Higher Order Sliding Mode controllers with optimal reaching

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Abstract—Higher order sliding mode (HOSM) control design is considered for systems with a known permanent relative degree. In this paper, we introduce the Robust Fuller’s Problem that is a robust generalization of the Fuller’s problem, a standard optimal control problem for a chain of integrators with bounded control. By solving the Robust Fuller’s Problem it is possible to obtain feedback laws that are HOSM algorithms of generic order and, in addition, provide optimal finite-time reaching of the sliding manifold. A common difficulty in the use of existing HOSM algorithms is the tuning of design parameters: our methodology proves useful for the tuning of HOSM controller parameters in order to assure desired performances and prevent instabilities. The convergence and stability properties of the proposed family of controllers are theoretically analyzed. Simulation evidence demonstrates their effectiveness.

Index Terms—Uncertain systems, Higher order sliding mode, Optimal control, Variable Structure Systems

I. INTRODUCTION

Sliding mode controllers [1]–[3] are powerful tools to robustly control uncertain systems. They are able to achieve finite time reaching and exact keeping of a suitably chosen sliding manifold in the state space by means of a discontinuous control. The choice of the sliding manifold is strictly related to the control objective to be attained. In standard formulations, the controlled system state is first steered to the sliding manifold in finite time, and then maintained confined to the manifold itself, giving rise to the so-called sliding mode, so that the equilibrium point corresponding to the fulfillment of the control objective is made asymptotically stable. Indeed, standard sliding mode controllers can only deal with systems having relative degree equal to one. In addition, they can induce undesirable chattering effects, i.e. high frequency oscillations of the controlled variable, which cannot be tolerated in some practical applications, especially mechanical and electro-mechanical ones [4].

Higher order sliding mode (HOSM) controllers are effective in extending the good properties of standard sliding mode controllers to systems with higher relative degree, and can be used also to reduce the chattering effect [5], [6]. Other methods, such as the so-called “dynamic sliding modes” have been proposed to improve the performances of standard sliding mode controllers. A joint analysis of higher order sliding controller and conventional sliding mode with dynamics can be found in [7]. HOSM controllers have the capability of stabilizing to zero in finite time not only the sliding variable, but also a number of its time derivatives. Second order sliding mode controllers have been widely studied, leading to well-established algorithms like the super-twisting controller [8], and the sub-optimal controller [9], [10], also proposed for the multi-input case [11]. However, very few methods have been proposed for the third [12] and generic order case [13], [14]. See also [15], where the so-called “quasi-continuous controller” is presented. Recently, an arbitrary order sliding mode algorithm has been proposed that relies on the idea of integral sliding modes, and exploits optimal linear quadratic control of the auxiliary system with a final state constraint [16]. The key element in the study of HOSM algorithms is the so-called auxiliary system for an uncertain system, that is a perturbed chain of integrators built relying on the output variable and its time derivatives. The goal of any HOSM control problem is to stabilize the auxiliary system state to the origin in a finite time, see e.g. [17].

In this paper, the connection between the construction of HOSM algorithms and the problem of stabilizing in finite time a chain of integrators with bounded control is studied. It is shown that the solution to the so-called Fuller’s problem can be adapted to generate HOSM algorithms that guarantee optimal reaching of the sliding manifold. First, the simple second order case is considered, where it is possible to obtain an interesting geometrical interpretation. Then, the attention is focused on third order algorithms and an explicit formula for the controller is provided. Finally, an HOSM algorithm of arbitrary order is derived. Several already known controllers are shown to belong to this general framework. Within our methodology, it is easy to tune HOSM controller parameters in order to assure desired performances and prevent instabilities. The performances of the proposed algorithms are theoretically analyzed and verified relying on simulation.

The paper is organized as follows. In Section II, higher order sliding modes are reviewed with reference to a dynamical system affected by bounded uncertainty. It is shown that finite-time stabilization of an uncertain system can be obtained by solving a rather general class of homogeneous problems for a chain of integrators. Then, the “Robust Fuller Problem” is introduced, that is the main tool used in this paper for design of HOSM controllers. In Section III, second, third and generic order algorithms are derived. First, a complete convergence analysis for a class of second order controllers that switch over branch of parabolas is carried out. Then, a third order algorithm that is based on the solution to a specific version of the Robust Fuller Problem is proposed. Finally, the procedure is generalized in order to obtain an arbitrary order sliding mode algorithm. In Section IV, the behavior of the proposed algorithms is tested relying on a simulation benchmark. Some conclusions (Section V) end the paper.
II. Preliminaries

Consider a SISO dynamic system affine in the control variable
\[
\begin{align*}
\dot{z}(t) &= a(z, t) + b(z, t)u(t), \\
y(t) &= \sigma(z, t),
\end{align*}
\]
where \(z \in \mathbb{R}^n, u \in \mathbb{R}, \sigma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\) is a sufficiently smooth output function. We say that (1) is an uncertain system if the system order \(n\) and functions \(a(z, t), b(z, t)\) are unknown. However, we assume that the relative degree of the system (1) will be a crucial factor in the subsequent analysis. For autonomous systems, \(r\) is the minimum order \(i\) of the time derivatives \(\sigma^{(i)}\) in which the control \(u\) appears explicitly. For suitable functions \(h(z, t)\) and \(g(z, t)\), one has
\[
y^{(r)}(t) = h(z(t), t) + g(z(t), t)u(t).
\]
Functions \(h(z(t), t)\) and \(g(z(t), t)\) are assumed to be bounded. More precisely, it is assumed that there exist positive constants \(C, K_m, K_M\), such that
\[
-C \leq h(z(t), t) \leq C, \quad 0 < K_m \leq g(z(t), t) \leq K_M.
\]

Let
\[
\sigma^{(i)} := \frac{d^i}{dt^i}u.
\]
The goal of any \(r\)-th order sliding mode control is to attain and keep the manifold
\[
\sigma^{(0)}(z, t) = \sigma^{(1)}(z, t) = \cdots = \sigma^{(r-1)}(z, t) = 0,
\]
in finite time. By introducing
\[
\sigma := \left(\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(r-1)}\right)^T,
\]
condition (4) can be rewritten as \(\sigma = 0\).

