Effects of Information Disclosure under First- and Second-Price Auctions in a Supply Chain Setting

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April 28, 2006

Abstract

We consider a two-stage supply chain comprising two retailers and a single supplier. Each retailer receives a signal of the consumer demand, and bids for the capacity of the supplier. The supplier sells his capacity as a bundle, and announces the winner as well as the auction price. Both retailers can get additional units in a procurement market when the auction closes, and then engage in a Cournot competition in the consumer market.

We analyze the impact of the information elicited by the supplier in the early stage of the game: In the first-price auction, the winning bid is announced, while in the second-price auction, the losing bid is revealed. As a consequence of the different information revealed under the two auction formats, the retailers’ expected payoffs get affected.

We characterize sufficient conditions for the existence of equilibrium behavior, derive the equilibrium bidding functions under both first- and second-price auctions, and prove that they are lower than the corresponding ones for a single shot auction with no resale. Our computational experiments with uniform signals show that both the supplier and retailers are better off by running a second-price auction. We also prove that consumers benefit if retailers have very different signals on the total demand. These results suggest that traditional auctions may have a significant impact in the context of a supply chain because of the information asymmetry introduced by announcing their results.

Keywords: supply chain, auctions, information asymmetry, demand effort

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1. Introduction

The use of auctions and the advances in information technology have been two of the most significant innovations incorporated in supply chains since the middle nineties, and have redefined the traditional relationships among different parties. In particular, online auctions have experienced a great explosion mainly due to reduced transaction costs, and easier accessibility to more participants, both bidders and sellers (e.g., see the online auction overviews by Lucking-Reiley (2000), and Pinker et al. (2003)). These virtual markets create opportunities for firms to reinvent their procurement processes, services, and general subcontracting of production.

Auctions have spread out to a huge variety of markets, from fresh flowers to industrial procurement, and from public-work contracts to the sale of natural-resource rights. They take place among manufacturers themselves, between manufacturers and retailers, and between retailers and consumers. The realization of auctions enforces information revelation among players in a supply chain environment through the disclosure of bids and auction prices. If we focus on a supplier/retailer relationship, since the auction results will at least be partially revealed, retailers may obtain some information advantage (or disadvantage) with respect to their competitors in the consumer market, and consequently adopt different purchasing/selling strategies. In practice, a variety of auction-related announcement policies are followed; a regular one being the publication of the closing price.

The most common sealed-bid auction formats are the first- and second-price auctions. In a first-price single unit auction, the highest bidder wins and pays her own bid. In the second-price single unit auction, the highest bidder wins but pays the second highest bid. In the online practice, open auctions are more common, with the ascending or English auction being by far the most widely used mechanism. ¹

The announcement of the closing price at the end of a single unit auction induces a clear information asymmetry: In the second-price auction the highest losing bid is revealed, while in the first-price auction the winning bid is disclosed. This is irrelevant in the traditional single-shot auction with no resale. However, if the bidders compete among themselves after the auction, this difference becomes crucial.

The impact of the information elicited under a first- and second-price auction is, indeed, the main focus of this paper. We consider a supply chain setting where a single supplier auctions his capacity to two retailers as a bundle. Both retailers can get additional units in a procurement market before engaging in a Cournot competition in the consumer market. How much the retailers value the auctioned capacity depends on the expected revenue to be collected from the end consumers. The

¹The English auction turns out to be strategically equivalent to the second-price auction in our two-bidder model (see discussion in Section 2.2)). Regarding online first-price auctions, Beam and Segev (1998) reported seven websites that were implementing it; with two of them announcing the winner’s payment.
supplier conducts either a first- or second-price auction, and announces the winner and the payment in each case. We examine the influence of this announcement policy on bidding functions, supplier’s and retailers’ expected payoffs, and transactions in the consumer market under both auction formats.

1.1 Literature review

A vast body of work on auction theory has been published since Vickrey (1961), including the influential paper of Milgrom and Weber (1982), the mechanism design approaches of Myerson (1981) and Maskin and Riley (1989), the survey by Klemperer (1999), and the recent book by Krishna (2002).

Gupta and Lebrun (1999), Krishna (2002, Section 4.4), and Haile (2003) consider first-price auctions with resale, but the resale occurs among the same set of bidders, and their assumptions on the announcement policies are quite different from ours: After the auction, all bidders’ values are announced in Gupta and Lebrun (1999), whereas in Krishna (2002) all bids are made public. In Haile (2003), bidders receive new signals after the auction and hence all bidders still hold private information. A recent paper by Benoit and Dubra (2005) considers the information revelation issue in auctions as well, but their concentration is on whether bidders are willing to disclose their private information before submitting their bids. Moresi (2000) develops a model of information acquisition prior to an open auction, in which the common value of the item has two distinct components. Each of two bidders must conduct some research and specialize in one component independently, and finally decide how much to bid. The model provides a rationale for bidders to differentiate themselves by conducting different lines of research.

Several papers have addressed the impact of a supplier’s announcement policy and the derived information sharing in more traditional supply chain contexts (e.g., see the surveys by Cachon (2003), Chen (2003), and references therein). There are also papers on the competitive interaction between supply chain parties; most of them first show that competition hurts the supply chain performance and then propose coordination mechanisms that bring the supply chain performance closer to the centralized optimum. Among them, Li (2002), and Padmanabhan and Png (1997) consider horizontal competition among retailers. Different streams of research that relate supply chain management and auction theory are the design of procurement contracts (e.g., see Dasgupta and Spulber (1989), Chen (2004), and the survey by Elmaghraby (2000)), and the structure of joint inventory policies and auction design decisions (e.g., van Ryzin and Vulcano (2004)). However, in all these papers, there is no integration of procurement, auctions, and horizontal competition among retailers.

In our setting, retailers compete horizontally in the consumer market. We consider a linear demand model based on retailers’ promotional effort. In a competitive retailing environment, the demand-enhancing effort of one retailer may increase the demand at other retailers. These spillovers may lead to free riding, where one retailer enjoys a higher demand due to the efforts of others without exerting the own effort. Some papers in the marketing literature have addressed this topic (e.g.,
see Bergen and John (1997), Lal (1990), and Nault and Dexter (1994)). Hula (1993) develop two profit-maximization models of firm behavior that incorporate industry-demand externality effects of firm price changes, advertising, and research and development expenditures. On the operational side, Krishnan et al. (2004) study mechanisms to achieve coordination in a simple supply chain setting with one manufacturer and one retailer, accounting for the retailer promotional effort.

Some research on treasury bill auctions is also related to our work, e.g., see Bikhchandani and Huang (1989), Nyborg and Sundaresan (1996), Chatterjea and Jarrow (1998), and references therein. These papers model the primary market of stock/bonds as an auction, where traders operate with private information about future values of the financial assets. The purpose of trading in the primary market is to resell the securities afterwards to other uninformed traders or market makers. In particular, Bikhchandani and Huang (1989) consider a multi-unit auction setting with a resale market, but the resale price is determined by the expectation of securities’ true values from uninformed traders. They discuss the impact of announcing the winning bid and the highest losing bid on the market equilibrium and traders’ behavior, but there is no oligopolistic competition in the resale market.

