Preprints of the 18th IFAC World Congress
Milano (Italy) August 28 - September 2, 2011

Practical Robust Positive Invariance for Large–Scale Discrete Time Systems ⋆

Saša V. Raković * ** **Benjamin Kern** * **Rolf Findeisen** *

* *Institute for Automation Engineering, Otto–von–Guericke–Universität Magdeburg, Germany(e-mail: {sasa.rakovic,benjamin.kern,rolf.findeisen}@ovgu.de).*
** *Deparment of Engineering Sciences, University of Oxford, UK*

Abstract: This note introduces practical set invariance notions for physically interconnected, discrete–time systems, subject to additive but bounded disturbances. The developed approach provides a decentralized, non–conservative and computationally tractable way to study desirable robust positive invariance and stability notions for the overall system as well as to guarantee safe and independent operation of the constituting subsystems. These desirable properties are inherited, under mild assumptions, from the classical stability and invariance properties of the associated vector–valued dynamics which capture in a simple but appropriate and non–conservative way the dynamical behavior induced by the underlying set–dynamics of interest.

Keywords: Large-scale Systems, Set Invariance, Stability, Set–Dynamics.

1. INTRODUCTION

The analysis of large–scale systems offers many challenging system–theoretic problems. For an in depth overview of control synthesis– and analysis–methods, see the important works [Šiljak, 1978, Šiljak and Zečević, 2005, Bakule, 2008] and references therein. In general, the modular analysis and controller synthesis of interconnected systems benefits greatly from the use of vector Lyapunov–functions [Bellman, 1962, Lakshmikantham et al., 1991]. Motivated by this notion and the recent results on set invariance under state and output feedback utilizing set–dynamics [Artstein and Raković, 2008, 2010], in Raković et al. [2010] a novel set invariance notion for interconnected systems was introduced and investigated. In particular, it was shown in Raković et al. [2010], that several "dynamically interconnected positively invariant sets" for the constituting subsystems lead to desirable invariance properties for the whole system, providing that an appropriately designed dynamics of suitably parametrized sets is stable. Our prime objective is to generalize the set invariance and stability notions of Raković et al. [2010] in order to provide flexible and practicable notions allowing for safe, stable and independent operation of all subsystems despite the presence of constraints, restrictions on the amount of information available locally (this "informational restriction" is inevitably induced by the decentralization of the original system) as well as the presence of additive but bounded disturbances. A more detailed discussion and all the proofs for the results in this note are given in Raković et al. [2011].

Paper Structure: Section 2 provides preliminaries. Section 3 reviews briefly practical set invariance in the nominal case. Section 4 discusses the extension to the robust case and delivers the main contribution of this work. Section 5 addresses the corresponding convergence and stability issues. Section 6 comments on computational aspects and delivers an illustrative example. Section 7 closes this note with a few concluding remarks.

Basic Nomenclature and Definitions: The sets of positive, non–negative integers and non-negative reals are denoted by $\mathbb{N}_+, \mathbb{N}$ and $\mathbb{R}_+$. For each positive integer $q \in \mathbb{N}_+$, $\mathbb{N}_{[1:q]} := \{1, 2, \ldots, q\}$. For $r \in \mathbb{N}_{[1:q]}$, we denote $\mathbb{N}_{(q,r)} := \{1, \ldots, r-1, r+1, \ldots, q\} = \mathbb{N}_{[1:q]} \backslash \{r\}$. A set X is non–trivial, if it is a proper, non–empty subset of $\mathbb{R}^n$ and not a singleton. Given two sets X, Y in $\mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y \ : \ x \in X, y \in Y\}$. For a set sequence $\{X_i \subset \mathbb{R}^n\}_{i=a}^{b}$, where $a, b \in \mathbb{N}$, $b > a$, we define $\bigoplus_{i=a}^{b} X_i := X_a \oplus \cdots \oplus X_b$. Given a set X in $\mathbb{R}^n$ and a real matrix M of compatible dimension (possibly a scalar), we define the image of X under M by $MX := \{Mx \, : \, x \in X\}$ and the preimage of X under M by $M^{-1}X := \{x \, : \, Mx \in X\}$. For any matrix $M \in \mathbb{R}^{n \times n}$, $\rho(M)$ denotes the largest absolute value of its eigenvalues. A set $X \subset \mathbb{R}^n$ is a C–set, if it is compact, convex and contains the origin. A set $X \subset \mathbb{R}^n$ is a proper C–set (PC–set) if it is a C–set, which contains the origin in its (non–empty) interior. A collection of sets $\{X_i \subset \mathbb{R}^n \, : \, i \in \mathbb{N}_{[1:q]}\}$ is a PC–collection if each X_i is a PC–set. A set $X \subseteq \mathbb{R}^n$ is symmetric, with respect to the origin in $\mathbb{R}^n$, if $X = -X$. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces and a *polytope* is the closed and bounded *polyhedron.* The family of all subsets of $\mathbb{R}^n$ is denoted by $2^{\mathbb{R}^n}$. The family of non–empty compact subsets in $\mathbb{R}^n$ is denoted by $\mathrm{Com}(\mathbb{R}^n)$. For any closed convex set Y in $\mathbb{R}^n$, $s(Y, x) := \sup_y\{x^T y \, : \, y \in Y\}$ for $x \in \mathbb{R}^n$ is called the support function. For $X \in \mathrm{Com}(\mathbb{R}^n)$ and $Y \in \mathrm{Com}(\mathbb{R}^n)$, the *Hausdorff semi–distance* and the *Hausdorff distance* of X and Y are given by $h(L, X, Y) := \min_\alpha\{\alpha \ : \ X \subseteq Y \oplus \alpha L, \ \alpha \geq 0\}$ and $H(L, X, Y) :=$

⋆ Saša V. Raković is the corresponding author (svr@sasavrakovic.com). The second author would like to acknowledge the support of the IMPRS, Magdeburg. This work was supported by the German research foundation (DFG–SPP 1305), project "Entwicklung asynchroner prädiktiver Regelungsverfahren für digital vernetzte Systeme".

$\max\{h(L, X, Y), h(L, Y, X)\}$, where L is a given, symmetric, proper C–set in $\mathbb{R}^n$.