Since the only available information about \(h(z, t)\) and \(g(z, t)\) are the bounds (2) and (3), the original dynamical system (1) implies the differential inclusion
\[
\dot{\sigma} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma^{(0)} \\ \sigma^{(1)} \\ \vdots \\ \sigma^{(r-1)} \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}u, \quad f_1(t) \in [-C, C], \quad f_2(t) \in [K_m, K_M], \quad \text{a.e.}
\]
in which the dependence of \(\sigma^{(i)}\) on the original state variable \(z\) has disappeared. For a background on differential inclusions see [20].

In the following, we will need to consider control inputs \(u\) obtained by discontinuous feedback laws \(u = U(\sigma)\). Solutions of differential inclusions are to be understood in the Filippov sense: they are absolutely continuous functions whose time derivative satisfy almost everywhere (a.e.) the inclusion (5) with a properly-enlarged right hand side [21]. More precisely, at each discontinuity point, the right hand side of (5) is replaced by the convex hull of the set of velocity vectors obtained by approaching \(\sigma\) from all the directions in \(\mathbb{R}^r\), while avoiding zero-measure sets.

The design of arbitrary order sliding mode controllers can be reduced to the design of controllers stabilizing the so-called auxiliary system (5) to the origin in finite time. Once the origin is reached, the controlled system is said to exhibit an \(r\)-th order sliding mode. The order of a sliding mode is the number, increased by one, of derivatives of the sliding variable which are steered to zero or, in other words, the dimension of vector \(\sigma\).

Stabilization of a chain of integrators to the origin has been widely studied, see e.g. [22], [23]. Herein, we are interested in controllers that are able to obtain finite-time stabilization robustly with respect to bounded perturbations of the control. In the following, all the functions are assumed to be defined on \(\mathbb{R}^+, L^\infty\) denotes the Banach space of essentially bounded functions whose norm is denoted by \(\| \cdot \|_\infty\).

The next Theorem is relative to a rather general class of optimal control problems, hereafter denoted as Problem 1, formulated on the unperturbed chain of integrators
\[
\dot{z} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}, \quad z \in \mathbb{R}^r, \quad \text{with bounded control } \|v\|_\infty \leq 1. \quad \text{Let}
\]
\[
c_1 = 1, \quad c_i \geq 0, \quad i = 2, \ldots, r,
\]
\[
L(z) := \sum_{i=1}^{r} c_i |z_i|^{\nu/(r+1-i)}, \quad \nu > 0.
\]

Problem 1:
\[
\begin{align*}
\min_{\|v\|_\infty \leq 1} & \int_{0}^{+\infty} L(z(t))dt, \\
\text{subject to } (6) \text{ and } z(0) = z_0.
\end{align*}
\]
In the following, the pair \((v^*, z^*)\) denotes a generic optimal solution (control, state trajectory).

Theorem 1: For all the initial conditions \(z_0 \in \mathbb{R}^r\), Problem 1 admits at least one optimal solution.

Proof: The proof of this Theorem can be found in [24].

There are two limiting cases for Problem 1. For \(\nu \to +\infty\) we obtain
\[
\min_{\|v\|_\infty \leq 1} \sum_{i=1}^{r} c_i |z_i(t)|_\infty,
\]
subject to (6), \(z(0) = z_0\). For \(\nu \to 0^+\), it can be shown that the solution of Problem 1 converges to the solution of the minimal-time problem defined as

Problem 2:
\[
\begin{align*}
\min_{\|v\|_\infty \leq 1} & \int_{0}^{T} 1dt, \\
\text{subject to } (6), \quad z(0) = z_0, \quad \text{and } z(T) = 0.
\end{align*}
\]
Notice that, while Problem 1 is formulated over an infinite horizon, Problem 2 is formulated on the compact interval \([0, T]\) and have the additional final constraint \(z(T) = 0\). Therefore, Problem 2 is not obtained just by letting \(\nu = 0\) in Problem
1, but is the result of a limiting process. More precisely, if $(v_\nu^*, z_\nu^*)$ denotes a generic optimal solution of Problem 1 with $\nu > 0$, and $(v_0^*, z_0^*)$ denotes a generic optimal solution of Problem 2, there exists a sequence $\nu_k \rightarrow 0^+$ such that $v_\nu^*$ converges to $v_0^*$ in the weak* topology [25] of $L^\infty$, and $z_\nu^*$ is point-wise convergent to $z_0^*$, convergence being uniform in all the compact subsets of $\mathbb{R}$. This is a consequence of weak* compactness of the the unit ball in $L^\infty$ (Banach-Alaoglu Theorem) and Theorem 2 in section III, which shows that, for any $\nu > 0$, the optimal trajectory $z_{\nu}^*$ vanish in finite time. This implies that the integral in the definition of the cost function of Problem 1 can be restricted to a finite interval. For an example of convergence of the corresponding feedback laws, the reader can refer to the case of second order sliding modes in subsection III-A, in which explicit expressions are available (notice that parameter $\beta$ converges to 1/2 when $\nu$ tends to zero.

By taking $c_i = 0, \forall i = 2, \ldots, n$ the so-called Fuller’s problem [26] is obtained:

**Problem 3 (Fuller’s Problem):**

$$\min_{\|v\|_\infty \leq 1} \int_0^{+\infty} |z_1(t)|\nu dt,$$

subject to (6) and $z(0) = z_0$.

By the bang-bang principle [27], the optimal feedback law for Problem 3 is a discontinuous function of the system state vector assuming values $-1$ and $+1$ almost everywhere. However, while in the minimal-time case the optimal state trajectory can be simply computed by integrating backward in time the system dynamics starting from the origin, in the general Fuller’s case this is not possible. This is because a certain kind of vibrations occurs near the origin: a detailed description of this phenomenon is given in [24].

Theorem 1 is relative to an unperturbed chain of integrators. However, in order to derive HOSM algorithms, it is necessary to achieve finite-time stabilization to the origin of the perturbed chain (5). Hence, we introduce a robust version of the Fuller’s problem, which can be named “Robust Fuller’s Problem” associated with the control amplitude $\alpha > 0$ and the output function $\sigma$.

**Problem 4 (Robust Fuller’s Problem):**

$$\min_{\|u\|_\infty \leq \alpha} \max_{K_M \leq f_2(t) \leq K_M} \int_0^{+\infty} |\sigma^{(0)}(t)|\nu dt,$$

subject to (5) and $\sigma(0) = \sigma_0$.