Finally, we complement this review with the computer science literature on trading agent design, a prominent application area in Artificial Intelligence. The main focus of this stream of research is on the automated decision-making processes of a supply chain agent in terms of procurement, selling, and production/inventroy management (e.g., see Buffett and Scott (2004), Pardoe and Stone (2004), and Wellman et al. (2003)).

1.2 Overview of main results

We characterize the retailers’ equilibrium bidding functions for our setting under first- and second-price auctions. We provide sufficient conditions for the existence of an incentive compatible equilibrium (i.e. an equilibrium where each retailer bids according to the demand signal that she got from the market), and show that the threat of revealing private information discourages retailers from bidding as high as they would bid in the conventional setting (i.e. auctions with no resale). The conditions for the existence of equilibrium bidding functions being monotonic in retailers’ own signal are more restrictive for the first-price auction than for the second-price auction.

We also show that consumers are better off (worse off) if retailers receive very different (similar) demand signals. In the extreme case where retailers receive very different signals, the first-price auction will raise the loser’s expectation of the product demand due to the disclosure of the winner’s bid, thus bringing the loser’s quantity back to the normal level. However, in the second-price auction, the loser has no access to the winner’s signal and hence underestimates the product demand, which results in an abnormally low quantity.

We verify numerically that the supplier’s expected revenue is lower in the first-price auction than in the second-price auction. Note that this result is in contrast with the Revenue Linkage Principle
(see Krishna (2002, Chapter 7)): In a conventional auction, if the signals are independent, and the valuations as function of the signals are interdependent (as assumed here), the supplier should receive the same revenue from both auction formats. In our setting, the winner of a first-price auction collects on average less profit from winning the auction, and hence she is unwilling to pay as much as in the second-price auction. In fact, the announcement of the winning bid translates into some information disadvantage; in this sense, winning brings “bad news”. This can be regarded as another sort of the so called winner’s curse –the possibility that the winner pays more than the “real value” of the object (Krishna, 2002, Chapter 6).

Our results imply that all parties in the supply chain are better off under the second-price auction. An explanation for this is that the combined first-mover and information advantage for the winning retailer reduces the tension between them, and hence the profit of the entire supply chain is driven up.

The rest of this paper is organized as follows. In Section 2 we describe our model setup, followed by a brief discussion of the model assumptions. We derive the equilibrium analysis of the two-stage game and relevant payoffs in Section 3. We provide numerical results in Section 4, and present our conclusions in Section 5. All proofs are in the Appendix.

2. Model description

A single supplier runs an auction to sell his capacity \( C \) to two competitive retailers who possess no initial capacity endowment. The retailers can get additional units in a procurement market right after the auction closes.

Once the retailers obtain their capacities, they engage in a Cournot (quantity) competition in the consumer market. The inverse demand function is common knowledge, and is described by:

\[
P(Q) = \theta - Q,
\]

where \( \theta \) is a random variable and \( Q = q_1 + q_2 \) is the total aggregated supply, i.e. the sum of the continuous quantities \( q_i \) provided by retailers \( i = 1, 2 \). This linear demand model is commonly adopted in the literature of economics, marketing, and operations, e.g., see Dixit (1979), Li (2002), McGuire and Staelin (1983), and Tirole (1995).

Even though the retailers face a stochastic demand, we neglect the nonnegativity constraint over the price for computational convenience. This assumption is plausible when the quantities chosen in equilibrium drive down the likelihood of a negative price to an ignorable level (e.g., see Li (2002)).

2.1 Sequence of events

For the ease of the presentation, we will divide the sequence of events into four periods:
Period 1: Retailer $i$ receives a private signal $s_i$ regarding the intercept $\theta$. We keep the following assumption throughout:

**Assumption 1.** The maximum possible demand is $\theta = \theta_0 + s_1 + s_2$, where $\theta_0$ is a common knowledge constant, and $s_1, s_2$ are independently and Unif[0, 1] distributed random variables.

Neither the other retailer nor the supplier knows the true value of retailer $i$’s signal $s_i$, but they do know that it is a draw from a Unif[0, 1] distribution. The contribution $s_i$ to the total demand is a proxy for retailer $i$’s demand effort (e.g., it could be related to advertising levels).

**Period 2:** At the beginning of this period, the supplier announces his capacity $C$ and sells it as a bundle through an auction. Therefore, although auctioning multiple objects, the supplier conducts in fact a single-unit auction. He announces the auction type $A$, which can be either a first- or second-price auction (i.e. $A = I$ or $A = II$ respectively). Then, the auction takes place. Both retailers participate in it, each one submitting a bid $b_i$, $i = 1, 2$. We assume that the individual rationality constraint of the retailers is verified (i.e. we are implicitly assuming that by not participating, a retailer can guarantee herself a null payoff).

**Period 3:** The auction closes and the supplier announces who the winner is and how much she should pay for the capacity $C$. Therefore, in the first-price auction, the winner’s bid is made available to the public\(^2\), while in the second-price auction, the losing bid is revealed.

**Period 4:** At the beginning of this period, retailers can pay a constant marginal cost $c$ for additional units from a procurement market with unlimited capacity. Finally, retailer $i$ puts $q_i$ units on the market. Demand is revealed and the market clears. The price is determined, and the retailers’ payoffs are realized.

The timing of this model is summarized in Figure 1. The information flow is presented in Figure 2: The supplier announces the auction type $A$ and the capacity $C$ to the retailers, and then retailers submit bids $b_1, b_2$ to the supplier. The retailers acquire demand information $s_1$ and $s_2$ from the final consumers before the auction, and provide quantities $q_1$ and $q_2$ to the consumer market after the procurement process takes place.

### 2.2 Discussion of the model assumptions

The Cournot competition is merely one possible way to model the retailing game. An alternative would be to explain it in terms of Bertrand (price) competition. However, in a Bertrand game with capacity constraints, we should consider a rationing rule because it is not clear what proportion of

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\(^2\)Although revealing the price in a first-price auction is not the most usual practice, there are some websites that implement it. Examples are Timeshare Resales International (http://www.tri-timeshare.com) and The Chicago Wine Company (http://www.tcwc.com).
Figure 1: Time flow diagram for the sequence of events.

Figure 2: Information flow diagram of relevant events.
consumers would choose retailer 1 over retailer 2 in case the demand is less than the capacity. It has been observed that different rationing rules, such as efficient-rationing and proportional-rationing, lead to considerably different market equilibria. Moreover, if the efficient-rationing rule is adopted between the two retailers, a pure-strategy equilibrium exists only if the capacity levels are extremely small or extremely large (see Kreps and Scheinkman (1983), and Tirole (1995, Chapter 5)).

