta}{2}\int d^3x J_{NY}^t(e,\omega)\}$. Mercuri has used this approach to derive the canonical formulation containing the Barbero-Immirzi parameter for a theory with spin-1/2 fermions [10]. This demonstrated for the first time, the role of Nieh-Yan density as the source of the quantization ambiguity reflected by the Barbero-Immirzi parameter. However, in this analysis the connection equation of motion has been used to express the

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$J_{\text{NY}}^t(e,\omega)$ in terms of the fermions. It is desirable to carry out this procedure, retaining the J_{NY}^t as in equation (3) in terms of the original geometric variables. Such a method then can be applied directly to a theory of gravity with or without matter.

In this paper, we work within a classical framework. In section II, we describe the Hamiltonian formulation based on the Lagrangian density (4) closely following the analysis carried out by Sa [5] for the Hilbert-Palatini gravity with the Holst term. In section III, we discuss the matter couplings, in particular the case of Dirac fermions. Coupling of any other matter can be done in an analogous and straight forward manner. Section IV contains a few concluding remarks.

II.   HAMILTONIAN ANALYSIS

We propose the Lagrangian density for pure gravity to be that given in equation (4), rewritten as:

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}e\Sigma_{IJ}^{\mu\nu}R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2}\left[e\Sigma_{IJ}^{\mu\nu}\tilde{R}_{\mu\nu}{}^{IJ}(\omega) + \epsilon^{\mu\nu\alpha\beta}D_\mu(\omega)e_\nu^I\,D_\alpha(\omega)e_{I\beta}\right]\\
&= \frac{1}{2}e\Sigma_{IJ}^{\mu\nu}R_{\mu\nu}^{(\eta)IJ}(\omega) + \frac{\eta}{2}\,\epsilon^{\mu\nu\alpha\beta}D_\mu(\omega)e_\nu^I D_\alpha(\omega)e_{I\beta}\quad,
\end{aligned}
\tag{5}
$$

where $R_{\mu\nu}^{(\eta)IJ}(\omega) := R_{\mu\nu}{}^{IJ}(\omega) + \eta\tilde{R}_{\mu\nu}{}^{IJ}(\omega)$ and we have used the identities,

$$
\Sigma_{IJ}^{\mu\nu}\tilde{R}_{\mu\nu}{}^{IJ}(\omega) = \tilde{\Sigma}_{IJ}^{\mu\nu}R_{\mu\nu}{}^{IJ}(\omega)\quad,\quad e\tilde{\Sigma}_{IJ}^{\mu\nu} := \frac{e}{2}\epsilon_{IJKL}\Sigma^{\mu\nu KL} = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\Sigma_{\alpha\beta IJ}\ .
\tag{6}
$$

Introducing the notation, $t_I^a := \eta\epsilon^{abc}D_b(\omega)e_{Ic}$ and $\epsilon^{abc} := \epsilon^{tabc}$, the 3+1 decomposition is expressed as:

$$
\mathcal{L} = e\Sigma_{IJ}^{ta}R_{ta}^{(\eta)IJ}(\omega) + \frac{e}{2}\Sigma_{IJ}^{ab}R_{ab}^{(\eta)IJ}(\omega) + t_I^a\left(D_t(\omega)e_a^I - D_a(\omega)e_t^I\right)
\tag{7}
$$

Defining $\omega_a^{(\eta)IJ} := \omega_a^{IJ} + \eta\tilde{\omega}_a^{IJ}$ and $\Sigma_{IJ}^{(\eta)ta} := \Sigma_{IJ}^{ta} + \eta\tilde{\Sigma}_{IJ}^{ta}$ we get,

$$
\begin{aligned}
\mathcal{L} &= e\Sigma_{IJ}^{ta}\partial_t\omega_a^{(\eta)IJ} + \omega_t^{IJ}D_a(\omega)\left(e\Sigma_{IJ}^{(\eta)ta}\right) + \frac{e}{2}\Sigma_{IJ}^{ab}R_{ab}^{(\eta)IJ}(\omega)\\
&\quad + t_I^a\partial_t e_a^I + \omega_t{}^{IJ}t_I^a e_{aJ} + e_t^I D_a(\omega)t_I^a - \partial_a\left(t_I^a e_t^I + e\Sigma_{IJ}^{(\eta)ta}\omega_t^{IJ}\right)
\end{aligned}
\tag{8}
$$

We parametrize the tetrad fields as:

$$
e_t^I = \sqrt{eN}M^I + N^a V_a^I\ ,\ e_a^I = V_a^I\ ;\ M_I V_a^I = 0\ ,\ M_I M^I = -1
\tag{9}
$$

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and then the inverse tetrad fields are:

$$e_I^t = -\frac{M_I}{\sqrt{eN}} \ , \ e_I^a = V_I^a + \frac{N^a M_I}{\sqrt{eN}} \ ;$$

$$M^I V_I^a := 0 \ , \ V_a^I V_I^b := \delta_a^b \ , \ V_a^I V_J^a := \delta_J^I + M^I M_J \tag{10}$$

Defining $q_{ab} := V_a^I V_{bI}$ and $q := \det q_{ab}$, leads to $e := det(e_\mu^I) = Nq$. We may thus trade the 16 tetrad fields with the 9 fields V_I^a ($M^I V_I^a = 0$), the 3 fields M^I ($M_I M^I = -1$) and the 4 fields N and N^a.