The underlying idea is the following: under the assumption of matched bounded disturbance, we seek the control law that guarantees optimality for the worst-case trajectory, that is the trajectory of the differential inclusion (5) such that the max with respect to $f_1$ and $f_2$ is attained in Problem 4. It turns out that, independently of $\nu$, the worst-case trajectory is such that $f_1 = -Csgn(u)$ and $f_2 = K_M$ almost everywhere (see the proof of Lemma 1). In other words, in the worst-case trajectory, the additive uncertainty $f_1$ always opposes the control $u$ with maximum amplitude, while the multiplicative uncertainty reduces the control amplitude $\alpha f_2$ to the smallest possible value $\alpha K_m$.

Thanks to the following lemma, it is possible to reduce Problem 4 to the standard form of Problem 3. It is useful to introduce the reduced control amplitude

$$\alpha_r := \alpha K_m - C.$$

**Lemma 1:** Suppose that $\alpha_r > 0$. Then, if $v^*$ is an optimal control for Problem 3 with $z_0 = \alpha_r^{-1} \sigma_0$, an optimal control $u^*$ for Problem 4 is given by

$$u^*(t) = \alpha_r \sigma^*(t).$$

Relation (7) still holds when $u^*$ and $v^*$ are the limiting optimal controls for $\nu \rightarrow 0^+$ of Problems 3 and 4, respectively.

**Proof:** Let

$$z := \alpha_r^{-1} \sigma.$$  

Apply the Pontryagin’s Maximum Principle [28]. The Hamiltonian function is

$$H(z, \lambda, f_1, f_2, u) = |z_1|\nu + \sum_{i=1}^{r-1} \lambda_i z_{i+1} + \lambda_r (f_1 + f_2 u).$$

The adjoint system is

$$\begin{align*}
\dot{\lambda}_1 &= -\nu |z_1|^{\nu-2} z_1^*, \\
\dot{\lambda}_2 &= -\lambda_1^*, \\
&\vdots \\
\dot{\lambda}_r &= -\lambda_{r-1}^*. 
\end{align*}$$

First, we show that there are no singular arcs, that is $\lambda_r^* = 0$ on a finite time interval cannot occur. Indeed, note that $\lambda_r^* = 0$ on a finite time interval would imply $\lambda_i^* = 0, (i = 1, \ldots, r - 1)$ on the same time interval. For $0 \leq \nu \leq 1$ (observe that the limiting case $\nu = 0$ is included), $\lambda_i^* = 0$ is not possible. For $\nu > 1$, the first equation of the adjoint system yields $z_1^* = 0$ so that $z_i^* = 0, (i = 1, \ldots, r)$ as well. Hence, $\lambda_i^* \neq 0$ almost everywhere. Maximization of $H$ with respect to $f_1$ and $f_2$ yields

$$f_1^* = Csgn(\lambda_1^*),$$

$$f_2^* = \begin{cases} K_m, & u\lambda_1^* < 0, \\
K_M, & u\lambda_r^* > 0. \end{cases}$$

Minimization with respect to $u$ yields $u^* = -\alpha sgn(\lambda_r^*)$. Thus,

$$f_1^* = -C sgn(u^*),$$

$$f_2^* = K_m.$$ 

Let $v := \alpha_r^{-1} (f_1 + f_2 u)$. Then, (5) reduces to (6). Since

$$v^* = \alpha_r^{-1} (-C sgn(u^*) + K_m \alpha sgn(u^*)) = sgn(u^*),$$

the optimization can be restricted to $\|v\|_\infty \leq 1$. Finally,

$$u^* = |u^*| sgn(u^*) = \alpha v^*.$$
Moreover, if \( V(z) \) is an optimal feedback law for Problem 3, then an optimal feedback law for Problem 4 is given by
\[
U(\sigma) = \alpha V \left( \frac{\sigma}{\alpha r} \right).
\]

**Proof:** By Lemma 1, optimal solutions to Problem 4 are associated with optimal solutions to Problem 3 via the correspondence (7)-(8). As a consequence, (9) holds. Existence of optimal solutions is given by Theorem 1.

III. HOSM CONTROLLERS WITH OPTIMAL REACHING

In this Section, it is first noticed that solutions to Problem 4 provide control laws that make the auxiliary system (5) finite-time stable. Then, solutions to the Robust Fuller Problem for second order systems are provided. A complete characterization for a class of second order sliding modes control laws that switch over branch of parabolas is carried out. Next, a generic \( r \)-th order algorithm is derived by solving the limiting case \( \nu \to 0^+ \) for the Robust Fuller Problem. In the third order case it is possible to obtain the solution in explicit form, while in the general case, the controller is characterized by a system of polynomial equations and inequalities. Observe that, in our approach to the HOSM design, the control amplitude \( \alpha \) is considered a physical constraint and is not to be chosen: this feature greatly simplifies the tuning of design parameters.

In the next result (Theorem 2), the finite-time stability property is proven exploiting homogeneity of the system (5) and the loss function of Problem 4. Our proof also provides a specific homogeneous Lyapunov function (that is the value function associated with Problem 4).

**Theorem 2:** Suppose that \( \alpha r > 0 \). Then, the origin is a uniformly global finite-time stable equilibrium of system (5) with the feedback law obtained by solving Problem 4.

**Proof:** Denote by \( \sigma(t; \sigma_0, f_1, f_2, u) \) a trajectory of (5) starting from \( \sigma_0 \), associated with measurable selections \( f_1(\cdot) \), \( f_2(\cdot) \) of the right hand side, and the control function \( u(\cdot) \). Let \( c \) denote a positive number. For any function \( f \in L^\infty \), \( f_c \) denotes a function such that
\[
f_c(t) = f(ct), \quad \text{a.e.}
\]

Introduce also the family of transformations
\[
T_c : \mathbb{R}^r \rightarrow \mathbb{R}^r
\]
\[
T_c(x_1, \ldots, x_r) = (c^r x_1, \ldots, c^r x_r).
\]

From (5), it is easy to see that
\[
\sigma(t; T_c^{-1}(\sigma_0), f_1, f_2, u) = T_c^{-1}(\sigma(ct; \sigma_0, (f_1)(c), (f_2)(c), u_c)).
\]

Consider the value function associated with Problem 4,
\[
W(\sigma_0) := \min_{\|u\|_\infty \leq \alpha} \max_{\|f_1\|_\infty \leq C} \int_0^{+\infty} |\sigma(0)(t)|^\nu dt
\]
subject to (5) and \( \sigma(0) = \sigma_0 \).