2. PRELIMINARIES

Throughout this note we consider a set of N discrete–time, time-invariant, linear interconnected systems given by:

$$\forall i \in \mathbb{N}_{[1:N]},\ x_i^+ = A_{(i,i)}x_i + \sum_{j\in\mathbb{N}_{(N,i)}} A_{(i,j)}x_j + w_i, \quad (1)$$

where $\forall i \in \mathbb{N}_{[1:N]}$, $x_i \in \mathbb{R}^{n_i}$ is the current state of the i^{th} subsystem, $A_{(i,i)} \in \mathbb{R}^{n_i\times n_i}$ is the state transition matrix of the i^{th} subsystem and $w_i \in \mathbb{W}_i$ denotes a disturbance acting on the i^{th} subsystem. The current overall state is given by $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^n$, with $n = \sum_{i\in\mathbb{N}_{[1:N]}} n_i$. For each $i \in \mathbb{N}_{[1:N]}$ and $j \in \mathbb{N}_{(N,i)}$, the i^{th} subsystem is affected by the j^{th} subsystem via the linear, physical interconnections specified by matrices $A_{(i,j)} \in \mathbb{R}^{n_i\times n_j}$. The variables x_i and w_i are subject to constraints:

$$\forall i \in \mathbb{N}_{[1:N]},\ x_i \in \mathbb{X}_i \text{ and } w_i \in \mathbb{W}_i. \quad (2)$$

In this note we work under the standard assumption:

Assumption 2.1. For all $i \in \mathbb{N}_{[1:N]}$, the sets $\mathbb{X}_i$ and $\mathbb{W}_i$ are proper C–sets and C–sets in $\mathbb{R}^{n_i}$, respectively.

Remark 2.1. Throughout this note we work in the autonomous setting due to space limitations. However, the methods presented here can be easily extended to a much broader setup involving decision making processes/control syntheses for each of the subsystems, see Raković et al. [2010] for more details.

The prime objective of this note is to introduce a non–conservative and practical robust positive invariance notion for the set of N discrete–time, time–invariant, linear interconnected systems given in (1). This naturally leads to the standard concept of robust positive invariance, cf. Aubin [1991], Blanchini [1999], Blanchini and Miani [2008]:

Definition 2.1. A set Ω is said to be robust positively invariant for the system $x^+ = Ax + w$ and the constraint set $(\mathbb{X}, \mathbb{W})$, if and only if $\Omega \subseteq \mathbb{X}$ and for all $x \in \Omega$ and $w \in \mathbb{W}$ it holds that $Ax + w \in \Omega$.

The most obvious way to obtain robust positive invariant sets for the subsystems in (1) is to consider the augmented form describing the overall system:

$$x^+ = Ax + w, \quad (3)$$

where $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^n$ is the overall current state, $w = (w_1, w_2, \ldots, w_N) \in \mathbb{R}^n$ is the overall current disturbance and $A \in \mathbb{R}^{n\times n}$ is the state transition matrix of the overall system which is composed from the matrices in $A_{(i,j)}$ appearing in (1). The overall state $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^n$ and disturbance $w = (w_1, w_2, \ldots, w_N) \in \mathbb{R}^n$ are, in view of (2) subject to hard constraints:

$$x \in \mathbb{X} \text{ and } w \in \mathbb{W},\ \text{where, } \mathbb{X} := \mathbb{X}_1 \times \mathbb{X}_2 \cdots \times \mathbb{X}_N,\ \text{and,}$$
$$\mathbb{W} := \mathbb{W}_1 \times \mathbb{W}_2 \cdots \times \mathbb{W}_N. \quad (4)$$

Indeed, the theory of the maximal and minimal robust positively invariant sets for the system (3) and constraint sets (4) is well studied and understood [Aubin, 1991, Kolmanovsky and Gilbert, 1998, Blanchini, 1999, Artstein and Raković, 2008, Blanchini and Miani, 2008]. Furthermore, the characterization and computation of the inner and outer robust positively invariant approximations of the maximal and minimal robust positively invariant sets, respectively, for the system (3) and constraint sets (4) is also well understood [Raković et al., 2005, Raković, 2007, Artstein and Raković, 2008, Raković and Fiacchini, 2008]. However, the corresponding computational procedures are severely limited by the dimension of the considered problem (hence, inapplicable to the case of large–scale systems). Furthermore, the classical robust positive invariance notions do not permit, in general, for the independent initialization and operation of the constituting subsystems in (1).

3. PRACTICAL POSITIVE INVARIANCE

A brief overview of the approach of Raković et al. [2010] relevant for the nominal case is now recalled. We consider systems of the form

$$\forall i \in \mathbb{N}_{[1:N]},\ \bar{x}_i^+ = A_{(i,i)}\bar{x}_i + \sum_{j\in\mathbb{N}_{(N,i)}} A_{(i,j)}\bar{x}_j, \quad (5)$$

where $\forall i \in \mathbb{N}_{[1:N]}, j \in \mathbb{N}_{(N,i)}$, $\bar{x}_i \in \mathbb{R}^{n_i}$, $A_{(i,i)} \in \mathbb{R}^{n_i\times n_i}$, $A_{(i,j)} \in \mathbb{R}^{n_i\times n_j}$. The local states $\bar{x}_i$, $i \in \mathbb{N}_{[1:N]}$ are subject to hard constraints:

$$\forall i \in \mathbb{N}_{[1:N]},\ \bar{x}_i \in \mathbb{X}_i, \quad (6)$$

where the constraint sets $\mathbb{X}_i$, $i \in \mathbb{N}_{[1:N]}$ satisfy Assumption 2.1. The main role in the examination of practical set invariance is played by the induced, independent, set–dynamics specified, for all $i \in \mathbb{N}_{[1:N]}$, by:

$$\bar{X}_i^+ = F_i(\bar{X}),\ \text{with}$$
$$F_i(\bar{X}) := A_{(i,i)}\bar{X}_i \oplus \bigoplus_{j\in\mathbb{N}_{[1:N]}} A_{(i,j)}\bar{X}_j, \quad (7)$$

where $\bar{X} = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_N) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \cdots \times 2^{\mathbb{R}^{n_N}}$. Within this setting, the exact set invariance problem requires the characterization and determination of an invariant family of sets. The complexity of the exact set invariance problem is in Raković et al. [2010] alleviated by the introduction of practical positive invariance notion. Namely, the notion that utilizes a parameterized family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ specified by:

$$\mathcal{S}(\mathbb{S}, \Theta) := \{(\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N)\ :\ \theta \in \Theta\}, \quad (8)$$