Our framework is an interdependent-value model with symmetric bidders. In an auction with $N$ bidders and interdependent values, a bidder’s valuation may depend on other bidders’ signals. Let $s_1, \ldots, s_N$ denote the bidders’ signals whose prior symmetric distribution $F(s_1, \ldots, s_N)$ is common knowledge. If we let $V_i$ denote bidder $i$’s valuation of the auction object, then $V_i$ can be expressed as $V_i = v_i(s_1, \ldots, s_N)$. Therefore, until bidder $i$ receives the signal $s_i$, she does not know exactly her own valuation, and knowing what other bidders also receive will change her expectation of her own true value.\(^3\)

The widely used (online) English auction is shown to be strategically equivalent to the second-price auction in the interdependent-value case with two bidders, since in this case the English auction ends as one bidder refuses to bid up the price (see Krishna (2002, Chapter 6)).\(^4\) Even though bidding your own value is a dominant strategy in the auctions where bidders possess private values, in an interdependent-value model bidders do consider the bidding function her opponent uses. This is because based on this she may be able to estimate her opponent’s signal and hence the “right value” of the auctioned object. Therefore, strategic interaction scatters away the hope of finding a dominant strategy equilibrium.

A well-established theorem for revenue comparison of conventional auctions under the interdependent-value setting is the so-called Revenue Linkage Principle (see Milgrom and Weber (1982), and Krishna (2002, Chapter 7)). An immediate application of this principle is to show that if the signals $s_1, \ldots, s_N$ are independent (even though the valuations $V_1, \ldots, V_N$ are interdependent, as in our case with $N = 2$), the seller’s expected revenue is the same under first- and second-price conventional auctions.

We indeed assume that signals are independently distributed (i.e. they are non-affiliated) and additive to demand. This is a plausible assumption in papers regarding information acquisition (e.g. see Froot et al. (1992) and Moresi (2000)), and its tractability enables us to formally prove our results via analytic calculations. This assumption corresponds to retailers putting independent efforts to increase market demand through strategies such as advertising and promotions, and to the fact that

\(^3\)More specifically, if $V_i = v_i(s_i)$, i.e., bidder $i$’s valuation depends only on her own signal, it degenerates to a private value model. If $V_i = v(s_1, \ldots, s_N)$, we call it a common value model. In this case, if bidders know all signals in the end, their valuations will become the same. An example for the common value model fit is the allocation of U.S. mineral rights. In our case though, a retailer expected payoff will be affected by the other retailer’s signal concerning demand.

\(^4\)Lucking-Reiley (1999) conducts experiments with some Internet auctions, comparing the revenues generation of English and second-price auctions. He verifies in practice the theoretical revenue equivalence between them, but under the assumption of private values.
the realization of her effort could be observed only by herself. Thus, the signals capture the retailers’ private information following their private actions.

We consider symmetric uniformly distributed signals. The symmetry in the bidders’ distribution of valuations for the private-value auction model is a strong but common assumption in the auction literature in operations management (e.g., see Pinker et al. (2003) and van Ryzin and Vulcano (2004)). Nevertheless, in our interdependent value model, symmetry in the signals is a reasonable assumption for a supply chain setting where retailers have comparable sizes and similar market influence.

In particular, the uniform distribution is commonly adopted in the existing auction literature, from the early paper by Vickrey (1961), to the more recent works by Pinker et al. (2003) and Krishna (2002, Chapters 6 and 8). The uniform distribution also brings the maximum entropy among the class of all distributions with finite support (see Arndt (2001, Chapter 15)), and hence can be regarded as the worst case belief if the seller has no other information except that the bidders’ signals have finite support.

In our model, we assume that retailers do not possess any capacity endowment before participating in the auction, and that their signals have the same precision. Thus, both retailers are ex ante symmetric. If their inventory endowments were different, their bids may depend not only on their signals but also on their inventory levels. It is known that in auctions with interdependent values, the asymmetry among bidders may destroy the Revenue Linkage Principle; see Krishna (2002, Chapter 8). If the precision of their signals concerning the realized demand varies across bidders, there may not exist a pure-strategy equilibrium; see Engelbrecht-Wiggans et al. (1983). By considering ex-ante symmetric retailers, we are really isolating the effect of information revelation that arises when they bid on the supplier’s capacity.

Although the supplier sells several units via an auction, this is a single object auction since we do not allow him to split his capacity. This is, of course, a strong assumption from both theoretical and practical viewpoints. However, it sticks out the effect of different announcement policies: If the seller were allowed to split his capacity, then the retailers’ signals may be at least partially revealed if they both receive some units. Nevertheless, there are some real world examples that work under this format, like the French timber sales\(^5\) and the European third generation (3G) auctions for telecommunication spectrum (see Binmore and Klemperer (2002) for a description for the British case). On the theoretical side, the paper by Anton and Yao (1989) also touches on the issue of divisible versus indivisible awards, for a complete information procurement auction. They show that under a split award procurement, the two bidders implicitly collude, and hence the auctioneer strictly prefers a single source award auction.

\(^5\)See Li and Perrigne (2003) for an empirical analysis on the French forest service, where local offices handle first-price sealed-bid auctions, and decide independently for the sizes of the timber sales. These auctions are conducted as single-unit supplies.
For ease of presentation, we consider only two retailers here to emphasize the different situations that a winner and a loser of the auction will face in the consumer market. If there were more than two participants in the auction, a similar approach would apply since only one bid would be disclosed.

Finally, in our model, the winning retailer can discard part of her capacity \( C \) without any penalty. She is allowed to purchase more from the procurement market with a fixed price \( c \) on her own will, but she cannot resell the awarded capacity to the procurement market. The assumption that the procurement market has a given price \( c \) is merely made for tractability, and is reasonable if these two retailers are relatively small with respect to the total aggregated market for this product. The prohibition of reselling capacity disallows speculation by the retailers (e.g. chasing financial arbitrage opportunities), and focuses retailers’ business on meeting demand. There are several examples in the literature along these lines: In Peleg et al. (2002), a firm is allowed to replenish its inventory via two channels: a long-term contract or an online search. Under the long-term contract, the price is fixed ex ante, while in the online search the price is random, reflecting different market situations. These two procurement alternatives, also labelled as two-source factor purchasing in Elmaghraby (2000), are similar to our procurement market and the auction. In fact, many suppliers provide auctions as a complementary sales channel besides their long-term relationship with retailers (e.g., see Grey et al. (2005)).

3. Equilibrium analysis of the game

Due to the sequential nature of our four-period game under incomplete information, we will characterize a Perfect Bayesian Equilibrium (PBE) (e.g., see Osborne and Rubinstein (1994)) for more details):

**Definition 1.** In a Perfect Bayesian Equilibrium, players’ strategies and beliefs satisfy the following conditions:

1. Bayesian updating: Players have correct initial beliefs, and after observing players’ actions at a stage, they use Bayes’ rule to update only the corresponding beliefs.

2. Sequential rationality: Given their beliefs, their actions must be best responses.

In the sequel, we will use backward induction to analyze the players’ behavior: First, we will derive the fourth-period equilibrium in the consumer market, assuming that a fully separating or strictly increasing equilibrium has been formed in period 2. Remember that period 2 is the auction game, where a retailer’s strategy is her bidding function that maps her signal to her bid, i.e., \( \beta : [0, 1] \rightarrow R_+ \).