Next using the identity,

$$\Sigma_{IJ}^{ab} = 2Ne\Sigma_{IK}^{t[a}\Sigma_{JL}^{b]t}\eta^{KL} + N^{[a}\Sigma_{IJ}^{b]t} \ , \tag{11}$$

and dropping the total space derivative terms,

$$\mathcal{L} = e\Sigma_{IJ}^{ta}\partial_t\omega_a^{(\eta)IJ} + t_I^a\partial_t e_a^I - NH - N^aH_a - \tfrac{1}{2}\omega_t^{IJ}G_{IJ} \tag{12}$$

where $2e\Sigma_{IJ}^{ta} = -\sqrt{q}M_{[I}V_{J]}^a$, $t_I^a := \eta\epsilon^{abc}D_b(\omega)V_{Ic}$ and

$$H = 2e^2\Sigma_{IK}^{ta}\Sigma_{JL}^{tb}\eta^{KL}R_{ab}^{(\eta)IJ}(\omega) - \sqrt{q}M^I D_a(\omega)t_I^a, \tag{13}$$

$$H_a = e\Sigma_{IJ}^{tb}R_{ab}^{(\eta)IJ}(\omega) - V_a^I D_b(\omega)t_I^b, \tag{14}$$

$$G_{IJ} = -2D_a(\omega)\left(e\Sigma_{IJ}^{(\eta)ta}\right) - t_{[I}^a V_{J]}a \ . \tag{15}$$

Introduce the fields,

$$E_i^a := 2e\Sigma_{0i}^{ta} \ , \ \chi_i := -M_i/M^0 \ , \ A_a^i := \omega_a^{(\eta)0i} - \chi_j\omega_a^{(\eta)ij} \ , \ \zeta^i := -E_j^a\omega_a^{(\eta)ij}. \tag{16}$$

In terms of these, we have $2e\Sigma_{ij}^{ta} = -E_{[i}^a\chi_{j]}$ and $e\Sigma_{IJ}^{ta}\partial_t\omega_a^{(\eta)IJ} = E_i^a\partial_t A_a^i + \zeta^i\partial_t\chi^i$, and the Lagrangian density is:

$$\mathcal{L} = E_i^a\partial_t A_a^i + \zeta^i\partial_t\chi^i + t_I^a\partial_t V_a^I - NH - N^aH_a - \tfrac{1}{2}\omega_t^{IJ}G_{IJ}, \tag{17}$$

where, now we need to re-express H, H_a and G_{IJ} ($G_{\text{boost}}^i := G_{0i}$, $G_{\text{rot}}^i := \tfrac{1}{2}\epsilon^{ijk}G_{jk}$) in terms of these new fields:

$$G_{\text{boost}}^i = -\partial_a\left(E_i^a - \eta\epsilon^{ijk}E_j^a\chi_k\right) + E_{[i}^a\chi_{k]}A_a^k + (\zeta^i - \chi\cdot\zeta\chi^i) - t_{[0}^aV_{i]a} \ , \tag{18}$$

$$G_{\text{rot}}^i = \partial_a\left(\epsilon^{ijk}E_j^a\chi_k + \eta E_i^a\right) + \epsilon^{ijk}\left(A_a^jE_k^a - \zeta_j\chi_k - t_j^aV_a^k\right) \tag{19}$$

$$H_a = E_i^b\left[R_{ab}^{(\eta)0i}(\omega) - \chi_j R_{ab}^{(\eta)ij}(\omega)\right] - V_a^I D_b(\omega)t_I^b \tag{20}$$

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$$= E_i^b \partial_{[a} A_{b]}^i + \zeta^i \partial_a \chi^i - V_a^I \partial_b t_I^b + t_I^b \partial_{[a} V_{b]}^I$$

$$-\frac{1}{1+\eta^2} \left[E_{[i}^b \chi_{l]} A_b^l + (\zeta_i - \chi \cdot \zeta \chi_i) - t_{[0}^b V_{i]b} - \eta \epsilon^{ijk}(A_b^j E_k^b - \zeta_j \chi_k - t_j^b V_b^k) \right] A_a^i$$

$$-\frac{1}{1+\eta^2} \left[\frac{1}{2} \epsilon^{ijk}(\eta G_{\text{boost}}^k + G_{\text{rot}}^k) - \chi^i (G_{\text{boost}}^j - \eta G_{\text{rot}}^j) \right] \omega_a^{(\eta)ij} \qquad (21)$$

$$H = -E_k^a \chi_k H_a - \frac{1}{2}(1 - \chi \cdot \chi) E_i^a E_j^b R_{ab}^{(\eta)ij}(\omega) - \left(E_k^a \chi_k V_a^I + \sqrt{q} M^I \right) D_b(\omega) t_I^b$$

$$= -E_k^a \chi_k H_a + (1 - \chi \cdot \chi) \left[E_i^a \partial_a \zeta_i + \frac{1}{2} \zeta_i E_i^a E_j^b \partial_a E_b^j \right]$$

$$+\frac{1 - \chi \cdot \chi}{2(1+\eta^2)} \zeta_i \left[-G_{\text{boost}}^i + \eta G_{\text{rot}}^i \right] - \left(E_k^a \chi_k V_a^I + \sqrt{q} M^I \right) \partial_b t_I^b$$