Clearly, we have
\[
W(\sigma_0) \geq 0,
\]
\[
W(\sigma_0) = 0 \Leftrightarrow \sigma_0 = 0,
\]
\[
W(T_c(\sigma_0)) = c^{(\nu r + 1)}W(\sigma_0).
\]

The last two properties also imply that \( W \) is coercive. Since \( \alpha r > 0 \), the system (5) is locally finite-time controllable to the origin. Then, for any \( \sigma_0 \) in a neighborhood of the origin, there exists a control function such that the integral at the right hand side of the definition of \( W \) can be restricted to a finite interval \([0, T]\). In that interval, \( \sigma(0) \) is bounded so that \( W \) is bounded in a neighborhood of the origin. Recalling that \( W \) is homogeneous,
\[
|W(\sigma_0)| \leq c^{-\nu r + 1}W(T_c(\sigma_0)), \quad \forall c > 0.
\]

Since \( W \) is bounded in a neighborhood of the origin, the last inequality implies that \( W \) is bounded on every compact set.

We now show that \( W \) is lower semi-continuous. Let \( \sigma^{(k)}_0 \) denote a sequence of initial conditions converging to \( \sigma_0 \) and \( u^{(k)}_1, f_1^{(k)}, f_2^{(k)} \) denote a corresponding sequence of optimal solutions. Since \( W \) is positive and bounded, we can apply the Fatou Lemma [29] to obtain
\[
\lim_{k \to +\infty} \inf \int_0^{+\infty} |\sigma(0)(t; \sigma_0^{(k)}, f_1^{(k)}, f_2^{(k)}, u^{(k)}_1)|^\nu dt \geq W(\sigma_0).
\]

This proves lower semi-continuity.

Now, denote by \( U(\sigma) \) an optimal feedback law obtained by solving Problem 4 and let
\[
L_k := \{ \sigma \in \mathbb{R}^r : W(\sigma) = 1/2^k \}.
\]

\[
T_k(\sigma_0, f_1, f_2) := \inf \{ T : \forall t \geq T, \sigma(t; \sigma_0, f_1, f_2, U(\sigma)) \in L_k \}.
\]

\[
\tau_k := \sup_{\sigma_0 \in L_k} \sup_{K_m \leq f_2(t) \leq K_M} \text{a.e.} \int_0^{+\infty} |\sigma(0)(t)|^\nu dt
\]

Above, \( L_k \) are the levels sets of \( W \), which are compact since \( W \) is lower semi-continuous and coercive. Scalars \( T_k \) are settling times within which the solution of (5) with a specific selection \( f_1, f_2 \) reaches \( L_{k+1} \) starting from \( \sigma_0 \), while \( \tau_k \) is the maximum time within which all the solutions of the differential inclusion (5) reach \( L_{k+1} \) starting from any point in \( L_k \).

Let \( c_0 = 2^{1/(1+\nu r)} \), \( \sigma_1 := T_{c_0}(\sigma_0) \), and observe that
\[
\sigma_1 \in L_k \Leftrightarrow \sigma_0 \in L_{k+1}.
\]

\[
W(\sigma(t; \sigma_0, f_1, f_2, U(\sigma))) = W(\sigma(t; T_{c_0}^{-1}(\sigma_0), f_1, f_2, U(\sigma)))
\]
\[
= W(T_{c_0}^{-1}(\sigma(0); \sigma_1, f_1(c_0), f_2(c_0), U(\sigma)))
\]
\[
= \frac{1}{2} W(\sigma(c_0(0); \sigma_1, f_1(c_0), f_2(c_0), U(\sigma)))
\]

Hence,
\[
T_{k+1}(\sigma_0, f_1, f_2) = c_0^{-1} T_k(\sigma_1, f_1, f_2).
\]
By definition of $\tau_k$,
\[
\tau_{k+1} = \sup_{\sigma_k \in L_k} \sup_{\|f_i\|_{\infty} \leq C} c_0^{-1} T_k(\sigma_1, f_1, f_2)
\]
\[
= c_0^{-1} \sup_{\sigma_1 \in L_k} \sup_{\|f_i\|_{\infty} \leq C} K_m \leq f_2(t) \leq K_M
\]
\[
= c_0^{-1} \tau_k.
\]

Now, it is sufficient to show that the control law obtained by solving Problem 4 makes the origin a globally asymptotically stable equilibrium of system (5). Indeed, this would imply $\tau_0 < +\infty$, so that
\[
\sum_{i=0}^{\infty} \tau_k = \frac{\tau_0}{1 - c_0^{-1}} < +\infty.
\]

Now, let us show that, by the definition of $W$ and the optimal feedback law, $W(\sigma_0) < +\infty$ implies uniform asymptotic stability. By Corollary 1 and the bang-bang principle [27], $U(\sigma)$ is equal to $\pm \alpha$ almost everywhere on $\mathbb{R}^r$ and is discontinuous on a zero-measure set. Thus, we can write $\mathbb{R}^r = \Sigma_1 \cup \Sigma_2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$, where $\Sigma_2$ is a zero measure set in which $U$ is discontinuous. Recalling that $W$ is bounded, assume, by contradiction, that for some initial value $\sigma_0$ and for some $f_1, f_2$, we have $\lim_{t \to +\infty} \sigma^{(0)}(t; \sigma_0, f_1, f_2, U(\sigma)) \neq 0$. This implies divergence of the integral in the definition of $W$ (the integrand is positive) contradicting $W(\sigma_0) < +\infty$. Hence, we have
\[
\lim_{t \to +\infty} \sigma^{(0)}(t; \sigma_0, f_1, f_2, U(\sigma)) = 0,
\]
for any $\sigma_0$ and any $f_1, f_2$.