**Definition 2.** An equilibrium is said to be strictly increasing if the equilibrium bidding function is strictly increasing in a bidder’s own signal.
Table 1: Information obtained by the winner and the loser when the auction closes.

<table>
<thead>
<tr>
<th></th>
<th>First-price auction</th>
<th>Second-price auction</th>
</tr>
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<tbody>
<tr>
<td><strong>Winner</strong></td>
<td>{s_w}, {s_l &lt; s_w}</td>
<td>{s_w}, {s_l}</td>
</tr>
<tr>
<td><strong>Loser</strong></td>
<td>{s_l}, {s_w}</td>
<td>{s_l}, {s_w &gt; s_l}</td>
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</table>

When the bidding function is strictly increasing, as players observe the bid, they can invert the bidding function to obtain a competitor’s signal.

As stated earlier, in the first-price auction, the winning bid is announced. Thus, the winning retailer’s private information becomes public. On the other hand, if a second-price auction is adopted, the supplier announces the winner’s payment, which is the loser’s bid. Table 1 summarizes the information that a bidder possesses at the end of period 3 (\(s_w\) and \(s_l\) are the signals received by the winner and the loser respectively).

As shown in Table 1, while competing in the consumer market, the loser has information advantage in the first-price auction; while the winner is the more informed retailer if the second-price auction is adopted. In what follows, we use superscript \(I\) to denote terms associated with the first-price auction, and superscript \(II\) for terms associated with the second-price auction. The subscripts \(w\) and \(l\) refer to the variables for the winner and the loser respectively.

### 3.1 The consumer market game

Given the auction format chosen by the supplier, there are two cases for the analysis of the consumer market game. Recall that at this stage of the game (period 4), the retailers’ decisions are about how many units, \(q_1\) and \(q_2\), they should provide to the consumer market.

#### 3.1.1 Second-price auction procurement case

If the supplier runs a second-price auction, the winning retailer aims at maximizing her own expected payoff, given that she knows both signals and that she has to pay \(c\) per unit in the procurement market (if her optimal quantity supplied to the consumer market exceeds the auctioned capacity \(C\)). Hence, the winner’s objective function is as follows:

\[
\max_{q_w} (\theta - q_w - q_l)q_w - c(q_w - C)^+, \\
\]

where \(a^+ \equiv \max\{a,0\}\) is the positive part of the real number \(a\). Similarly, the loser’s objective is

\[
\max_{q_l} E_{s_w}[((\theta - q_w - q_l - c)q_l | s_w > s_l, s_l].
\]

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Let $q^I_{IW}(sw, sl)$ and $q^I_{IL}(sl)$ denote the optimal quantities chosen by the winner and the loser respectively, where the arguments depict their corresponding information knowledge at the moment of decision. Recalling from Assumption 1 that $\theta = \theta_0 + s_1 + s_2$, the following proposition summarizes the equilibrium quantities provided by these two retailers in the consumer market:

**Proposition 1.** In the consumer market after a procurement second-price auction, there exists a unique Nash equilibrium in which

$$q^I_{IW}(sw, sl) = \begin{cases} \frac{1}{2} \left( \theta_0 + sw + sl - q^I_{IL}(sl) \right), & \text{if } sw < S^I_{II}(sl) \\ C, & \text{if } S^I_{II}(sl) \leq sw \leq S^I_{II}(sl) \\ \frac{1}{2} \left( \theta_0 + sw + sl - q^I_{IL}(sl) - c \right), & \text{if } sw > S^I_{II}(sl), \end{cases}$$

(2)

$$q^I_{IL}(sl) = \frac{1}{2} \left( E_{sw}[\theta | sw > sl, sl] - E_{sw}[q_w(sw, sl) | sw > sl] - c \right),$$

where the two thresholds $S^I_{II}(sl)$ and $S^I_{II}(sl)$ are

$$S^I_{II}(sl) = \min \left\{ \max\{sl, 2C - \theta_0 - sl + q^I_{IL}(sl)\}, 1 \right\},$$

$$S^I_{II}(sl) = \min \left\{ \max\{sl, 2C - \theta_0 - sl + q^I_{IL}(sl) + c\}, 1 \right\}. \hspace{1cm} (3)$$

Note that these two thresholds are greater than $sl$, and when $S^I_{II}(sl)$ has not hit the boundary 1, $S^I_{II}(sl) = S^I_{II}(sl) + c$, i.e. the second threshold is higher. Now we provide the structural property of the points at which the capacity constraint is binding. Figure 3 illustrates the general shape of $S^I_{II}(sl)$ and $S^I_{II}(sl)$ as functions of $sl$.

![Figure 3: An example to show the shape of $S^I_{II}(s)$ and $S^I_{II}(s)$ under the second-price auction procurement case](image)

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Proposition 2. The threshold function $S_{II}^1(s_l)$ can be divided into three regions by two values $s_1^*$ and $s_1^{**}$, where $s_1^* \leq s_1^{**}$.

If $0 \leq s_l \leq s_1^*$, $S_{II}^1(s_l) = 1$, i.e., the capacity constraint is not binding for all $s_w$.

If $s_1^* \leq s_l \leq s_1^{**}$, $S_{II}^1(s_l)$ decreases from 1 until it hits the 45-degree line, and

$$S_{II}^1(s_1) - S_{II}^1(s_2) < s_2 - s_1, \forall s_1 < s_2.$$  \hspace{1cm} (4)

Finally, when $s_1^{**} \leq s_l \leq 1$, $S_{II}^1(s_l) = s_l$, i.e. $q_{wII}(s_w, s_l) \geq C$ for all $s_w$.

Likewise, there exist two corresponding thresholds $s_2^*$ and $s_2^{**}$ for $S_{II}^2(s_l)$ such that

$$s_2^* \geq s_1^*, \quad s_2^{**} \geq s_1^{**}, \quad s_2^{**} \geq s_2^*,$$

$$S_{II}^2(s_1) - S_{II}^2(s_2) < s_2 - s_1, \forall s_1 < s_2.$$  \hspace{1cm} (5)

Note that according to the values of some parameters, a region could be undistinguishable in the graph.

In Figure 3, as $s_l$ becomes larger, the threshold function $S_{II}^1(s_l)$ first stays at the upper bound, and then starts to decrease until it hits the 45-degree line; after that, $S_{II}^1(s_l)$ coincides with $s_l$. $S_{II}^2(s_l)$ has a similar graph, except that the thresholds $s_2^*$ and $s_2^{**}$ occur at higher values. Note also that within region III, $S_{II}^2(s_l)$ could also stay at 1, depending on the ordering of $s_2^*$ and $s_1^{**}$.

3.1.2 First-price auction procurement case

In the first-price auction, since the winner’s bid is announced, the loser knows both signals while the winner only knows her own signal. The retailers’ objective functions are respectively

$$\text{Winner: } \max_{q_w} \mathbb{E}_{s_l}[(\theta - q_w - q_l)q_w - c(q_w - C)^+ |s_w, s_l < s_w],$$

$$\text{Loser: } \max_{q_l} (\theta - q_w - q_l - c)q_l.$$  \hspace{1cm} (6)

After differentiating the objective functions and applying the same argument as in Proposition 1, we obtain the retailers’ best responses. The proof involves routine algebra and hence is omitted.