$$-\frac{1 - \chi \cdot \chi}{1+\eta^2} \left[\frac{1}{2} E_{[i}^a E_{j]}^b A_a^i A_b^j + E_i^a A_a^i \chi \cdot \zeta + \eta \epsilon^{ijk} \zeta_i A_a^j E_k^a \right.$$

$$\left. +\frac{3}{4}(\chi \cdot \zeta)^2 - \frac{3}{4}(\zeta \cdot \zeta) + \frac{1}{2} \zeta_i t_{[0}^a V_{i]a} - \frac{\eta}{2} \zeta_i \epsilon^{ijk} t_j^a V_a^k \right]$$

$$+\frac{1 - \chi \cdot \chi}{1+\eta^2} \left[\frac{1}{\sqrt{E}} A_a^i t_i^a + \frac{1}{2} V_a^i \left(\zeta \cdot \chi t_i^a - \chi_i \zeta_j t_j^a + \eta \epsilon^{ijk} \zeta_j t_k^a \right) \right]$$

$$+\frac{1 - \chi \cdot \chi}{1+\eta^2} \left[-\frac{1}{\sqrt{E}} \chi_i t_j^b + \frac{\eta}{2\sqrt{E}} \epsilon^{ijk} t_k^b + (1+\eta^2) E_i^a \partial_a E_j^b + E_i^a \chi_j E_m^b A_a^m \right.$$

$$\left. -\eta \epsilon^{imn} E_m^a E_n^b A_a^j - \frac{\eta}{4} \left(\epsilon^{ijm} E_n^b + \epsilon^{ijn} E_m^b \right) \chi_m \zeta_n \right] u_b^{ij}$$

$$+\frac{1 - \chi \cdot \chi}{2(1+\eta^2)} \left(\chi_m \chi_n - \delta_{mn} \right) E_j^a E_i^b u_a^{im} u_b^{jn} \qquad (22)$$

In the above, $E_a^i := \sqrt{E} V_a^i$ is the inverse of E_i^a i.e. $E_a^i E_i^b = \delta_a^b$, $E_i^a E_a^j = \delta_j^i$ and $E^{-1} = q(M^0)^2$ equals $\det E_i^a$. Furthermore we have also set $u_a^{ij} := \omega_a^{(\eta)ij} - \frac{1}{2} E_a^{[i} \zeta^{j]}$. Notice that $E_i^b u_b^{ij} = 0$. The six independent fields in u_a^{ij} may be parametrized in terms a symmetric matrix M^{ij} as, $u_a^{ij} := \frac{1}{2} \epsilon^{ijk} E_a^l M^{kl}$ [5].

We have replaced the original 16 tetrad fields with 16 new fields: E_i^a, χ_i, N and N^a. In place of the original 24 connection fields ω_μ^{IJ} we use the new set of 24 fields $A_a^i, \zeta_i, M^{kl}, \omega_t^{ij}$ and ω_t^{0i}. The fields V_a^I and t_I^a are not independent; these are given in terms of the fundamental fields as: $V_a^I = v_a^I$ and $t_I^a = \tau_I^a$ where

$$v_a^0 := -\frac{1}{\sqrt{E}} E_a^i \chi_i \quad , \quad v_a^i := \frac{1}{\sqrt{E}} E_a^i \qquad (23)$$

$$\tau_0^a := \eta \epsilon^{abc} D_b(\omega) V_{0c} = \eta \sqrt{E} E_m^a \left[G_{\text{rot}}^m - \frac{\chi_l}{2} \left(\frac{2 f_{ml} + N_{ml}}{1+\eta^2} + \epsilon_{mln} G_{\text{boost}}^n \right) \right] \qquad (24)$$

$$\tau_k^a := \eta \epsilon^{abc} D_b(\omega) V_{ck} = -\frac{\eta}{2} \sqrt{E} E_m^a \left[\frac{2 f_{mk} + N_{mk}}{1+\eta^2} + \epsilon_{kmn} G_{\text{boost}}^n \right] \qquad (25)$$

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where,

$$2f_{kl} := \epsilon_{ijk}E_i^a\left[(1+\eta^2)E_b^l\partial_a E_j^b + \chi_j A_a^l\right] + \eta\left(E_l^a A_a^k - \delta^{kl}E_m^a A_a^m - \chi_l\zeta_k\right) + (l \leftrightarrow k) \tag{26}$$

$$N_{kl} := \epsilon^{ijk}(\chi_m\chi_j - \delta_{mj})E_i^a u_a^{lm} + (l \leftrightarrow k) \tag{27}$$

$$= (\chi\cdot\chi - 1)(M_{kl} - M_{mm}\delta_{kl}) + \chi_m\chi_n M_{mn}\delta_{kl} + \chi_l\chi_k M_{mm} - \chi_m(\chi_k M_{ml} + \chi_l M_{mk})$$