Letting $\Lambda_1 := \{\sigma \in \mathbb{R}^r : \sigma^{(0)}(t) = 0\}$, we have
\[
\lim_{t \to +\infty} \sigma(t; \sigma_0, f_1, f_2, U(\sigma)) \in \Lambda_1.
\]

Thus, $\Lambda_1$ contains an asymptotically stable set. Assume $\sigma \in \Lambda_1$. For $\sigma^{(1)} \neq 0$, any selection of the velocity vector (the right hand side) of system (5) strictly outside the manifold $\Lambda_1$.

Thus, the asymptotically stable set is entirely contained in $\Sigma_1 := \{\sigma \in \mathbb{R}^r : \sigma^{(0)}(t) = 0, \sigma^{(1)}(t) = 0\}$. By applying this argument $r - 1$ times, we conclude that the asymptotically stable set reduces to the origin
\[
\Lambda_r := \{\sigma \in \mathbb{R}^r : \sigma^{(i)} = 0, i = 0, \ldots, r - 1\}.
\]

Now, assuming $0 \in \Sigma_2$, by (5) we have either $(U(0) = -\alpha)$
\[
\dot{\sigma} \in (0, 0, \ldots, [-C - \alpha K_M, C - \alpha K_M]^T),
\]
or $(U(0) = \alpha)$
\[
\dot{\sigma} \in (0, 0, \ldots, [-C + \alpha K_M, C + \alpha K_M]^T).
\]

In both cases, since $\alpha_r > 0$, any selection of the velocity vector points outside the origin. Thus, $0 \in \Sigma_2$. Over $\Sigma$, $U(\sigma)$ is discontinuous so that, according to the notion of Filippov solution, the right hand side of (5) must be replaced by the convex hull of the set of velocity vectors obtained by approaching $\sigma$ from all the directions in $\mathbb{R}^r$. For system (5), the convexification leads to
\[
\dot{\sigma} \in (0, 0, \ldots, [-C + \alpha K_M, C + \alpha K_M])^T.
\]

Now, it is possible to select $0$ for the right-hand side, so that the origin is the only uniformly asymptotically stable point.

An alternative way to conclude the proof of Theorem 2 can be devised by invoking a recent result [30]. However, this would require to prove that the feedback law obtained by solving Problem 4 is homogeneous. A discussion on the role of homogeneity in higher order sliding modes can be found in [31], where robustness and asymptotic accuracy issues are analyzed.

### A. Second Order Sliding Mode Algorithms

Consider Problem 3 in a two dimensional state space. It can be shown [26] that an optimal feedback law with $\nu > 0$ is given by
\[
V(x_1, x_2) = -\text{sgn} (x_1 + \beta(\nu) x_2 x_2), \quad \beta(\nu) \in \left(\frac{1}{4}, \frac{1}{2}\right).
\]

Different values for the continuous function $\beta(\nu)$ are tabulated in [32] where it can be seen that the curve $\beta(\nu)$ is decreasing from $1/2$ to $1/4$ as $\nu$ tends to $+\infty$. For $\nu \to +\infty$, we have $\beta(\nu) \to 1/4$, while for $\nu \to 0^+$ (minimal time problem), the optimal feedback law [27] is given by
\[
V(x_1, x_2) = -\begin{cases} 0, & (x_1, x_2) \in M_0, \\ \text{sgn}(x_2), & (x_1, x_2) \in M_1 \setminus M_0, \\ \text{sgn}(x_1 + \frac{1}{2} x_2 |x_2|), & \text{else} \end{cases}.
\]

By Corollary 1, an optimal family of second order switching curves can be derived from the solution to the Robust Fuller’s Problem (Problem 4):
\[
U(\sigma, \dot{\sigma}) = -\alpha \text{sgn} \left( \sigma + \frac{\beta(\nu)}{\alpha} \dot{\sigma} |\dot{\sigma}| \right), \quad \beta(\nu) \in \left[\frac{1}{4}, \frac{1}{2}\right).
\]

Theorem 2 ensures that these laws are indeed second-order sliding mode algorithms. Note that, when the value of $\dot{\sigma}$ is not available, one can rely on Levant’s exact differentiator [33] to retrieve $\dot{\sigma}$ in finite time, so that the proposed second order sliding mode control law can be applied by simply measuring the sliding variable $\sigma$.

Remarkably, it can be shown that stabilization in finite time is possible with all the values of $\beta > 1/4$. In order to see that stabilization in finite time is guaranteed also for $\beta > 1/2$, let us rewrite the control law (10) in a slightly different form. First, observe that, for any $x, y \in \mathbb{R}, \gamma > 0, p > 1$,
\[
x = -\gamma |y|^{p-2} \Rightarrow y = -\frac{1}{\gamma} |x|^{q-2}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]
In particular, for $p = 3$, we have
\[
x = -\gamma |y| \Rightarrow y = -\sqrt{\frac{1}{\gamma}} |x|^{\frac{1}{3}}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]
Hence, we can write
\[ U(\sigma, \dot{\sigma}) = -\alpha \text{sgn} \left( \dot{\sigma} + \sqrt{\frac{\alpha_r}{\beta(\nu)}} \text{sgn}(\sigma) \sqrt{\sigma} \right). \] (11)

Equation (11) coincides with the controller with a “prescribed convergence law” \cite{8}, \cite{34}:
\[
U_p(\sigma, \dot{\sigma}) = -\alpha \text{sgn} \left( \dot{\sigma} + \lambda \text{sgn}(\sigma) \sqrt{\sigma} \right),
\]
provided that
\[
\lambda = \sqrt{\frac{\alpha_r}{\beta(\nu)}}.
\]

For these controllers, it is easy to see that the condition
\[
\alpha > 0, \quad \lambda > 0, \quad \alpha_r > \lambda^2/2,
\]
implies finite-time stabilization of the auxiliary system via a sliding mode on the switching curve (see, for example, \cite{14}). This last condition is equivalent to \( \beta > 1/2 \). In fact, Theorem 2 shows that finite-time convergence is guaranteed under the less strict condition
\[
\alpha > 0, \quad \lambda > 0, \quad \alpha_r > \lambda^2/4.
\]

By putting together the previous observations, we can conclude that, for all \( \beta > 1/4 \), the controller (10) is a second order sliding mode algorithm. However, the behavior of the auxiliary system trajectories depends on \( \beta \). For \( 1/4 \leq \beta < 1/2 \), the system state may converge twisting around the origin with an infinite number of turns in a finite time-interval. Also, the control variable may have an infinite number of switches in a finite time interval. This phenomenon has been called “chattering” in the optimal control literature \cite{24}, although it is different from that arising in sliding mode control. We have already seen that the values \( 1/4 \leq \beta \leq 1/2 \) are particularly interesting being those associated with the Robust Fuller’s Problem. For \( \beta > 1/2 \), the optimal control interpretation is lost and all the trajectories reach the origin in finite time featuring a sliding mode on the switching curve. The three different switching regions are illustrated in Fig. 1.