Proposition 3. There exists a unique Nash equilibrium in the product market under a first-price auction. Define thresholds $S_1^I$ and $S_2^I$ as

$$S_1^I = \min \left\{ 1, \left( 2C - \frac{2}{3}c - \frac{2}{3}\theta_0 \right)^+ \right\},$$

$$S_2^I = \min \left\{ 1, \left( 2C + \frac{2}{3}c - \frac{2}{3}\theta_0 \right)^+ \right\}.$$  \hspace{1cm} (7)
In equilibrium, if \( s_w < S_1 \),

\[
q_w(s_w) = \frac{1}{3}\theta_0 + \frac{1}{2}s_w + \frac{1}{3}c,
\]

\[
q_l(s_l, s_w) = \frac{1}{3}\theta_0 + \frac{1}{4}s_w + \frac{1}{2}s_l - \frac{2}{3}c, \forall s_l \leq s_w.
\]

If \( S_1 \leq s_w \leq S_2 \), then

\[
q_w(s_w) = C,
\]

\[
q_l(s_l, s_w) = \frac{1}{2}(\theta_0 + s_w + s_l - C - c), \forall s_l \leq s_w.
\]

Finally, if \( s_w > S_2 \),

\[
q_w(s_w) = \frac{1}{3}\theta_0 + \frac{1}{2}s_w - \frac{1}{3}c,
\]

\[
q_l(s_l, s_w) = \frac{1}{3}\theta_0 + \frac{1}{4}s_w + \frac{1}{2}s_l - \frac{1}{3}c, \forall s_l \leq s_w.
\]

Note that \( S_1 \) and \( S_2 \) are regulated by 0 and 1 because the signals have finite support \([0, 1]\). In the first case, the winner’s optimum is feasible before hitting \( C \), and the loser chooses the corresponding best response. In the second case, the global optimum of the winner without considering marginal cost \( c \) exceeds her capacity \( C \), and the marginal revenue is less than \( c \) if we increase \( q_w \) above \( C \), therefore the equilibrium turns out to be a corner solution. In the third case, the optimal order quantity exceeds \( C \).

Observe that in the first case, the loser’s quantity \( q_l(s_l, s_w) \) can be expressed as

\[
q_l(s_l, s_w) = \frac{1}{3}(\theta_0 + s_w + s_l) + \frac{1}{6}\{(\theta_0 + s_w + s_l) - \mathbb{E}_{s_l}[\theta|s_w, s_l < s_w]\} - \frac{2}{3}c,
\]

where \( \frac{1}{6}\{(\theta_0 + s_w + s_l) - \mathbb{E}_{s_l}[\theta|s_w, s_l < s_w]\} \) is the adjusted term resulting from the winner’s bias on \( \theta \) expectation. In other words, although the loser knows both signals and henceforth has the best prediction of \( \theta \), her best response still contains the winner’s bias \( \mathbb{E}_{s_l}[\theta|s_w, s_l < s_w]\). Likewise, we observe the same phenomenon in the last case.

We can also see that \( q_w(s_w) \) is increasing in \( s_w \), and that \( q_l(s_l, s_w) \) is increasing in both \( s_l \) and \( s_w \). This is because the higher the signals are, the higher the demand the retailers can expect, and therefore they will provide higher quantities to the consumer market.

### 3.2 The auction game

Using backward induction, we proceed to a previous stage of the game (Period 2). At this point, the retailers’ decision is about how to bid in the auction. We restrict ourselves to the symmetric
(and strictly increasing) equilibrium throughout this section, where an equilibrium in the auction mechanism \( A \) is said to be symmetric if there exists a bidding function \( \beta^A(s) \) such that a retailer who receives a signal \( s \) submits the bid \( \beta^A(s) \), independent of the retailer’s identity. We will establish in the sequel the uniqueness of such an equilibrium under both first- and second-price auctions.

Suppose a retailer receives a signal \( s \). If she participates in an auction whose format is \( A \) and pretends as if her signal were \( z \), we denote by \( \Pi^A(z|s) \) her ex ante expected payoff. A retailer’s objective is to choose a strategy that maximizes it, and therefore a truth-telling equilibrium requires

\[
s \in \arg\max_z \left\{ \Pi^A(z|s) : 0 \leq z \leq 1 \right\}.
\]

In order to verify this fact for both auction mechanisms, we introduce the quantities \( \pi^A_w(\cdot) \) and \( \pi^A_l(\cdot) \): the partial payoffs under auction mechanism \( A \) in case of winning and losing respectively. By \textit{partial payoff}, we refer to the expected revenue in the consumer market minus the procurement cost in the procurement market (i.e., we do not account for the procurement cost through the auction).

### 3.2.1 Second-price auction

If the supplier sells the capacity through a second-price auction, the expected payoff of a type-\( s \) retailer who behaves as a type-\( z \) retailer, assuming that her opponent adopts an increasing equilibrium bidding function \( \beta^{II}(y) \), is

\[
\Pi^{II}(z|s) = \int_0^z (\pi^{II}_w(s, y) - \beta^{II}(y)) dy + \int_z^1 \pi^{II}_l(z, s, y) dy,
\]

where \( z \) and \( s \) belong to the inspected retailer, and \( y \) refers to the opponent’s type. The first term represents the event that she wins the auction, in which case she gets the object and pays the opponent’s bid \( \beta^{II}(y) \). Due to the monotonicity of \( \beta^{II}(y) \), this event happens when the opponent’s signal is less than \( z \). Her partial payoff \( \pi^{II}_w(s, y) \) is independent of her own bid \( \beta^{II}(z) \) since in the second-price auction, the supplier does not announce the winning bid. The second term corresponds to the event that she loses in the auction while pretending to be type-\( z \): \( \pi^{II}_l(z, s, y) \) is the partial payoff of a loser whose signal is \( s \) and plays as a type-\( z \), given that her opponent, i.e., the winner, receives signal \( y \). Note that if she loses while playing as a type-\( z \), her bid \( \beta^{II}(z) \) will be revealed and therefore the opponent will infer a wrong type \( z \) and will choose the best response accordingly.