We can upgrade V_a^I and t_I^a as independent fields through terms containing the Lagrange multiplier fields ξ_I^a and ϕ_a^I in the Lagrangian density:

$$\mathcal{L} = E_i^a\partial_t A_a^i + \zeta^i\partial_t\chi^i + t_I^a\partial_t V_a^I - \mathcal{H}$$
$$\mathcal{H} := NH + N^a H_a + \frac{1}{2}\omega_t^{IJ}G_{IJ} + \xi_I^a(V_a^I - \upsilon_a^I) + \phi_a^I(t_I^a - \tau_I^a) \tag{28}$$

where υ_a^I and τ_I^a are defined in equations (23 - 25). We have 24 pairs of canonically conjugate independent field variables $(E_i^a, A_b^j), (\zeta^i, \chi^j), (t_I^a, V_a^I)$. The remaining fields, namely, $N, N^a, \omega_t^{IJ}, \xi_I^a, \phi_a^I$ and M^{kl} have no conjugate momenta since in the Lagrangian their velocities do not appear. Preservation of these constraints (vanishing of the variation of the Hamiltonian with respect to the fields) leads to the secondary constraints. From the variations with respect to fields $\omega_t^{0i}, \omega_t^{ij}, N^a, N, \xi_I^a$ and ϕ_a^I, we get the constraints:

$$G_{\text{boost}}^i \approx 0 \ , \ G_{\text{rot}}^i \approx 0 \ ; \ H_a \approx 0 \ , \ H \approx 0 \ ; \tag{29}$$

$$V_a^I - \upsilon_a^I \approx 0 \ , \ t_I^a - \tau_I^a \approx 0. \tag{30}$$

From the variation with respect to M^{kl} or equivalently u_a^{ij}, we get:

$$\frac{\delta\mathcal{H}}{\delta M^{kl}}\delta M^{kl} \approx \frac{\delta H}{\delta M^{kl}}\delta M^{kl} = \frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}[(\eta t_k^a - \epsilon^{ijk}\chi_i t_j^a)V_a^l + f_{kl} + \frac{1}{2}N_{kl}]\delta M^{kl} \approx 0.$$

This leads to

$$(\eta t_k^a - \epsilon^{ijk}\chi_i t_j^a)V_a^l + f_{kl} + \frac{1}{2}N_{kl} + (k \leftrightarrow l) \approx 0 \tag{31}$$

Using constraints (30), and the expressions (23 - 25) for τ_I^a, υ_a^I, equation (31) implies :

$$\left(\eta\epsilon^{ijk}\chi_i + \delta_{kj}\right)(2f_{jl} + N_{jl}) + \eta(1+\eta^2)\left(\delta^{kl}\chi_m G_{\text{boost}}^m - \chi_l G_{\text{boost}}^k\right) + (k \leftrightarrow l) \approx 0$$

Using (29), this in turn implies the constraint:

$$2f_{kl} + N_{kl} \approx 0 \tag{32}$$

where, f_{kl} and N_{kl} are given in (26, 27). This constraint can be solved for M_{kl}. Furthermore it implies, from the definitions (24, 25), that $\tau_I^a \approx 0$ and hence,

$$t_I^a \approx 0 \ . \tag{33}$$

Implementing this constraint then reduces the Hamiltonian density to

$$\mathcal{H} \ = \ NH + N^a H_a + \frac{1}{2}\omega_t^{IJ} G_{IJ} \tag{34}$$

where now,

$$G_{\text{boost}}^i \ = \ -\partial_a \left(E_i^a - \eta\epsilon^{ijk} E_j^a \chi_k \right) + E_{[i}^a \chi_{k]} A_a^k + (\zeta^i - \chi \cdot \zeta \zeta^i) \ \approx \ 0 \ , \tag{35}$$

$$G_{\text{rot}}^i \ = \ \partial_a \left(\epsilon^{ijk} E_j^a \chi_k + \eta E_i^a \right) + \epsilon^{ijk}(A_a^j E_k^a - \zeta_j \chi_k) \ \approx \ 0 \ , \tag{36}$$

$$H_a \ = \ E_i^b \partial_{[a} A_{b]}^i + \zeta_i \partial_a \chi_i \tag{37}$$
$$-\frac{1}{1+\eta^2} \left[E_{[k}^b \chi_{l]} A_b^l + \zeta_i - \chi \cdot \zeta \chi^i - \eta\epsilon^{ijk}(A_a^j E_k^a - \zeta_j \chi_k) \right] A_a^i$$
$$-\frac{1}{1+\eta^2} \left[\frac{1}{2}\epsilon^{ijk}(\eta G_{\text{boost}}^k + G_{\text{rot}}^k) - \chi^i(G_{\text{boost}}^j - \eta G_{\text{rot}}^j) \right] \omega_a^{(\eta)ij} \ \approx \ 0 \ ,$$

$$H \ = \ -E_k^a \chi_k H_a + (1 - \chi \cdot \chi) \left[E_i^a \partial_a \zeta_i + \frac{1}{2}\zeta_i E_i^a E_j^b \partial_a E_b^j \right] \tag{38}$$
$$+\frac{(1 - \chi \cdot \chi)}{2(1+\eta^2)}\zeta_i[-G_{\text{boost}}^j + \eta G_{\text{rot}}^j]$$
$$-\frac{(1 - \chi \cdot \chi)}{1+\eta^2} \left[\frac{1}{2}E_{[i}^a E_{j]}^b A_a^i A_b^j + E_i^a A_a^i \chi \cdot \zeta + \eta\epsilon_{ijk}\zeta_i A_a^j E_k^a + \frac{3}{4}(\chi \cdot \zeta)^2 - \frac{3}{4}(\zeta \cdot \zeta) \right]$$
$$+\frac{(1 - \chi \cdot \chi)}{2(1+\eta^2)} \left[f_{kl}M^{kl} + \frac{1}{4}(\chi \cdot \chi - 1)(M^{kl}M^{kl} - M^{kk}M^{ll}) \right.$$
$$\left. +\frac{1}{2}\chi_k\chi_l(M^{pp}M^{kl} - M^{kp}M^{lp}) \right] \ \approx \ 0 \ .$$