where $\mathbb{S}$ is a prescribed collection of sets given by:

$$\mathbb{S} := \{S_i\ :\ i \in \mathbb{N}_{[1:N]}\},\ (S_1, \ldots, S_N) \in 2^{\mathbb{R}^{n_1}} \times \ldots \times 2^{\mathbb{R}^{n_N}}, \quad (9)$$

and $\Theta \subseteq \mathbb{R}_+^N$ is a suitably defined set. More specifically, it was shown in Raković et al. [2010] that, under Assumption 2.1, the collection of sets $\mathbb{S}$ can be taken to be a PC–collection of sets. Simple linear functions specified, for all $i \in \mathbb{N}_{[1:N]}$, by:

$$\theta_i^+ = \sum_{j\in\mathbb{N}_{[1:N]}} \mu_{(i,j)}\theta_j,\ \text{with } \forall(i,j) \in \mathbb{N}_{[1:N]} \times \mathbb{N}_{[1:N]},$$
$$\mu_{(i,j)} := \min_{\mu\geq 0}\{\mu : A_{(i,j)}S_j \subseteq \mu S_i\} \quad (10)$$

are associated with the parameterized family of sets $\mathcal{S}(\mathbb{S}, \Theta)$. These linear functions capture the dynamics of the scaling factors $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ which takes the form of a linear autonomous discrete–time system:

$$\theta^+ = M\theta, \quad (11)$$

where $\theta \in \mathbb{R}_+^N$ is the current value of the scaling factors, $\theta^+ \in \mathbb{R}_+^N$ is the successor value of the scaling factors and $M \in \mathbb{R}_+^{N\times N}$ is the matrix composed from the scalars

$\mu_{(i,j)} \in \mathbb{R}_+$, $(i,j) \in \mathbb{N}_{[1:N]} \times \mathbb{N}_{[1:N]}$ defined in (10). We note that, by construction, given a current value of the scaling factors $\theta \in \mathbb{R}_+^N$ it is guaranteed that, for all $i \in \mathbb{N}_{[1:N]}$, it holds that $F_i((\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N)) \subseteq \theta_i^+ S_i$. The dynamical behavior of the outer–bounding approximate set–dynamics is fully characterized by the simple θ–dynamics given by (11) which are standard vector–valued dynamics. The satisfaction of state constrains (6) is enforced by imposing the constraints on the scalings θ as specified by:

$$\Theta_0 := \{\theta \in \mathbb{R}_+^N : \forall i \in \mathbb{N}_{[1:N]}, \theta_i S_i \subseteq \mathbb{X}_i\}. \tag{12}$$

The set Θ appearing in the parameterization of the family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ is taken to be a positively invariant set (potentially the maximal positively invariant set) for the θ–dynamics and constraint set Θ_0, i.e. the set Θ satisfies:

$$\Theta \subseteq \Theta_0 \text{ and } M\Theta \subseteq \Theta. \tag{13}$$

The notion of practical positive invariance as introduced and analyzed in Raković et al. [2010] is specified by:

Definition 3.1. A family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ specified in (8) with $\mathbb{S}$ given by (9) and $\Theta \subseteq \mathbb{R}_+^N$ is said to be a *positively invariant family of sets* for the system (5) and constraint set (6), if for all $i \in \mathbb{N}_{[1:N]}$ and all $(\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N) \in \mathcal{S}(\mathbb{S}, \Theta)$ it holds that, $\theta_i S_i \subseteq \mathbb{X}_i$, $F_i((\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N)) \subseteq \theta_i^+ S_i$, and additionally $(\theta_1^+ S_1, \theta_2^+ S_2, \ldots, \theta_N^+ S_N) \in \mathcal{S}(\mathbb{S}, \Theta)$.

It was demonstrated in Raković et al. [2010] that, under Assumption 2.1 and when $\mathbb{S}$ is a PC–collection of sets, the sufficient condition for the existence of a non–trivial positively invariant family of sets for the interconnected system (5) and constraint set (6) is the strict stability of the matrix $M \in \mathbb{R}_+^{N \times N}$ inducing the scaling dynamics in (11). In fact, it was also shown in Raković et al. [2010] that, under these assumptions, the system (5) can be initiated and operated in an independent fashion while guaranteeing the constraint satisfaction of the overall system as well as stability of the origin with the well–defined domain of attraction (which is induced from the collection of sets $\mathbb{S}$ and the corresponding positively invariant set Θ).

4. PRACTICAL ROBUST POSITIVE INVARIANCE

Motivated by theoretical interest and practical relevance, we proceed to extend the framework of Raković et al. [2010] to the robust case. As in the disturbance free case, the requirement for the independent operation of the set of N subsystems affected by the additive uncertainty as specified in (1) leads naturally to induced, independent, set–dynamics given for all $i \in \mathbb{N}_{[1:N]}$ and all $X = (X_1, X_2, \ldots, X_N) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots \times 2^{\mathbb{R}^{n_N}}$, by

$$X_i^+ = F_i(X, \mathbb{W}), \tag{14}$$

where $\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2 \times \ldots \mathbb{W}_N$ and

$$F_i(X, \mathbb{W}) = A_{(i,i)} X_i \oplus \bigoplus_{j \in \mathbb{N}_{(N,i)}} A_{(i,j)} X_j \oplus \mathbb{W}_i. \tag{15}$$

Within the framework of set–dynamics (14)–(15) several robust positive invariance notions can be considered. In this sense, a potentially interesting alternative is to look for a robust positively invariant set of a particular form consistent with the decentralized form of the overall system. Namely, to aim for the characterization and computation of a robust positively invariant set Ω taking the form $\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_N$ where, for all $i \in \mathbb{N}_{[1:N]}$, $\Omega_i \subseteq \mathbb{R}^{n_i}$ as reflected via the following definition:

Definition 4.1. A collection of sets $\boldsymbol{\Omega} := \{\Omega_i : i \in \mathbb{N}\}$, is a robust positively invariant collection of sets for the system (1) and the constraint sets (2), if and only if, for all $i \in \mathbb{N}_{[1:N]}$ it holds that $\Omega_i \subseteq \mathbb{X}_i$ and $F_i((\Omega_1, \Omega_2, \ldots, \Omega_N), \mathbb{W}) \subseteq \Omega_i$.