The winning partial payoff \( \pi^{II}_w(s, y) \) is as follows:

\[
\pi^{II}_w(s, y) = (\theta_0 + s + y - q^{II}_w(s, y) - q^{II}(y)) q^{II}_w(s, y) - c(q^{II}_w(s, y) - C)^+,
\]

where \( q^{II}_w(s, y) \) and \( q^{II}(y) \) are given by equation (2). It is simply the product of the realized price and the quantity, minus the purchasing cost in the procurement market.
Now we consider the losing partial payoff $\pi_{II}^l(z, s, y)$, where now the opponent’s type $y$ is the winning type. From Proposition 1, given $q_{II}^w(y, z)$, the retailer’s best response is

$$q_{II}^l(z, s) = \frac{1}{2} \left( E_y[\theta|y > z, s] - \int_z^1 q_{II}^w(y, z) \frac{1}{1-y} \, dy - c \right).$$

(7)

The loser does not know the winner’s signal $y$ since only the losing bid $\beta_{II}^l(z)$ is announced. Expanding $E_y[\theta|y > z, s]$ in equation (7), we obtain that $q_{II}^l(z, s) = q_{II}^l(z) + \frac{1}{2}(s - z)$, where $q_{II}^l(z)$ is the equilibrium quantity if the bidder’s type is $z$, and the extra term $\frac{1}{2}(s - z)$ captures the noise introduced by the misreported type. The loser’s partial payoff in this case is

$$\pi_{II}^l(z, s, y) = q_{II}^l(z, s)(\theta_0 + y + s - q_{II}^w(y, z) - q_{II}^l(z, s) - c).$$

Similarly to Milgrom and Weber (1982), we focus on the cases where the expected (total) payoff is unimodal in the reported type. The unimodal pattern of the expected payoff can be rationalized by the following argument: As the bidder increases her bid, her payoff will first increase due to the higher probability of winning. However, the bidder cannot increase her bid unboundedly because at some point she will pay too much to the seller. A sufficient condition to make the expected payoff unimodal is $q_{II}^w(s, s) \geq q_{II}^l(s)$, $\forall s$. However, differently from Milgrom and Weber (1982), we need an even stronger assumption to guarantee the monotonicity of the retailer bidding function:

**Assumption 2.** $q_{II}^w(s, s) \geq \frac{8}{5}q_{II}^l(s) + \frac{3}{16}$, $\forall s$.

This condition simply requires that a truth-telling type-$s$ retailer provide a somewhat significant higher quantity when she gets the capacity than when she loses. Note that we could express this condition in terms of the primitive values $C, c$ and $\theta_0$ by using Propositions 1 and 2, but it would become harder to read and less intuitive to understand.

Now we are ready to characterize the equilibrium bidding function of the second-price auction. Assuming that the objective function is differentiable and there exists an interior solution to the first-order condition of $\Pi_{II}(z|s)$ in equation (6), we have the following result:

**Theorem 1.** Suppose Assumption 2 holds. The unique symmetric, strictly increasing equilibrium bidding function $\beta_{II}^l(s)$ in the second-price auction is

$$\beta_{II}^l(s) = \pi_{II}^w(s, s) - \pi_{II}^l(s, s) s, + \int_s^1 \frac{\partial}{\partial v} \pi_{II}^l(v, s, y)|_{v=s} \, dy.$$

Moreover, $\beta_{II}^l(s) \leq \pi_{II}^w(s, s) - \pi_{II}^l(s, s, s)$.

Note that $\pi_{II}^w(s, s) - \pi_{II}^l(s, s, s)$ can be regarded as the equilibrium bidding function in the second-price auction with no resale: A bidder with signal $s$ is asked to bid an amount such that if she were
to win the auction with that bid, she would just “breakeven” (e.g., see Krishna (2002, Chapter 6)). Theorem 1 tells us that the bidding function here is lower than that in the conventional auction. This is because the bid of a type-s retailer will be announced with probability \(1 - P(s)\) (i.e., when she loses), and hence the threat of revealing her private information lowers her bid.

### 3.2.2 First-price auction

If the supplier uses a first-price auction in period 2, the expected payoff of a type-s retailer who behaves as a type-z retailer, assuming that her opponent adopts the same equilibrium bidding function, is

\[
\Pi^f(z|s) = \int_0^z (\pi^f_w(z, s, y) - \beta^f(z)) \, dy + \int_z^1 \pi^f(s, y) \, dy,
\]

(8)

The first term accounts for the case where the bidder wins the auction, which happens when the opponent’s signal \(y\) is less than \(z\), and therefore her payment is her bid \(\beta^f(z)\). In this case, the opponent believes that the other bidder receives the signal \(z\), and then both bidders start to play the Cournot game in the consumer market. Therefore, \(\pi^f_w(z, s, y)\) depends on the bidder’s reported \(z\). The second term corresponds to the event that she loses in the auction, in which case the partial payoff \(\pi^f_l(s, y)\) is independent of the losing bid since the winner does not observe it.

We first present the equilibrium quantities. In all cases, the formulas for \(q^f_l(y, z)\) are obtained from Proposition 3, and following them we calculate \(q^f_w(z, s)\) as the corresponding best responses.

**Proposition 4.** If a type-s bidder that behaves as a type-z wins the auction, then:

If \(z \leq S^f_1\),

\[
q^f_l(y, z) = \frac{1}{3} \theta_0 + \frac{1}{4} z + \frac{1}{2} y - \frac{2}{3} c,
\]

\[
q^f_w(z, s) = \begin{cases} 
\frac{1}{2} \theta_0 + \frac{1}{2} s + \frac{3}{4} c, & s \leq 2 (C - \frac{1}{3} \theta_0 - \frac{1}{3} c), \\
C, & 2 (C - \frac{1}{3} \theta_0 - \frac{1}{3} c) \leq s \leq 2 (C - \frac{1}{3} \theta_0 + \frac{1}{6} c), \\
\frac{1}{4} \theta_0 + \frac{1}{2} s - \frac{1}{6} c, & s \geq 2 (C - \frac{1}{3} \theta_0 + \frac{1}{6} c).
\end{cases}
\]

If \(S^f_1 \leq z \leq S^f_2\), then

\[
q^f_l(y, z) = \frac{1}{2} (\theta_0 + z + y - C - c),
\]

\[
q^f_w(z, s) = \begin{cases} 
\frac{1}{4} \theta_0 + \frac{1}{2} s - \frac{1}{2} z + \frac{1}{4} C + \frac{1}{4} c, & s \leq \frac{3}{2} C - \frac{1}{2} \theta_0 + \frac{1}{2} z - \frac{1}{2} c, \\
C, & \frac{3}{2} C - \frac{1}{2} \theta_0 + \frac{1}{2} z - \frac{1}{2} c \leq s \leq \frac{3}{2} C - \frac{1}{2} \theta_0 + \frac{1}{2} z + \frac{1}{2} c, \\
\frac{1}{4} \theta_0 + \frac{1}{2} s - \frac{1}{2} z + \frac{1}{4} C - \frac{1}{4} c, & s \geq \frac{3}{2} C - \frac{1}{2} \theta_0 + \frac{1}{2} z + \frac{1}{2} c.
\end{cases}
\]
Finally, if \( z \geq S^I \),

\[
q^I_w(y, z) = \frac{1}{3} \theta_0 + \frac{1}{4} z + \frac{1}{2} y - \frac{1}{3} c,
\]

\[
q^I_w(z, s) = \begin{cases} 
\frac{1}{3} \theta_0 + \frac{1}{2} s + \frac{1}{6} c, & s \leq 2C - \frac{2}{3} \theta_0 - \frac{1}{3} c, \\
C, & 2C - \frac{2}{3} \theta_0 - \frac{1}{3} c \leq s \leq 2C - \frac{2}{3} \theta_0 + \frac{2}{3} c, \\
\frac{1}{3} \theta_0 + \frac{1}{2} s - \frac{1}{3} c, & s \geq 2C - \frac{2}{3} \theta_0 + \frac{2}{3} c.
\end{cases}
\]