**B. Third Order Sliding Mode Algorithm**

Solution to the general Robust Fuller Problem in the third-order case has a very complicated form. Hence, we restrict the analysis to the case \( \nu \to 0^+ \), corresponding to the minimal time problem for the worst-case trajectory. The idea of using time-optimal control in the third order problem is present also in \cite{12}. It is possible to analyze the general case \( \nu > 0 \) by adapting the results of \cite{35} and \cite{36} to the Robust Fuller Problem. The minimal time case is probably the most interesting one, in that it provides a bound for the worst-case reaching time while it is sufficient for the finite-time stabilization of the auxiliary system.

The solution to Problem 3 with \( \nu \to 0^+ \) for a chain of three integrators is known since \cite{37}. By applying Corollary 1, the solution to the Robust Fuller Problem 4 with \( \nu \to 0^+ \) in the third order case is obtained.

\[
U(\sigma) = -\alpha \begin{cases}
0, & \sigma \in M_0, \\
\text{sgn}(\sigma), & \sigma \in M_1 \setminus M_0, \\
\text{sgn} \left( \sigma + \frac{\tilde{\sigma}^2 u_1}{2\alpha_r} \right), & \sigma \in M_2 \setminus M_1,
\end{cases}
\]

(12)

where
\[
s(\sigma, \dot{\sigma}, \ddot{\sigma}) := \sigma + \frac{\tilde{\sigma}^3}{3\alpha_r^2} + u_2 \left[ \frac{1}{\sqrt{\alpha_r}} \left( u_2 \dot{\sigma} + \frac{\tilde{\sigma}^2}{2\alpha_r} \right) \right]^\frac{3}{2} + \frac{\ddot{\sigma}}{\alpha_r}
\]

and the manifolds \( M_0, M_1, M_2 \) are defined by
\[
M_0 := \left\{ (\sigma, \dot{\sigma}, \ddot{\sigma}) \in \mathbb{R}^3 : \sigma = \dot{\sigma} = \ddot{\sigma} = 0 \right\},
\]
\[
M_1 := \left\{ (\sigma, \dot{\sigma}, \ddot{\sigma}) \in \mathbb{R}^3 : \sigma - \frac{\tilde{\sigma}^3}{6\alpha_r^2} = 0, \dot{\sigma} + \frac{\ddot{\sigma} \sqrt{\sigma}}{2\alpha_r} = 0 \right\},
\]
\[
M_2 := \left\{ (\sigma, \dot{\sigma}, \ddot{\sigma}) \in \mathbb{R}^3 : s(\sigma, \dot{\sigma}, \ddot{\sigma}) = 0 \right\}.
\]

It is interesting to observe that the proposed controller has no free parameters to be tuned (except for the control amplitude \( \alpha \)).

**C. Generic Order Sliding Mode Algorithms**

The construction of the previous sections can be generalized to obtain an \( r \)-th order HOSM algorithm with optimal
reaching. In order to solve the Robust Fuller’s Problem with \( \nu \to 0^+ \), the computation of the switching hyper-surfaces associated with the standard time-optimal control is required. Again, the time-optimal control has a nested structure similar to (12). However, for practical purposes it is sufficient to evaluate the “outer” switching surface. This can be accomplished by a rather standard procedure that consists in the integration of the system equations (6) backward in time, changing the control sign in \( r \) time instants, starting from either \( +1 \) or \( -1 \). By parameterizing the state trajectory with the switching instants, two systems of polynomial equations are obtained, the first associated with the control sequence \( \{+1, -1, \ldots\} \), the second associated with \( \{-1, +1, \ldots\} \). If the system state is not exactly on the switching surface, only one of these two systems admits a solution satisfying \( t_i \leq 0, i = 1, \ldots, r \). By knowing which of these two systems is solvable in the real negative orthant, it is possible to decide the sign of the optimal control.

Via a suitable change of variable (the procedure is summarized in the Appendix), it is possible to show that the solution to the Robust Fuller Problem with \( \nu \to 0^+ \) is given by the following \( r \)-th order HOSM algorithm:

\[
U(\sigma) = \alpha(-1)^{r+1} \tilde{U} (\sigma),
\]

where \( \tilde{U} \in \{-1, +1\} \) is such that the system of equation and inequalities

\[
z_i^{r-i+1} + 2 \sum_{n=2}^{r} (-1)^{n+1} z_n^{r-i+1} = \tilde{U} \frac{(r - i + 1)!}{\alpha r} \sigma^{i-1},
\]

\[
i = 1, \ldots, r,
\]

\[
z_1 \leq \ldots \leq z_r \leq 0.
\]

admits solutions.

The complexity of the switching surface grows very fast with \( r \). It would be an interesting issue to determine if it is possible to obtain exact tests for the switching hyper-surface exploiting recent algorithms from computational algebra [38]. Alternatively, numerical methods can be employed in order to check whether (14)-(15) admit solutions. It is still an open issue the derivation of efficient methods (numerical or exact-algebraic) to deal with this problem.

**IV. SIMULATION RESULTS**

In this section, we illustrate the behavior of the algorithms proposed in the previous sections, by means of a simulated example.