However, the conditions of Definition 4.1 are, in general case, rather strong and lead to restrictive and unrealistic requirements for the existence of a collection of robust positively invariant sets. A non–naive and exact robust positive notion is possible but it requires the utilization of a suitable family of sets as specified by:

Definition 4.2. A family of sets $\mathcal{X} \subseteq 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots 2^{\mathbb{R}^{n_N}}$ is said to be a *robust positively invariant family of sets* for the system (1) and constraint sets (2) if and only if, for all $X = (X_1, X_2, \ldots, X_N) \in \mathcal{X}$ and all $i \in \mathbb{N}_{[1:N]}$ it holds that, $X_i \subseteq \mathbb{X}_i$, $F_i((X_1, X_2, \ldots, X_N), \mathbb{W}) \subseteq X_i^+$ and $X^+ = (X_1^+, X_2^+, \ldots, X_N^+) \in \mathcal{X}$.

The concept of robust positively invariant family of sets in Definition 4.2 is fully compatible with the induced, independent, set–dynamics specified in (14)–(15) and allows us to capture appropriately the physical interconnection effects by directly relating them to the dynamical change of the sets $X \in \mathcal{X}$. From a conceptual point of view, Definition 4.2 provides a framework, in which the theoretical analysis can be carried out quite conveniently. However, our equally important aim is to provide computationally tractable robust positive invariance notions and to this end we proceed to analyze a sensible trade–off between naive and conservative notions of Definition 4.1 and general and flexible notions of Definition 4.2. The main idea is to reach a reasonable trade–off by considering a parameterized family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ specified by:

$$\mathcal{S}(\mathbb{S}, \Theta) := \{(\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N) \ : \ \theta \in \Theta\}, \tag{16}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}_+^N$, $\Theta \subseteq \mathbb{R}_+^N$ and, for all $i \in \mathbb{N}_{[1:N]}$, $S_i \in 2^{\mathbb{R}^{n_i}}$, and by invoking the following concept of practical robust positive invariance:

Definition 4.3. Given a collection of sets $\mathbb{S} = \{S_i : i \in \mathbb{N}_{[1:N]}\}$ with $(S_1, S_2, \ldots, S_N) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots 2^{\mathbb{R}^{n_N}}$ and a set $\Theta \subseteq \mathbb{R}_+^N$, the family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ specified by (16) is said to be a practical *robust positively invariant family of sets* for the system (1) and the constraint sets (2) if and only if, for all $i \in \mathbb{N}_{[1:N]}$ and all $(\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N) \in \mathcal{S}(\mathbb{S}, \Theta)$, it holds that:

$$\theta_i S_i \subseteq \mathbb{X}_i, \tag{17a}$$

$$F_i((\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N), \mathbb{W}) \subseteq \theta_i^+ S_i, \text{ and,} \tag{17b}$$

$$(\theta_1^+ S_1, \theta_2^+ S_2, \ldots, \theta_N^+ S_N) \in \mathcal{S}(\mathbb{S}, \Theta). \tag{17c}$$

The main rationale behind the practical robust positive invariance notions of Definition 4.3 is to restrict the exact set–dynamics (14)–(15) to a suitably parameterized family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ as specifed in (16) and employ, for computational simplicity, outer–bounding set–dynamics induced from the dynamics of the scaling factors θ. The basic assumption on the collection of sets $\mathbb{S} = \{S_i : i \in \mathbb{N}_{[1:N]}\}$ is, as in the nominal case [Raković et al., 2010], invoked without loss of generality due to Assumption 2.1:

Assumption 4.1. The collection of sets $\mathbb{S} = \{S_i : i \in \mathbb{N}_{[1:N]}\}$ is a PC–collection of sets.

The requirement (17b) and the computational practicability motivates the utilization of the collection of simple, affine functions $\{\mu_i(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ specified, for all $i \in \mathbb{N}_{[1:N]}$ and all $\theta \in \mathbb{R}^N_+$, by:

$$\mu_i(\theta) := \sum_{j\in\mathbb{N}_{[1:N]}} \mu_{(i,j)}\theta_j + \alpha_i, \tag{18}$$

where, for each i and j,

$$\mu_{(i,j)} := \min_{\mu\geq 0}\{\mu \ : \ A_{(i,j)}S_j \subseteq \mu S_i\}, \text{ and}, \tag{19a}$$

$$\alpha_i := \min_{\alpha\geq 0}\{\alpha \ : \ \mathbb{W}_i \subseteq \alpha S_i\}. \tag{19b}$$

The collection of affine functions $\{\mu_i(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ given via (18)–(19), leads to the introduction of the dynamics of scaling factors taking the form of a simple affine, discrete–time autonomous system:

$$\theta^+ = M\theta + \alpha, \tag{20}$$

where θ is the vector of current scaling factors, θ^+ is the vector of successor scaling factors, and the θ–transition matrix $M \in \mathbb{R}^{N\times N}_+$ as well as α, the offset vector, are composed, respectively, from the scalars $\mu_{(i,j)} \in \mathbb{R}_+$, $(i,j) \in \mathbb{N}_{[1:N]} \times \mathbb{N}_{[1:N]}$ and the scalars α_i, $i \in \mathbb{N}_{[1:N]}$ specified in (19). Evidently, the system–theoretic analysis of the affine dynamics of the scaling factors is simple and, hence, it further enhances computational practicability. Clearly, by construction, it follows that for any $\theta \in \mathbb{R}^N_+$ and any $X = (X_1, X_2, \ldots, X_N) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots 2^{\mathbb{R}^{n_N}}$ such that, for all $i \in \mathbb{N}_{[1:N]}$, $X_i \subseteq \theta_i S_i$ it holds that $F_i((X_1, X_2, \ldots, X_N), \mathbb{W}) \subseteq F_i((\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N), \mathbb{W}) \subseteq \theta_i^+ S_i$ with $\theta_i^+ = \mu_i(\theta)$. Hence, the condition (17b) in the practical robust positive invariance notions of Definition 4.3 is ensured by considering the θ–dynamics as specified in (20). In order to ensure the satisfaction of the condition (17a) it is necessary to restrict the values of the scaling factors θ to an adequate subset Θ of the set of constraint admissible scaling factors Θ_0 given by:

$$\Theta_0 := \{\theta \in \mathbb{R}^N_+ \ : \ \forall i \in \mathbb{N}_{(N,i)}, \ \theta_i S_i \subseteq \mathbb{X}_i\}. \tag{21}$$

We note that the set Θ_0 is, under Assumptions 2.1 and 4.1, a convex and non–empty compact subset of $\mathbb{R}^N_+$. In order to guarantee the non–emptiness of a suitable set $\Theta \subseteq \Theta_0$ as well as to ensure the satisfaction of the condition (17c) we provide the following set of natural sufficient conditions:

Assumption 4.2. (i) The matrix M inducing the θ–dynamics in (20) is strictly stable, i.e. $\rho(M) < 1$,

(ii) the unique fixed point of the equation $\theta = M\theta + \alpha$ say $\bar{\theta}$ is such that $\bar{\theta} \in \Theta_0$, and,

(iii) the set $\Theta \subseteq \mathbb{R}^N_+$ is a non–trivial convex and compact positively invariant set for the system (20) and constraint set (21), i.e. $\forall \theta \in \Theta \subseteq \Theta_0$, $M\theta + \alpha \in \Theta$.