In the last equation we have M^{kl} given by the constraint $2f_{kl} + N_{kl} = 0$, which can be solved as:

$$(1 - \chi \cdot \chi)M_{kl} = 2f_{kl} + (\chi_m\chi_n f_{mn} - f_{mm})\delta_{lk} + (\chi_m\chi_n f_{mn} + f_{mm})\chi_k\chi_l - 2\chi_m(\chi_l f_{mk} + \chi_k f_{ml}) \tag{39}$$

This is the same set of equations as those obtained by Sa [5] in his analysis of the action containing Holst term.

We may fix the boost gauge transformations (time gauge) by imposing $\chi^i \approx 0$ which together with the $G_{\text{boost}}^i \approx 0$ forms a second class pair. Solving the boost constraint with $\chi^i = 0$ yields,

$$\zeta_i \ = \ \partial_a E_i^a \tag{40}$$

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In this gauge we then recover a canonical Hamiltonian formulation in terms of real $SU(2)$ gauge fields A_a^i which reduces to the Barbero formulation for $\eta = 1$ [5].

To summarize, like the Holst term, the Nieh-Yan term leads to an $SU(2)$ gauge theoretic formulation. But, it is only the coefficient of the Nieh-Yan term that has a topological character.

III. MATTER COUPLING

As stated earlier, the matter can now be coupled to gravity in a straight forward manner. As an example, we consider a spin-$\frac{1}{2}$ Dirac fermion with its usual *minimal* coupling to gravity. The Lagrangian density is[1],

$$\mathcal{L} = \frac{1}{2} e \, \Sigma_{IJ}^{\mu\nu} \, R_{\mu\nu}{}^{IJ}(\omega) \; + \; \frac{\eta}{2} \, I_{NY} \; + \; \frac{ie}{2} \, [\bar{\lambda}\gamma^\mu D_\mu(\omega)\lambda \; - \; \overline{D_\mu(\omega)\lambda}\gamma^\mu\lambda] \tag{41}$$

where,

$$D_\mu(\omega)\lambda := \partial_\mu\lambda + \frac{1}{2}\,\omega_{\mu IJ}\,\sigma^{IJ}\,\lambda \;\;,\;\; \overline{D_\mu(\omega)\lambda} := \partial_\mu\bar{\lambda} - \frac{1}{2}\,\bar{\lambda}\,\omega_{\mu IJ}\,\sigma^{IJ}$$

Notice that, unlike earlier attempts of setting up a theory of fermions and gravity with Barbero-Immirzi parameter [7, 8] where the Holst term was modified to include an additional non-minimal term for the fermions, the Lagrangian density here containing the Nieh-Yan density does not require any further modification, just the usual minimal fermion terms suffice. This is so because the Nieh-Yan term is topological.

We expand the fermion terms as

$$\mathcal{L}(F) := \frac{ie}{2}\,[\bar{\lambda}\gamma^\mu D_\mu(\omega)\lambda \; - \; \overline{D_\mu(\omega)\lambda}\gamma^\mu\lambda]$$

$$= [\partial_t\bar{\lambda}\Pi \; - \; \bar{\Pi}\partial_t\lambda] \; - \; NH(F) \; - \; N^a H_a(F) \; - \; \frac{1}{2}\,\omega_t^{IJ}\,G_{IJ}(F) \tag{42}$$

where $\bar{\Pi}$ and Π are canonically conjugate momenta fields associated with λ and $\bar{\lambda}$ respectively. Explicitly [2],

$$\bar{\Pi} = -\frac{ie}{2}\bar{\lambda}\gamma^t = \frac{i\sqrt{q}}{2}M_I\bar{\lambda}\gamma^I \;\;,\;\; \Pi = -\frac{ie}{2}\gamma^t\lambda = \frac{i\sqrt{q}}{2}M_I\gamma^I\lambda \tag{43}$$

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[1] Our Dirac matrices satisfy the Clifford algebra: $\gamma^I\gamma^J + \gamma^J\gamma^I = 2\eta^{IJ}$, $\eta^{IJ} := \mathrm{diag}(-1,1,1,1)$. The chiral matrix $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{IJ} := \frac{1}{4}\,[\gamma^I,\gamma^J]$.

[2] The fermions are Grassmann valued and the functional differentiation is done on the left factor which accounts for the signs in the definitions of the conjugate momenta in (43).

$$G^{IJ}(F) = \bar{\Pi}\sigma^{IJ}\lambda + \bar{\lambda}\sigma^{IJ}\Pi \tag{44}$$

$$H_a(F) = \overline{D_a(\omega)\lambda}\,\Pi - \bar{\Pi}D_a(\omega)\lambda \tag{45}$$

$$H(F) = (-2\,e\,\Sigma_{IJ}^{ta})\left[\overline{D_a(\omega)\lambda}\,\sigma^{IJ}\,\Pi + \bar{\Pi}\,\sigma^{IJ}\,D_a(\omega)\lambda\right] \tag{46}$$