Nonholonomic car motion planning [39] is a typical application area for testing the performances of HOSM algorithms. We use a benchmark problem [13], [15], [16] that gives us the possibility to test the behavior of our algorithms and obtain a comparison with existing approaches. The system is described by

\[
\dot{x} = v \cos \varphi, \quad \dot{y} = v \sin \varphi, \quad \dot{\varphi} = \frac{v}{\ell} \tan \theta, \quad \dot{\theta} = u,
\]

where \((x, y)\) are the cartesian coordinates of the rear-axle middle point, \(\varphi\) is the orientation angle, \(\theta\) is the steering angle and \(u\) is the control variable, see Fig. 3. Parameters \(v = 10 \text{ m/s}, \ell = 5 \text{ m}\) represents the longitudinal velocity and the distance between the two axles, respectively. The goal of the control system is to steer the car from a given
initial position \((x_0, y_0, \varphi_0, \theta_0) = (0, 0, 0, 0)\) to the trajectory
\(y = 10 \sin(x/20) + 5\). Thus, a sliding variable \(\sigma\) is defined as
\[
\sigma = y - 10 \sin(x/20) - 5.
\]
It is easy to check that the relative degree of this system is 3:
\[
\begin{align*}
\dot{\sigma} &= v \left( \sin \varphi - \frac{1}{2} \cos \left( \frac{x}{20} \right) \cos \varphi \right), \\
\dot{\varphi} &= \frac{v^2}{\ell} \left[ \tan \theta \left( \cos \varphi + \frac{1}{2} \cos \left( \frac{x}{20} \right) \sin \varphi \right) \\
&+ \frac{\ell}{40} \sin \left( \frac{x}{20} \right) \cos^2 \varphi \right], \\
\ddot{\sigma} &\approx \frac{v^2}{\ell} \sec^2 \theta \left( \cos \varphi + \frac{1}{2} \cos \left( \frac{x}{20} \right) \sin \varphi \right) \\
&+ \frac{v^3}{\ell^2} \tan^2 \theta \left( -\sin \varphi + \frac{1}{2} \cos \left( \frac{x}{20} \right) \cos \varphi \right) \\
&- \frac{v^3}{40\ell} \sin \left( \frac{x}{20} \right) \cos \varphi \sin \varphi (2 \tan \theta + 1) \\
&+ \frac{y^3}{20} \cos \left( \frac{x}{20} \right) \cos^3 \varphi.
\end{align*}
\]
Note that, for small angles \(\varphi \approx 0, \theta \approx 0\), we have
\[
\ddot{\sigma} \approx \frac{v^2}{\ell} + \frac{v^3}{20} \cos \left( \frac{x}{20} \right) = 20u + 50 \cos \left( \frac{x}{20} \right).
\]
Hence, under the assumption of small angles
\[
\ddot{\sigma} \in [-50, 50] + 20u,
\]
so that \(K_m \approx K_M \approx 20, C \approx 50\). However, since moderately ample angles can manifest in the transient, we conservatively let \(K_m = 5, K_M = 5000, C = 90\). Thus, we consider the larger inclusion
\[
\ddot{\sigma} \in [-90, 90] + [5, 5000] u.
\]
The feasibility condition \(\alpha_r = \alpha K_m - C > 0\) yields \(\alpha > 18\).

Three different 3-th order controllers are considered: the third order algorithm proposed in [13], the recently proposed quasi-continuous third order sliding mode algorithm [15], and our third order law (12). As previously observed, since the switching surface is a zero-measure set, it is sufficient to use the “outer” switching surface when implementing (12). The controller laws are denoted by \(U^3_L\) (Levant 3-th order), \(U^3_{QC}\) (Quasi-Continuous 3-th order) and \(U^3_{OR}\) (Optimal Reaching 3-th order):
\[
\begin{align*}
U^3_L(\sigma) &= -\alpha \text{sgn} \left[ \dot{\sigma} + \beta_2 (|\dot{\sigma}| + |\dot{\sigma}|^{\frac{3}{2}}) \text{sgn}(\dot{\sigma} + \beta_1 |\dot{\sigma}|^{\frac{3}{2}} \text{sgn}(\sigma)) \right], \\
U^3_{QC}(\sigma) &= -\alpha \left( \dot{\sigma} + \beta_2 (|\dot{\sigma}| + |\dot{\sigma}|^{\frac{3}{2}}) \right) (\dot{\sigma} + \beta_1 |\dot{\sigma}|^{\frac{3}{2}} \text{sgn}(\sigma)) \frac{\dot{\sigma}}{|\dot{\sigma}| + \beta_2 (|\dot{\sigma}| + \beta_1 |\dot{\sigma}|^{\frac{3}{2}} \text{sgn}(\sigma))^{\frac{1}{2}}}, \\
U^3_{OR}(\sigma) &= -\alpha \text{sgn} \left[ \sigma + \frac{\ddot{\sigma}^3}{3 \alpha_r^2} + \frac{u_2}{\sqrt{\alpha_r}} \left( u_2 \dot{\sigma} + \frac{\ddot{\sigma}^2}{2 \alpha_r} \right) + \frac{u_2 \ddot{\sigma}}{\alpha_r} \right],
\end{align*}
\]
\[
u_2 := \text{sgn} \left( \dot{\sigma} + \beta_1 |\dot{\sigma}| \right) \right).
\]
For the sake of a fair comparison, the amplitude \(\alpha\) is considered a physical constraint and thus fixed for all the controllers to 20. Moreover, to make the comparison depend only on the control laws, the use of state observers has been avoided by directly calculating derivatives of the sliding variable. For what concerns the choice of design parameters in \(U_L\) and \(U_{QC}\), the author suggests setting \(\beta_1 = 1, \beta_2 = 2, 13, 15\). Note that in \(U^3_{OR}\) the only needed parameter is the reduced control amplitude \(\alpha_r = \alpha K_m - C = 10\).

Simulations have been carried out in the time horizon 0–20 seconds. The controllers have been enabled only at \(t = 0.5s\) (in the first half-second are set to zero) in order to better compare the results with [13], [15]. Results are summarized in Fig. 4-6: dotted lines are relative to \(U_L\), continuous lines to \(U_{QC}\) and thick-continuous lines to \(U_{OR}\). In Fig. 4, the car trajectory obtained with the three controllers is shown together with the reference trajectory \(y = 10 \sin(x/20) + 5\). In Fig. 5 the auxiliary system state \((\sigma, \dot{\sigma}, \ddot{\sigma})\) is plotted against time for all the controllers. Finally, in Fig. 6 the dynamics of the steering angle \(\theta\) is reported.