Remark 4.1. We note that Assumption 4.2 (*i*) is an usual requirement which is also utilized in the nominal case [Raković et al., 2010]. Assumption 4.2 (*ii*) is also a reasonable condition since $\rho(M) < 1$, so that the fixed point $\bar{\theta} = \sum_{k=0}^{\infty} M^k\alpha = (I - M)^{-1}\alpha$ is a globally stable attractor for the θ–dynamics specified in (20). Finally, the satisfaction of Assumption 4.2 (*iii*) is guaranteed under Assumptions 2.1, 4.2 (*i*) and 4.2 (*ii*). In particular, the corresponding set Θ as specified in Assumption 4.2 (*iii*) can be obtained by the standard set iteration:

$$\forall k \in \mathbb{N}, \ \Theta_{k+1} = M^{-1}(\{-\alpha\} \oplus \Theta_k) \bigcap \Theta_0,$$

where M is specified in (20) and Θ_0 given by (21). Under Assumptions 2.1, 4.2 (*i*) and 4.2 (*ii*), the set iteration above results in a sequence of convex and compact sets $\{\Theta_k\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$, $\{\bar{\theta}\} \subseteq \Theta_{k+1} \subseteq \Theta_k$. In fact, the set sequence $\{\Theta_k\}_{k\in\mathbb{N}}$ admits the limit, say Θ_∞, with respect to the Hausdorff distance which is the maximal positively invariant set for the system (20) and constraint set (21). Furthermore, the standard results [Kolmanovsky and Gilbert, 1998, Raković and Fiacchini, 2008] imply that the limit Θ_∞ of the set sequence $\{\Theta_k\}_{k\in\mathbb{N}}$ is, under Assumptions 2.1, 4.2 (*i*) and 4.2 (*ii*), finitely determined, i.e. there exists a finite integer k^* such that $\Theta_{k^*} = \Theta_{k^*+1} = \Theta_\infty$; in addition, the set Θ_∞ is a polytopic subset of $\mathbb{R}^N_+$ which is guaranteed to contain the set $[0, \bar{\theta}_1] \times [0, \bar{\theta}_2] \times \ldots \times [0, \bar{\theta}_N]$ (here $\bar{\theta}_i$ are the coordinates of the corresponding fixed point $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, \ldots \bar{\theta}_N)$).

Utilizing the preceding construction and invoked assumptions we are able to offer our first main result:

Proposition 4.1. Suppose Assumptions 2.1, 4.1 and 4.2 hold. Then the family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ given by (16) is a non-trivial, robust positively invariant family of sets.

5. CONVERGENCE AND STABILITY

We now turn attention to the issues related to convergence and stability. Under our assumptions, as already indicated, for any $\theta \in \Theta$ and any $X = (X_1, X_2, \ldots, X_N) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots 2^{\mathbb{R}^{n_N}}$ such that, for all $i \in \mathbb{N}_{[1:N]}$, $X_i \subseteq \theta_i S_i$ it holds that:

$$F_i((X_1, X_2, \ldots, X_N), \mathbb{W}) \subseteq F_i((\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N), \mathbb{W}), \tag{22a}$$

$$F_i((\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N), \mathbb{W}) \subseteq \theta_i^+ S_i, \text{ with}, \tag{22b}$$

$$\theta_i^+ = \mu_i(\theta) \in \Theta. \tag{22c}$$

Motivated by this fact we proceed to demonstrate how the stability properties of the θ–dynamics in (20) can be utilized to obtain guaranteed robust stability properties of the exact induced, independent set–dynamics in (14)–(15) as well as the original set of N systems specified in (1).

For the subsequent analysis, let $\boldsymbol{X}(X_0)$ denote, for any $X_0 = (X_{(0;1)}, X_{(0;1)}, \ldots, X_{(0;N)}) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots 2^{\mathbb{R}^{n_N}}$, the sequence $\{X_k = (X_{(k;1)}, X_{(k;2)}, \cdots X_{(k;N)})\}_{k\in\mathbb{N}}$, generated by (14)–(15), i.e. for all $k \in \mathbb{N}$ and all $i \in \mathbb{N}_{[1:N]}$,

$$X_{(k+1;i)} = F_i(X_k, \mathbb{W}), \tag{23}$$

where the maps $F_i(\cdot,\cdot)$, $i \in \mathbb{N}_{[1:N]}$ are given by (15). Let also $\boldsymbol{Y}(Y_0)$ denote, for any $\theta_0 = (\theta_{(0;1)}, \theta_{(0;2)}, \ldots, \theta_{(0;N)}) \in \mathbb{R}^N_+$ and $Y_0 = (\theta_{(0;1)}S_1, \theta_{(0;2)}S_2, \ldots, \theta_{(0;N)}S_N)$, the sequence $\{Y_k = (\theta_{(k;1)}S_1, \theta_{(k;2)}S_2, \cdots \theta_{(k;N)}S_N)\}_{k\in\mathbb{N}}$, where for all $k \in \mathbb{N}$ and all $i \in \mathbb{N}_{[1:N]}$, we have $\theta_k = (\theta_{(k;1)}, \theta_{(k;2)}, \ldots, \theta_{(k;N)})$ and

$$\theta_{k+1} = M\theta_k + \alpha. \tag{24}$$

Our next main result discusses the corresponding convergence properties:

Theorem 5.1. Suppose Assumptions 2.1, 4.1 and 4.2 hold. Consider the family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ given by (16) and any sequence $\boldsymbol{Y}(Y_0)$ generated by (24) with $Y_0 \in \mathcal{S}(\mathbb{S}, \Theta)$. Then for all $k \in \mathbb{N}$, (i) $Y_k \in \mathcal{S}(\mathbb{S}, \Theta)$, (ii)

$$\sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, \theta_{(k;i)}S_i, \bar{\theta}_i S_i) \leq a^k b \sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, \theta_{(0;i)}S_i, \bar{\theta}_i S_i),$$

for some scalars $a \in [0,1)$ and $b \in (0,\infty)$, and, (iii) $H(L_i, \theta_{(k;i)}S_i, \bar{\theta}_i S_i) \to 0$ as $k \to \infty$, with the fixed point $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N)$ from Assumption 4.2 (*ii*).