Incorporating these fermionic terms in the pure gravity Lagrangian density given in equation (28), we write the full Lagrangian density as,

$$\begin{aligned}
\mathcal{L} =\ & E_i^a\,\partial_t A_a^i + \zeta^i\,\partial_t\chi_i + t_I^a\partial_t V_a^I + \partial_t\bar{\lambda}\,\Pi - \bar{\Pi}\,\partial_t\lambda - NH' - N^a H_a' - \frac{1}{2}\omega_t^{IJ}\,G_{IJ}' \\
& - \xi_I^a(V_a^I - v_a^I) - \phi_a^I(t_I^a - \tau_I^a)
\end{aligned} \tag{47}$$

where now

$$G_{IJ}' = G^{IJ} + G^{IJ}(F)\ ,\quad H_a' = H_a + H_a(F)\ ,\quad H' = H + H(F)\ , \tag{48}$$

with G^{IJ}, H_a and H as the contributions from the pure gravity sector as given by the equations (13 – 15) or equivalently by the equations (19 – 22).

The various quantities above can then be rewritten in terms of the basic fields as:

$$\begin{aligned}
G_{\text{boost}}'^i =\ & -\partial_a(E_i^a - \eta\epsilon^{ijk}E_j^a\chi_k) + E_{[i}^a\chi_{k]}A_a^k + (\zeta^i - \chi\cdot\zeta\,\chi^i) - t_{[0}'^a V_{i]a} \\
& + \left[\bar{\Pi}(1+i\eta\gamma_5)\sigma_{0i}\lambda + \bar{\lambda}(1+i\eta\gamma_5)\sigma_{0i}\Pi\right]\ ; \tag{49}
\end{aligned}$$

$$\begin{aligned}
G_{\text{rot}}'^i =\ & \partial_a(\epsilon^{ijk}E_j^a\chi_k + \eta E_i^a) + \epsilon^{ijk}(A_a^j E_k^a - \zeta_j\chi_k - t_j'^a V_a^k) \\
& + \left[\bar{\Pi}(i\gamma_5 - \eta)\sigma_{0i}\lambda + \bar{\lambda}(i\gamma_5 - \eta)\sigma_{0i}\Pi\right]\ ; \tag{50}
\end{aligned}$$

$$\begin{aligned}
H_a' =\ & E_i^b\partial_{[a}A_{b]}^i + \zeta_i\partial_a\chi_i - \partial_b(t_I'^b V_a^I) + t_I'^b\partial_a V_b^I \\
& + \left[\partial_a\bar{\lambda}(1+i\eta\gamma_5)\Pi - \bar{\Pi}(1+i\eta\gamma_5)\partial_a\lambda\right] - \left[\bar{\lambda}\sigma_{0i}\Pi + \bar{\Pi}\sigma_{0i}\lambda\right]A_a^i \\
& - \frac{1}{1+\eta^2}\left[E_{[i}^b\chi_{l]}A_b^l + \zeta_i - \chi\cdot\zeta\chi^i - t_{[0}'^b V_{i]b} - \eta\epsilon^{ijk}\,(A_a^j E_k^b - \chi_j\zeta_k - t_j'^b V_b^k)\right]A_a^i \\
& - \frac{1}{1+\eta^2}\left[\frac{1}{2}\epsilon^{ijk}\left(\eta G_{\text{boost}}'^k + G_{\text{rot}}'^k\right) - \chi^i(G_{\text{boost}}'^j - \eta G_{\text{rot}}'^j)\right]\omega_a^{(\eta)ij} \tag{51}
\end{aligned}$$

$$\begin{aligned}
H' =\ & -E_k^a\chi_k H_a' - (E_k^a\chi_k V_a^I + \sqrt{q}M^I)\,\partial_b t_I'^b + (1-\chi\cdot\chi)\left[E_i^a\partial_a\zeta_i + \frac{1}{2}\zeta_i E_i^a E_j^b\partial_a E_b^j\right] \\
& - \frac{(1-\chi\cdot\chi)}{1+\eta^2}\left[\frac{1}{2}E_{[i}^a E_{j]}^b A_a^i A_b^j + E_i^a A_a^i\chi\cdot\zeta + \eta\epsilon_{ijk}\zeta_i A_a^j E_k^a + \frac{3}{4}(\chi\cdot\zeta)^2 - \frac{3}{4}(\zeta\cdot\zeta)\right] \\
& + \frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}\left[\zeta_i - 2V_b^i(t_0^b - t_0'^b)\right]\left[-G_{\text{boost}}'^i + \eta G_{\text{rot}}'^i - t_{[0}'^a V_{i]a} + \eta\epsilon_{ijk}t_j'^a V_a^k\right] \\
& + \frac{(1-\chi\cdot\chi)}{\sqrt{E}(1+\eta^2)}\left[t_m'^b A_b^m + \frac{1}{2}E_b^i t_{[i}'^b\chi_{j]}\zeta_j + \frac{\eta}{2}\epsilon_{ijk}t_i'^b E_b^j\zeta_k\right] \\
& - 2e\,\Sigma_{IJ}^{ta}\left[\partial_a\bar{\lambda}(1+i\eta\gamma_5)\sigma^{IJ}\Pi + \bar{\Pi}(1+i\eta\gamma_5)\sigma^{IJ}\partial_a\lambda\right]
\end{aligned}$$

$$+ E_k^a \chi_k \left[\partial_a \bar{\lambda}(1 + i\eta\gamma_5)\Pi - \bar{\Pi}(1 + i\eta\gamma_5)\partial_a\lambda \right]$$