The payoffs \( \pi^I_w(z, s, y) \) and \( \pi^I_l(s, y) \) in both cases can be expressed as follows:

\[
\pi^I_w(z, s, y) = q^I_w(z, s) (\theta_0 + s + y - q^I_w(z, s)) - c(q^I_w(z, s) - C)^+, \\
\pi^I_l(s, y) = q^I_l(s, y) (\theta_0 + y + s - q^I_l(y, y)) - q^I_l(s, y) - c).
\]

(9)

In deriving the equilibrium bidding function for the first-price auction, we would like to focus on the case where the expected (total) payoff is unimodal and the bidding function increasing, hence the first-order condition can be applied. We first characterize a sufficient condition similar to Assumption 2:

**Assumption 3.** \( q^I_w(s, s) \geq 4q^I_l(s, s), \forall s. \)

Note that the condition is stronger than Assumption 2. In Milgrom and Weber (1982), the sufficient equilibrium condition for the first-price auction is also stronger than the one for the second-price auction. In the second-price auction, they only require the valuation \( v(s, y) \) to be strictly increasing in both \( s \) and \( y \), while in the first-price auction, they need the signal affiliation to obtain the unimodality of bidders’ payoffs.

The next theorem characterizes the equilibrium bidding function.

**Theorem 2.** Suppose Assumption 3 holds. In the first-price auction, the unique symmetric, strictly increasing equilibrium bidding function is

\[
\beta^I(s) = \frac{1}{s} \int_0^s \left( \pi^I_w(y, y, y) - \pi^I_l(y, y) + \int_0^y \frac{\partial}{\partial u} \pi^I_w(u, y, v) \right|_{u=y} dv \right) dy.
\]

Moreover, \( \beta^I(s) \leq \frac{1}{s} \int_0^s (\pi^I_w(y, y, y) - \pi^I_l(y, y)) dy \).

The term \( \frac{1}{s} \int_0^s (\pi^I_w(y, y, y) - \pi^I_l(y, y)) dy \) can be regarded as the equilibrium bidding function in the conventional first-price auction by a slight modification of Milgrom and Weber (1982). Thus, Theorem 2 shows that the winner’s expected payment here is lower than the winner’s expected payment in the conventional counterpart. This is because the bid of a type-\( s \) retailer will be announced with probability \( P(s) \), and hence the threat of revealing her private information lowers her bid.

The monotonicity conditions of the bidding functions are also worth noting. \( \pi^I_w(s, s, s) - \pi^I_l(s, s, s) \) and \( \pi^I_w(l, s, s) - \pi^I_l(s, s, s) \) are the differences of equilibrium payoffs of a type-\( s \) retailer between the
cases where she wins and where she loses in the auctions, provided that her opponent also receives the same signal. Because our bidding functions $\beta^I(s)$ and $\beta^f(s)$ have an extra integral term, namely $\int_s^1 \frac{\partial}{\partial u} \pi^I(v, s, y)|_{v=s} dy$ and $\int_0^s \frac{\partial}{\partial u} \pi^f(u, y, v)|_{u=y} dv$ respectively, it is not enough to ensure that the difference of payoffs is increasing. This is different from Milgrom and Weber (1982). In fact, it constitutes a slight generalization for the type-dependent participation constraint. Put in terms of their notation, $v_l(s, y) \neq 0$, where $s$ is the bidder’s own signal, $y$ is her opponent’s signal, and $v_l$ is the utility when the bidder does not win the object. More specifically, when $v_l(s, y) = 0$, $\forall s$, $\forall y$, Milgrom and Weber (1982) show that $\beta^I(s) = v_w(s, s)$, where $v_w(s, s)$ is the expected value of the object if both bidders receive signal $s$. With a Unif[0, 1] signal distribution, the equilibrium bidding function under the first-price auction is $\beta^f(s) = \int_0^s v_w(y, y) \frac{1}{2} dy$. While allowing $v_l(s, y)$ to depend on the signals as well, it can be verified that $\beta^I(s) = v_w(s, s) - v_l(s, s)$ and $\beta^f(s) = \int_0^s (v_w(y, y) - v_l(y, y)) \frac{1}{s} dy$. In both auctions, a sufficient condition for the monotonicity of the bidding function is that $v_w(s, s) - v_l(s, s)$ is increasing in $s$.

In the first-price auction, we observe in the proof of Theorem 2 that the third term in the integrand is either $-\frac{1}{2} \int_0^y q_w(u, u) du$ or $-\frac{1}{4} \int_0^y q_w(u, u) du$; in either case, it is decreasing in $y$. This implies that we need a stronger condition to guarantee the monotonicity of $\beta^f(s)$, i.e., $\pi^f_w(s, s, s) - \pi^f_l(s, s)$ being strictly increasing does not suffice. More specifically, we require the integrand $\pi^f_w(y, y, y) - \pi^f_l(y, y) + \int_0^y \frac{\partial}{\partial y} \pi^f_w(u, y, v)|_{u=y} dv$ to be strictly increasing in $y$.

On the contrary, in the second-price auction, we can find situations where the integral $\int_s^1 \frac{\partial}{\partial u} \pi^I(v, s, y)|_{v=s} dy$ is non-monotonic (in region II for example). Thus, in some cases the extra integral term actually helps to sustain the monotonicity of $\beta^I(s)$. We conclude that it is more difficult to sustain a fully revealing bidding equilibrium in the first-price auction than in the second-price auction.

Finally, note that the retailers’ participation constraint is implicitly taken into consideration under both auction mechanisms: If a retailer does not want to participate, she can always join and submit a type-0 bid (i.e. $\beta^I(0)$ or $\beta^f(0)$). In this way, she will lose, and her true signal will not be revealed.

3.3 Outcome of the game

3.3.1 Supplier’s expected revenue

Under a second-price auction, the supplier’s expected revenue is $E[R^I] = \int_0^1 \beta^I(s) f_{(2)}(s) ds$, where $f_{(2)}(s)$ is the probability density function of $\min\{s_1, s_2\}$, since the payment is the highest losing bid. Given the independently uniformly distributed $s_1$ and $s_2$, $f_{(2)}(s) = 2(1 - s)$.

Under a first-price auction, when the payment equals the winning bid, the supplier’s expected revenue is $E[R^I] = \int_0^1 \beta^I(s) f_{(1)}(s) ds$, where $f_{(1)}(s)$ is the density function of $\max\{s_1, s_2\}$, and in the context of our model, $f_{(1)}(s) = 2s$. 19
3.3.2 Quantities in the consumer market

Given the inverse linear demand function (1), the total aggregated consumers’ surplus given auction format A is \( \Pi_A(s_1, s_2) = \frac{1}{2} (Q_A(s_1, s_2))^2 \), where \( Q_A(s_1, s_2) \) is the total quantity that retailers provide to the consumer market.\(^6\) Note that for a given pair of signals \((s_1, s_2)\), \( \Pi_A(s_1, s_2) \) is strictly increasing in \( Q_A(s_1, s_2) \). Therefore, to compare consumers’ surplus in both auctions, it suffices to consider the equilibrium quantities.