The relationships indicated in (22) allow for a direct utilization of Theorem 5.1 to discuss guaranteed invariance and convergence issues for the exact induced, independent set–dynamics in (14)–(15).

Corollary 5.1. Suppose Assumptions 2.1, 4.1 and 4.2 hold. Consider the family of sets $\mathcal{S}(\mathbb{S},\Theta)$ given by (16) and any two sequences $\boldsymbol{X}(X_0)$, $\boldsymbol{Y}(Y_0)$ generated by (23) and (24) with, for all $i \in \mathbb{N}_{[1:N]}$, $X_{(0;i)} \subseteq Y_{(0;i)}$ for some $Y_0 \in \mathcal{S}(\mathbb{S},\Theta)$. Then for, all $k \in \mathbb{N}$ and all $i \in \mathbb{N}_{[1:N]}$, (i) $X_{(k;i)} \subseteq Y_{(k;i)}$, (ii) $X_{(k;i)} \subseteq \mathbb{X}_i$, (iii) $h(L_i, X_{(k;i)}, \bar{\theta}_i S_i) \to 0$ as $k \to \infty$, with the fixed point $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_N)$ from Assumption 4.2 (*ii*).

Remark 5.1. Coming back to our initial problem specified in Section 2, stability and convergence properties are now easily discussed. Utilizing Theorem 5.1 and Corollary 5.1 we can analyze the behavior of the set of state sequences $\{x_{(k;i)}\}_{k\in\mathbb{N}}$, $i \in \mathbb{N}_{[1:N]}$ of the N physically, interconnected systems that satisfy, for all $i \in \mathbb{N}_{[1:N]}$ and $k \in \mathbb{N}$,

$$x_{(k+1;i)} \in A_{(i,i)} x_{(k;i)} + \sum_{j\in\mathbb{N}_{(N,i)}} A_{(i,j)} x_{(k;j)} \oplus \mathbb{W}_i, \quad (25)$$

with given initial conditions $x_{(0;i)}$, $i \in \mathbb{N}_{[1:N]}$. As a consequence, we can also deduce from Theorem 5.1 and Corollary 5.1 that for all $k \in \mathbb{N}$ and all $i \in \mathbb{N}_{[1:N]}$, it holds that $x_{(k;i)} \in X_{(k;i)} \subseteq \theta_{(k;i)} S_i$. More precisely, any set of state sequences $\{x_{(k;i)}\}_{k\in\mathbb{N}}$, $i \in \mathbb{N}_{[1:N]}$ that satisfies (25) is such that, for all $k \in \mathbb{N}$ and all $i \in \mathbb{N}_{[1:N]}$, it holds that

$$x_{(k;i)} \in \theta_{(k;i)} S_i \subseteq \mathbb{X}_i$$

providing, of course, that $x_{(0;i)} \in \theta_{(0;i)} S_i$, $i \in \mathbb{N}_{[1:N]}$ and $\theta_0 = (\theta_{(0;1)}, \theta_{(0;2)}, \dots, \theta_{(0;N)}) \in \Theta$ and $\{\theta_k\}_{k\in\mathbb{N}}$ is generated via (24). Furthermore, we also have that, for all $k \in \mathbb{N}$ and all $i \in \mathbb{N}_{[1:N]}$, it holds that, $\sum_{i\in\mathbb{N}_{[1:N]}} h(L_i, \{x_{(k;i)}\}, \bar{\theta}_i S_i) \leq \sum_{i\in\mathbb{N}_{[1:N]}} h(L_i, \theta_{(k,i)} S_i, \bar{\theta}_i S_i)$ and, hence,:

$$\sum_{i\in\mathbb{N}_{[1:N]}} h(L_i, \{x_{(k;i)}\}, \bar{\theta}_i S_i) \leq a^k b \sum_{i\in\mathbb{N}_{[1:N]}} h(L_i, \theta_{(0,i)} S_i, \bar{\theta}_i S_i)$$

for some scalars $a \in [0,1)$ and $b \in (0,\infty)$ and where $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, \dots \bar{\theta}_N)$ is the fixed point specified in Assumption 4.2 (*ii*). In turn, it also follows that $h(L_i, \{x_{(k,i)}\}, \bar{\theta}_i S_i) \to 0$ as $k \to \infty$. In addition, the initialization of the subsystems (1) is also simplified, since each individual subsystem does not need the exact knowledge of the initial states of the other subsystems; in other words the only requirement for the safe and independent operation of the interconnected systems specified in (1) under uncertainty is the condition that $x_{(0;i)} \in \theta_{(0;i)} S_i$, with $\theta_0 = (\theta_{(0;1)}, \theta_{(0;2)}, \dots, \theta_{(0;N)}) \in \Theta$, for all $i \in \mathbb{N}_{[1:N]}$.