$$- 2e \, \Sigma_{IJ}^{ta} \left[-\bar{\lambda}\sigma_{0l} \, \sigma^{IJ}\Pi + \bar{\Pi}\sigma^{IJ} \, \sigma_{0l}\lambda \right] A_a^l - E_k^a \chi_k \left[\bar{\Pi}\sigma_{0l}\lambda + \bar{\lambda}\sigma_{0l}\Pi \right] A_a^l$$

$$+ \frac{(1 - \chi \cdot \chi)}{2(1 + \eta^2)} \left[\left(\eta t_k'^a - \epsilon^{ijk}\chi_i t_j'^a \right) V_l^a + f_{kl} + (1 + \eta^2)J_{kl} + \frac{1}{4}N_{kl}(M) \right] M^{kl} \qquad (52)$$

where as earlier, $2e \, \Sigma_{0i}^{ta} = E_i^a$, $2e \, \Sigma_{ij}^{ta} = -E_{[i}^a\chi_{j]}$ and f_{kl} , $N_{kl}(M)$ are given by equations (26, 27) respectively. Also,

$$t_I'^a := t_I^a - \eta e \, \Sigma_{IJ}^{ta}\bar{\lambda}\gamma_5\gamma^J\lambda \qquad (53)$$

$$= t_I^a + \frac{i\eta}{\sqrt{q}} \, e \, \Sigma_{IJ}^{ta} \left[M^J(\bar{\Pi}\gamma_5\lambda - \bar{\lambda}\gamma_5\Pi) + 2M_L(\bar{\Pi}\gamma_5\sigma^{LJ}\lambda + \bar{\lambda}\gamma_5\sigma^{LJ}\Pi) \right]$$

$$2J_{kl} := \frac{1}{2\sqrt{E}} \, \bar{\lambda}\gamma_5 \left(\chi_k\gamma_l + \chi_l\gamma_k + 2\delta_{kl}\frac{M^I\gamma_I}{M^0} \right) \lambda \qquad (54)$$

$$= \frac{i}{2}(\delta_{kl} + M_k M_l)(\bar{\Pi}\gamma_5\lambda - \bar{\lambda}\gamma_5\Pi) + iM_l M^J(\bar{\Pi}\gamma_5\sigma_{Jk}\lambda + \bar{\lambda}\gamma_5\sigma_{Jk}\Pi) + (k \leftrightarrow l)$$

The Hamiltonian density now reads:

$$\mathcal{H} = NH' + N^a H_a' + \frac{1}{2} \, \omega_t^{IJ} \, G_{IJ}' + \xi_I^a(V_a^I - v_a^I) + \phi_a^I(t_I^a - \tau_I^a) \qquad (55)$$

The constraints associated with the fields $N^a, N, \omega_t^{0i}, \omega_t^{ij}, \xi_I^a$ and ϕ_a^I respectively are:

$$H_a' \approx 0 \ , \quad H' \approx 0 \ , \quad G_{\text{boost}}'^i \approx 0 \ , \quad G_{\text{rot}}'^i \approx 0 \qquad (56)$$

$$V_a^I - v_a^I \approx 0 \ , \quad t_I^a - \tau_I^a \approx 0. \qquad (57)$$

The remaining fields M^{kl}, from $\frac{\delta H'}{\delta M^{kl}}\delta M^{kl} \approx 0$, lead to the constraint,

$$\left(\eta t_k'^a - \epsilon^{ijk}\chi_i t_j'^a \right)V_a^l + f_{kl} + \frac{1}{2}N_{kl} + (1 + \eta^2)J_{kl} + (k \leftrightarrow l) \approx 0 \qquad (58)$$

Using $t_I^a \approx \tau_I^a$, we write

$$t_k'^a \approx -\frac{\eta}{2}\sqrt{E} \, E_l^a \left[\frac{2f_{kl} + N_{kl}}{1 + \eta^2} + 2J_{kl} + \epsilon_{kln}G_{\text{boost}}'^n \right]$$

$$t_0'^a \approx \eta\sqrt{E} \, E_l^a \left[G_{\text{rot}}'^l - \frac{\chi_k}{2} \left(\frac{2f_{kl} + N_{kl}}{1 + \eta^2} + 2J_{kl} + \epsilon_{kln} \, G_{\text{boost}}'^n \right) \right] \qquad (59)$$

Using (59) in (58), leads to

$$2f_{kl} + N_{kl} + 2(1 + \eta^2)J_{kl} \approx 0 \qquad (60)$$

generalizing the constraint (32) of the pure gravity case. This in turn implies

$$t_I'^a \approx 0 \qquad (61)$$

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corresponding to the constraint (33) for pure gravity. Implementing this constraint along with those in (57) reduces the Hamiltonian density to

$$\mathcal{H} = NH' + N^a H'_a + \frac{1}{2}\omega_t^{IJ} \, G'_{IJ} \tag{62}$$

where the final set of constraints are obtained from equations (49 – 52) by substituting $t_I'^a = 0$ and dropping the terms containing $G''^i_{\text{boost}}, G''^i_{\text{rot}}$ in H'_a, H'. The M_{kl} is given by the solution of the constraint (60).