From Fig. 5 we see that \(U_{OR}\) performs very well in terms of reaching time: in this respect, it consistently outperforms the other two controllers. From the analysis of Fig. 6, it can be seen that \(U_{OR}\) produces quite effective steering angles, while avoiding dangerous vibration (differently from \(U_L\)). The controller \(U_{QC}\) produces a quite smooth evolution of the steering angle and also avoid vibrations, but is definitely slower than \(U_{OR}\) and a bit more unpredictable. It is interesting to observe that the speed of convergence of \(U_{QC}\) is rather independent of the control amplitude: in [15], the author obtained a reaching time similar to that of this experiment (for \(U_{QC}\)) by using only \(\alpha = 1\). This also shows that our bounds on \(C\) and \(K_m\) are very conservative. In our setting, using \(\alpha = 1\) corresponds to \(C < K_m\) (feasibility condition). Notice that, to compare the controllers with such value of the control amplitude, parameter \(\alpha_r\) should be reduced accordingly in \(U_{OR}\). We have made additional simulations by fixing \(\alpha = 1\) and trying several values of \(\alpha_r\) in the interval \([1, 4]\). \(U_{OR}\) still obtains the best reaching times.
V. CONCLUSIONS

The connection between the design of high order sliding modes algorithms and the solution to some optimal control problems for a perturbed chain of integrators has been investigated in the paper. The Robust Fuller Problem, which is a suitable generalization of the so-called Fuller’s problems for a chain of integrators, has been formulated. It is shown that the optimal control law solving the Robust Fuller’s Problem is able to attain finite-time stabilization of the auxiliary system associated with arbitrary order sliding modes. In this way, a rather general family of HOSM controllers is obtained which guarantees optimal reaching of the sliding manifold with respect to a suitable optimality index. This family includes several already known HOSM control algorithms. Special cases, namely second and third order algorithms are analyzed in the paper. The proposed controllers have simple rules to select the design parameters and exhibit state of the art performances.

APPENDIX

Denote by $e_i$ the $i$-th vector of the canonical basis of $\mathbb{R}^r$ (all zeros, with 1 in position $i$):

$$ e_i = \begin{pmatrix} 0 & \ldots & 1 & \ldots & 0 \end{pmatrix}^T. $$

System (6) can be written as

$$ \dot{x} = Ax + Bv $$

with

$$ A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad B = e_r. $$

We have

$$ e^{At} = \sum_{n=1}^{+\infty} \frac{A^n t^n}{n!} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \ldots & \frac{t^{r-1}}{(r-1)!} \\ 0 & 1 & t & \ldots & \frac{t^{r-2}}{(r-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & t \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. $$

For the considered system, it is possible to reach the origin from any initial state with at most $r - 1$ switches of the control variable $v$. Since linear systems are reversible, the characterization of the optimal state trajectories can be made using a standard techniques from optimal control of linear systems [27], that is backward integration in time from the final state (in this case, the origin). More precisely, we integrate the system equations backward in time in $r$ adjacent time intervals, starting from the origin, changing the control sign at each step. Let $\tilde{U} = \pm 1 = \tilde{U}^{-1}$ denote the first control sign, and

$$ \tilde{x}_k := x(t_1, \ldots, t_k), \quad A_k := e^{A t_k}, $$

$$ b_k := \tilde{U} (-1) t_{k+1} e^{A t_k} \int_0^{t_k} e^{A \tau} e_r d\tau. $$

Then, we obtain

$$ \dot{x}_{k+1} = A_k x_k + b_{k+1}, \quad \dot{x}_0 = 0, $$

with

$$ b_k = \tilde{U} (-1) t_{k+1} \begin{pmatrix} t_k^{r-1} \\ t_{k+1}^{(r-1)!} \\ \vdots \\ t_k \end{pmatrix}. $$

Now we can compute the state variables as a function of the switching instants $t_i$:

$$ x_k = \sum_{n=1}^{k-1} \left( \prod_{j=n+1}^{k} A_j \right) b_n + b_k = \sum_{n=1}^{k-1} e^{A \sum_{j=n+1}^{k} t_j} b_n + b_k. $$

We have

$$ e^{A \sum_{j=n+1}^{k} t_j} b_n e_t = \tilde{U} (-1)^{n+1} \frac{\sum_{s=1}^{r} \left( \frac{\sum_{j=n+1}^{k} t_j}{s} \right)^{s-i} t_n^{r-s+1}}{(r-s+1)!} $$

$$ \tilde{U} (-1)^{n+1} \frac{\sum_{s=0}^{r} \left( \frac{\sum_{j=n+1}^{k} t_j}{s} \right)^{r-i+1-s}}{(r-i+1)!} $$

$$ \tilde{U} (-1)^{n+1} \frac{\sum_{s=0}^{r} \left( \frac{\sum_{j=n+1}^{k} t_j}{s} \right)^{r-i+1}}{(r-i+1)!}. $$

Introduce the variables

$$ z_n^k := \sum_{j=n}^{k} t_j, \quad z_n := z_n^r, $$

so that

$$ e^{A \sum_{j=n+1}^{k} t_j} b_n e_t = \tilde{U} (-1)^{n+1} \frac{(z_n^k)^{r-i+1} - (z_{n+1}^k)^{r-i+1}}{(r-i+1)!}. $$

Now,

$$ (x_k)_i = \sum_{n=1}^{k-1} \tilde{U} (-1)^{n+1} \frac{(z_n^k)^{r-i+1} - (z_{n+1}^k)^{r-i+1}}{(r-i+1)!} $$

$$ + \tilde{U} (-1)^{k+1} \frac{(z_k^k)^{r-i+1}}{(r-i+1)!} $$

$$ = \tilde{U} (z_1^k)^{r-i+1} + 2 \sum_{n=2}^{k} (-1)^{n+1} (z_n^k)^{r-i+1}. $$

In particular, letting $k = r$, we have

$$ x_i = \tilde{U} (z_i^r)^{r-i+1} + 2 \sum_{n=2}^{r} (-1)^{n+1} z_n^r^{r-i+1}. $$

Since we are integrating backward in time, we have

$$ t_i \leq 0, \quad i = 1, \ldots, r, $$

which implies

$$ z_1 \leq \ldots \leq z_r \leq 0. $$

By applying Corollary 1, controller (13)-(14)-(15) is obtained.
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