6. COMPUTATIONAL REMARKS

In this section we consider the case when the state and disturbance constraint sets are, respectively, PC– and C–polytopic sets and are given, for all $i \in \mathbb{N}_{[1:N]}$, via their irreducible representations:

$$\mathbb{X}_i := \{x_i \in \mathbb{R}^{n_i} : \forall j \in \mathbb{N}_{[1:q_i]},\ l_{(i,j)}^T x_i \leq 1\}, \text{ and,} \quad (26a)$$

$$\mathbb{W}_i := \{w_i \in \mathbb{R}^{n_i} : \forall k \in \mathbb{N}_{[1:r_i]},\ m_{(i,k)}^T w_i \leq t_{(i,k)}\}, \quad (26b)$$

where, for all $j \in \mathbb{N}_{[1:q_i]}$ and $k \in \mathbb{N}_{[1:r_i]}$, $l_{(i,j)} \in \mathbb{R}^{n_i}$, $m_{(i,k)} \in \mathbb{R}^{n_i}$ and $t_{(i,k)} \in \mathbb{R}_+$, respectively, for all $i \in \mathbb{N}_{[1:N]}$. With the requirement of the independent operation and the objective to detect and compute locally the candidate sets S_i, $i \in \mathbb{N}_{[1:N]}$ in mind, a meaningful way is to employ the so–called λ–contractive sets [Blanchini and Miani, 2008]. Namely, the underlying idea is to consider, for all $i \in \mathbb{N}_{[1:N]}$, the set of N subsystems whose dynamics and constraints take the form:

$$x_i^+ = A_{(i,i)} x_i + w_i, \text{ and,} \quad (27a)$$

$$x_i \in \mathbb{X}_i \text{ and } w_i \in \mathbb{W}_i. \quad (27b)$$

The interconnection terms $\sum_{j\in\mathbb{N}_{(N,i)}} A_{(i,j)} x_j$ are neglected in (27), but their effect is compensated for indirectly by computing the maximal or just λ–contractive sets $S_i, i \in \mathbb{N}_{[1:N]}$. The corresponding λ–contractive sets $S_i, i \in \mathbb{N}_{[1:N]}$ are such that:

$$A_{(i,i)} S_i \oplus \mathbb{W}_i \subseteq \lambda_i S_i \text{ and } S_i \subseteq \mathbb{X}_i, \quad (28)$$

where the contraction factors $\lambda_i, i \in \mathbb{N}_{[1:N]}$ satisfy, for all $i \in \mathbb{N}_{[1:N]}$, $\lambda_i \in [0,1)$. The computation of the λ–contractive sets $S_i, i \in \mathbb{N}_{[1:N]}$ for the subsystems and constraints sets can be realized efficiently by employing the standard methods [Blanchini and Miani, 2008] or more recent results on the parameterized λ–contractive sets [Raković and Barić, 2009, 2010]. In fact, under these conditions, it follows that the maximal λ–contractive sets are computable in finite time. Furthermore, under our assumptions, they are PC–polytopic sets, when non–empty, and, hence, admit irreducible representations given, for all $i \in \mathbb{N}_{[1:N]}$, by:

$$S_i := \{s_i \in \mathbb{R}^{n_i} : \forall p \in \mathbb{N}_{[1:\tau_i]},\ \phi_{(i,p)}^T s_i \leq 1\}, \quad (29)$$

where, for all $p \in \mathbb{N}_{[1:\tau_i]}$, $\phi_{(i,p)} \in \mathbb{R}^{n_i}$. The computation of the scalars $\mu_{(i,j)}$ forming the matrix M is efficiently done by using the basic properties of the support function [Rockafellar, 1970, Schneider, 1993]. Namely, the set inclusions $A_{(i,j)} S_i \subseteq \mu_{(i,j)} S_i$ are satisfied, for all $i \in \mathbb{N}_{[1:N]}$ and all $j \in \mathbb{N}_{[1:N]}$, if and only if, for all $p \in \mathbb{N}_{[1:\tau_i]}$, it holds that $s(S_j, A_{(i,j)}^T \phi_{(i,p)}) \leq \mu_{(i,j)}$. In turn, it follows that, for all $i \in \mathbb{N}_{[1:N]}$ and all $j \in \mathbb{N}_{[1:N]}$, the corresponding scalars $\mu_{(i,j)}$ are given by:

$$\mu_{(i,j)} := \max_p \{s(S_j, A_{(i,j)}^T \phi_{(i,p)}) \ : \ p \in \mathbb{N}_{[1:\tau_i]}\} \quad (30)$$

and can be computed by solving a sequence of well–defined linear programs. Likewise, the set inclusions $W_i \subseteq \alpha_i S_i$ are satisfied, for all $i \in \mathbb{N}_{[1:N]}$, if and only if, for all $p \in \mathbb{N}_{[1:\tau_i]}$, it holds that $s(\mathbb{W}_i, \phi_{(i,p)}) \leq \alpha_i$. Hence, for all $i \in \mathbb{N}_{[1:N]}$, the corresponding scalars α_i are given by:

$$\alpha_i := \max_p \{s(\mathbb{W}_i, \phi_{(i,p)}) \ : \ p \in \mathbb{N}_{[1:\tau_i]}\} \quad (31)$$

and, as above, can be computed by solving a sequence of well–defined linear programs. Finally, the set Θ_0 specified in (21) takes the polytopic form given by:

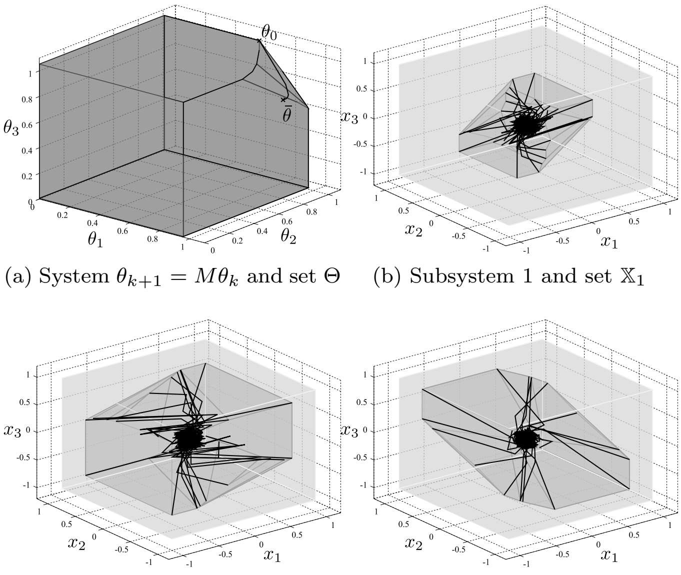
$$\Theta_0 = \{\theta \in \mathbb{R}_+^N : \forall i \in \mathbb{N}_{[1:N]},\ \theta_i \leq \max_{j\in\mathbb{N}_{[1:q_i]}} s(S_i, l_{(i,j)})\}, \quad (32)$$

and, consequently, the polytopic set Θ (see Assumption 4.2 (*iii*)) can be computed as indicated in Remark 4.1.