Time gauge:

We may now make the gauge choice $\chi_i = 0$ and solve the boost constraint $G''^i_{\text{boost}} = 0$ to obtain

$$\zeta_i = \partial_a E^a_i - i\eta \left[\bar{\Pi}\gamma_5\sigma_{0i}\lambda + \bar{\lambda}\gamma_5\sigma_{0i}\Pi\right] \tag{63}$$

Thus we have a canonical Hamiltonian formulation for a theory of gravity with fermions in terms of real $SU(2)$ gauge fields A^i_a with the following constraints:

$$
\begin{aligned}
G''^i_{\text{rot}} &= \eta\,\partial_a E^a_i + \epsilon^{ijk} A^j_a E^a_k + i[\,\bar{\Pi}\gamma_5\sigma_{0i}\lambda + \bar{\lambda}\gamma_5\sigma_{0i}\Pi] \;\approx\; 0; \\
H'_a &= E^b_i\partial_{[a}A^i_{b]} + [\,\partial_a\bar{\lambda}(1+i\eta\gamma_5)\Pi - \bar{\Pi}(1+i\eta\gamma_5)\partial_a\lambda\,] \\
&\quad - \frac{1}{1+\eta^2}\left[\partial_a E^a_i - \eta\epsilon_{ijk}A^j_b E^b_k - i\eta\left(\bar{\Pi}\gamma_5\sigma_{0i}\lambda + \bar{\lambda}\gamma_5\sigma_{0i}\Pi\,\right)\right] A^i_a \;\approx\; 0; \\
H' &= [E^a_i\partial_a\zeta_i + \frac{1}{2}\zeta_i E^a_i E^b_j\partial_a E^j_b] - \frac{1}{1+\eta^2}\left[\frac{1}{2}E^a_{[i}E^b_{j]}A^i_a A^j_b + \eta\epsilon_{ijk}\zeta_i A^j_a E^a_k - \frac{3}{4}\zeta\cdot\zeta\right] \\
&\quad + 2E^a_i\left[\partial_a\bar{\lambda}(1+i\eta\gamma_5)\sigma_{0i}\Pi + \bar{\Pi}(1+i\eta\gamma_5)\sigma_{0i}\partial_a\lambda\right] + E^a_i\,\left[\bar{\lambda}\sigma_{il}\Pi + \bar{\Pi}\sigma^{il}\lambda\right] A^l_a \\
&\quad + \frac{1}{2(1+\eta^2)}\left[\left\{f_{kl} + (1+\eta^2)J_{kl}\right\}M^{kl} - \frac{1}{4}\left(M^{kl}M^{kl} - M^{kk}M^{ll}\right)\right] \approx\; 0 \tag{64}
\end{aligned}
$$

where ζ^i are given by (63) and

$$M^{kl} = 2\left[f_{kl} + (1+\eta^2)J_{kl}\right] - \delta_{kl}\left[f_{mm} + (1+\eta^2)J_{mm}\right] \tag{65}$$

with

$$
\begin{aligned}
2f_{kl} &= (1+\eta^2)\epsilon^{ijk}E^a_i E^l_b\partial_a E^b_j + \eta\left(E^a_k A^l_a - \delta^{kl}E^a_m A^m_a\right) + (k\leftrightarrow l) \\
2J_{kl} &= i\delta_{kl}\left[\bar{\Pi}\gamma_5\lambda - \bar{\lambda}\gamma_5\Pi\right] \tag{66}
\end{aligned}
$$

This completes our discussion of a fermion minimally coupled to gravity including the Nieh-Yan term. This analysis can now be extended in an analogous manner to a theory with any matter content with any couplings.

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IV.   CONCLUSIONS

We have demonstrated that inclusion of Nieh-Yan topological density in the Lagrangian density of a theory of gravity allows us, in the time gauge, to describe gravity in terms of a real $SU(2)$ connection. The set of constraints so obtained in the Hamiltonian formulation, for $\eta = 1$, is the same as that in the Barbero formulation. For other real values of this parameter, we have the Immirzi formulation with Barbero-Immirzi parameter $\gamma = \eta^{-1}$. Thus the parameter η has similar interpretation as the θ-parameter of QCD. Like the topologically non-trivial vacuum structure of QCD, which reflects itself in terms of presence of the θ-parameter, the η-parameter in the theory of gravity should indicate a rich vacuum structure of gravity which needs further and thorough investigation.

Like the θ-term in QCD, the Nieh-Yan term in gravity is also universal, i.e., it does not need to be changed when various kinds of matter are coupled to the theory. We have discussed this in detail for spin $\frac{1}{2}$ matter coupled to gravity. For other matter, for example, in the theories involving an anti-symmetric tensor gauge field, and also theories of supergravity, the same Nieh-Yan topological term allows a description in terms of a theory of a real $SU(2)$ gauge connection in the time gauge. This is to be contrasted with the case of Holst modification of Hilbert-Palatini action, where for different matter couplings, the corresponding Holst term in the Lagrangian density needs to be changed on a case by case basis so as to keep the equations of motion unaltered [7, 8]. It is worth emphasizing that the Nieh-Yan density is entirely made up of geometric quantities while the modified Holst terms contain matter fields as well. The two get related only after using the connection equation of motion.

In a complete theory of gravity, besides the Nieh-Yan topological term, we need to include two other topological terms, the Pontryagin density and the Euler density. This introduces two additional topological parameters associated with such topological terms, besides the parameter η we have discussed here. Any quantum theory of gravity should have all these three CP-violating topological couplings.

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