6.1 Illustrative Example

We consider a nine dimensional, discrete–time linear system, $x^+ = Ax + w$, where

$$A = \begin{pmatrix} 0.1 & 0.3 & 0 & -0.1 & 0 & 0 & 0.1 & 0.1 & 0 \\ 0.1 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0.1 \\ -0.3 & 0.3 & 0.3 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & -0.1 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & -0.1 & 0.2 & 0.3 & 0 & 0 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0.1 & 0 \\ 0.1 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0.1 & 0 & 0 & 0.1 & 0.3 & -0.1 & 0.2 \end{pmatrix},$$



(a) System $\theta_{k+1} = M\theta_k$ and set Θ (b) Subsystem 1 and set $\mathbb{X}_1$

(c) Subsystem 2 and set $\mathbb{X}_2$ (d) Subsystem 3 and set $\mathbb{X}_3$

Fig. 1. Sets of sample trajectories initialized in the sets $\theta_{(0;i)}S_i$ for $i \in \mathbb{N}_{[1:3]}$ with initial condition θ_0.

and treat it as three, 3–dimensional, interconnected systems with the matrices $A_{(i,i)}$, being on the diagonal of the matrix A. The corresponding state and disturbance constraint sets are $\mathbb{X}_i = \{x \in \mathbb{R}^3 : |x|_\infty \leq 1\}$ and $\mathbb{W}_i = \{w \in \mathbb{R}^3 : |w|_\infty \leq 0.1\}$ for $i \in \mathbb{N}_{[1:3]}$. The collection of sets $\{S_1, S_2, S_2\}$ is obtained by computing PC–polytopic, λ-contractive sets, for the subsystems $x_i^+ = A_{(i,i)}x_i$, for $i \in \mathbb{N}_{[1:3]}$, with the contraction factors $(0.595, 0.589, 0.459)$. The λ–contractive sets S_i, for $i \in \mathbb{N}_{[1:3]}$ are shown in Figure 1. The matrix M and vector α are computed by using relations (30) and (31); the matrix M is strictly stable. The set Θ_0 is given by $\Theta_0 = \{\theta \in \mathbb{R}^3_+ : |\theta_1| \leq 1.202, |\theta_2| \leq 1.2045, |\theta_3| \leq 1.2786\}$ while the set Θ is computed according to Remark 4.1 and is depicted in Figure 1 (a). The corresponding fixed point $\bar{\theta}$ is, in this case, given by $\bar{\theta} = (0.8986, 0.7444, 0.6891)$ and it lies in the interior of the set $\Theta \subseteq \Theta_0$. Hence, all our assumptions are satisfied. We select $\theta_0 = (0.6328, 1.0097, 1.0469)$, which is an extreme point of the set Θ, and simulate several sets of state trajectories, with initial conditions being equal to the extreme points of the sets $\theta_{(0;i)}S_i$, for $i \in \mathbb{N}_{[1:3]}$. As expected in view of Theorem 5.1 and Corollary 5.1, the state constraints are satisfied, and the trajectories converge to the sets $\bar{\theta}_i S_i$ in a stable fashion.

7. CONCLUSIONS

In this paper, we discussed exact and practical robust positive invariance notions for physically interconnected, discrete–time systems. For computational tractability, we provided a relaxation of the exact notions, by employing a suitably parametrized family of sets. It was shown, that under mild and natural assumptions, the derived notions allow for the safe, independent operation for each of the physically interconnected subsystems, regardless the decentralization effects and the presence of the additive bounded disturbances.

REFERENCES

Z. Artstein and S.V. Raković. Feedback and Invariance under Uncertainty via Set Iterates. *Automatica*, 44(2): 520–525, 2008.

Z. Artstein and S.V. Raković. Set Invariance under Output Feedback: A Set–Dynamics Approach. *International Journal of Systems Science*, 2010.

J.P. Aubin. *Viability Theory.* Birkhäuser, Boston, 1991.

L. Bakule. Decentralized control: An overview. *Ann. Rev. in Contr.*, 32(1):87–98, 2008.

R. Bellman. Vector Lyapunov Functions. *SIAM J. Contr.*, 1:32–34, 1962.

F. Blanchini. Set invariance in control. *Automatica*, 35 (11):1747–1767, 1999.

F. Blanchini and S. Miani. *Set-theoretic methods in control.* Birkhäuser, Boston, 2008.

I. Kolmanovsky and E.G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4(4): 317–363, 1998.

V. Lakshmikantham, V.M. Matrosov, and S. Sivasundaram. *Vector Lyapunov functions and stability analysis of nonlinear systems.* Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.

S.V. Raković. Minkowski Algebra and Banach Contraction Principle in Set Invariance for Linear Discrete Time Systems. In *Proceedings of 46th IEEE Conference on Decision and Control, CDC 2007*, New Orleans, LA, USA, December 2007.

S.V. Raković and M. Barić. Local Control Lyapunov Functions for Constrained Linear Discrete-Time Systems: The Minkowski Algebra Approach. *IEEE Transactions on Automatic Control*, 54(11):2686–2692, 2009.

S.V. Raković and M. Barić. Parameterized Robust Control Invariant Sets for Linear Systems: Theoretical Advances and Computational Remarks. *IEEE Transactions on Automatic Control*, 55(7):1599–1614, 2010.

S.V. Raković and M. Fiacchini. Invariant Approximations of the Maximal Invariant Set of "Encircling the Square". In *Proceedings of the 17th IFAC World Congress IFAC 2008*, Seoul, Korea, 2008.

S.V. Raković, E.C. Kerrigan, K.I. Kouramas, and D.Q. Mayne. Invariant approximations of the minimal robustly positively invariant sets. *IEEE Transactions on Automatic Control*, 50(3):406–410, 2005.

S.V. Raković, B. Kern, and R. Findeisen. Practical set invariance for decentralized discrete time systems. In *Proc. of the IEEE Conf. on Dec. and Contr., CDC'10*, Atlanta, USA, 2010.

S.V. Raković, B. Kern, and R. Findeisen. Practical robust positive invariance for large–scale discrete time systems. *Technical Report IFAT-SYS 1/2011*, 2011.

R.T. Rockafellar. *Convex Analysis.* Princeton University Press, 1970.

R. Schneider. *Convex bodies: the Brunn-Minkowski theory.* Cambridge Univ Pr, 1993.

D.D. Šiljak. *Large-scale dynamic systems: stability and structure.* North Holland, 1978.

D.D. Šiljak and A.I. Zečević. Control of large-scale systems: beyond decentralized feedback. *Ann. Rev. in Contr.*, 29:169–179, 2005.