According to the auction announcement policy, we obtain the equilibrium quantities \( Q_I(s_1, s_2) = q_I^w(s_w) + q_I^l(s_l, s_w) \) and \( Q^{II}(s_1, s_2) = q^{II}_w(s_w, s_l) + q^{II}_l(s_l) \), where \( s_w = \max\{s_1, s_2\} \) and \( s_l = \min\{s_1, s_2\} \).

We obtain that

**Theorem 3.** *If the difference between retailers’ signals is large, consumers are better off in the first-price auction. Otherwise, if the difference between retailers’ signals is small, consumers are better off in the second-price auction.*

In the extreme case where retailers receive very different signals \((s_1, s_2)\), the real demand is actually close to its mean. According to Theorem 3, in the first-price auction, the disclosure of the winner’s bid will raise the loser’s expectation of the product demand, and therefore bring the loser’s quantity back to the normal level. However, in the second-price auction, the loser has no access to the winner’s signal, and therefore she underestimates the product demand, which results in an abnormally low quantity. Thus in this case, consumers are better off in the first-price auction due to the correction of loser’s belief.

Figure 4 illustrates the quantity difference between the first-price and second-price auctions. Note that the graph is symmetric since these two retailers are symmetric, and the boundary between the two regions in the lower triangle is nondecreasing in \( s_1 \). Moreover, the closer a point is to the lower-right corner (equivalently, the upper-left corner), the larger the difference between \( Q_I(s_1, s_2) \) and \( Q^{II}(s_1, s_2) \).

4. Numerical Results

In this section, we provide several numerical examples to illustrate payoffs, quantities and bidding functions in equilibria. The parameters \( \theta_0, c \) and \( C \) are carefully chosen such that the nonnegativity of realized prices and quantities is satisfied in all subgames. In particular, unless explicitly mentioned, we fix \( \theta_0 = 3 \), \( C = 1.5 \), and \( c = 2.5 \) in all the following examples. In fact, given \( \theta_0 = 3 \), we observed similar qualitative results with different values of \( c \). Note that in all cases, we need Assumptions 2-4

\(^6\)Note that the for the inverse linear demand function \( P(Q) = \theta - Q \), the total aggregated consumers’ surplus is \( \int_0^\theta P(q) dq - P(Q) Q = \frac{1}{2} Q^2 \).
to hold in order to guarantee the unimodal property of the retailers’ payoffs and the monotonicity of the bidding functions.

Figure 5 compares the total quantities $Q_I(s_1, s_2)$ and $Q_{II}(s_1, s_2)$ provided to the consumer market. Since retailers are symmetric, the graph is symmetric; and $Q_I(s_1, s_2)$ turns out to be higher in the upper-left and lower-right corners. This is consistent with Theorem 3 and Figure 4 since larger quantities in the consumer market corresponds to lower prices for the consumers.

In Figure 6, we plot the bidding function of a type-$s$ retailer. Note that the bidding function in a first-price auction is lower than that in a second-price auction for all signals. Since in the first-price auction the bid is the payment when the retailer wins, she will decrease her bid to maintain her rent (similar results are reported in Krishna (2002, Chapter 6)). Moreover, the bidding functions inferred here are lower than the counterparts under the conventional auction (as mentioned in Section 3.2), and the difference becomes larger as the signal gets higher.

Figure 7 presents the supplier’s expected revenue under both mechanisms. The supplier’s expected revenue, derived in Section 3.3.1, equals the expected payment from the retailers while taking expectation with respect to their signals. This amount turns out to be higher in the second-price auction than in the first-price auction. Note that this result is in contrast with the Revenue Linkage Principle: In a conventional auction, if the signals are independent as in our case, the Revenue Linkage Principle says that the supplier should receive the same revenue from both first- and second-price mechanisms. Here, the foreseeable information disadvantage in the consumer market constitutes a sort of winner’s
Figure 5: Total aggregated quantity provided to the consumer market under both auction mechanisms.

curse in that the winner in the first-price auction collects on average less profit from winning the auction, and hence she is unwilling to pay as much as she would pay under the second-price auction. This result directly lowers the supplier’s expected revenue in the first-price auction.

From Figure 7, we also observe that as the supplier increases the capacity level $C$, his expected revenue gets saturated after certain thresholds. This is related to the maximum quantity that the market demand may request (recall the downward slopping demand function in equation (1)), and retailers are unwilling to pay more for excess capacity. Suppose that the auction mechanism is given. The supplier can decide his optimal capacity, taking into account his own cost of building capacity. Our results further suggest, for example, that providing capacity higher than $C = 2.5$ is a weakly dominated strategy, independent of supplier’s cost structure (as long as capacity cost is increasing). The fact that the saturation comes later in the second-price auction (at $C = 2.5$ in the figure) is also worth noting: it implies that retailers are still willing to pay for marginal increase of capacity $C$ due to the combined first-mover and information advantage.

Next, we investigate the retailer’s willingness-to-pay in both auctions as opposed to the given spot price $c$, where willingness-to-pay is defined as the expected payment given that the retailer wins. That is, in the first-price auction, the willingness-to-pay is the bidding function, whereas in the second-price auction, it becomes the expected losing bid. Figure 8 shows that the willingness-to-pay under both auctions increases linearly in $c$, and runs always below $c$ (the 45-degree line). The participation of retailers in the auction can be attributed to two reasons: gathering information about the competitor’s private signal, and lower expected procurement cost. Thus, a supplier with limited capacity would sell it at a lower price than the unlimited procurement market. This also justifies why dual sourcing is possible: neither source is dominated by the other.

Figure 9 exhibits the retailers’ expected profit under both auctions. That is, we draw two curves
Figure 6: Comparison of retailer bidding functions under both auction mechanisms.

Figure 7: Comparison of supplier’s revenues $E[R^I]$ and $E[R^{II}]$. 
that come from equations (6) and (8) respectively, when replacing $z$ by the true type $s$. The figure shows clearly that a retailer is better off in the second-price auction, and that the gap is expanded as the signal increases. This observation jointly with Figure 7 implies that the second-price auction in fact yields a higher combined payoff for the entire supply chain since the supplier and the retailers are all better off. Allowing the winner to get both the first-mover and the information advantage reduces the competition tension between downstream retailers, and hence ex ante the entire supply chain benefits. Note also that in the first-price auction, the retailers with low signals receive just their reservation utilities eventually, and hence bidders with low signals do not get information rent.

Finally, we want to emphasize that we have briefly reported our numerical experiments on sets over which the bidding functions remain monotonic. More specifically, let us take the parameters $(\theta_0 = 3, C = 1.5, c = 2.5)$ as the reference point, and change them one at a time. Fix $\theta_0 = 3$ and...
$C = 1.5$, the monotonicity holds for both auctions when $c \in [2.1, \infty)$; fix $\theta_0 = 3$ and $c = 2.5$, it holds when $C \in [0.83, \infty)$; and if $C = 1.5, c = 2.5$, the bidding functions are monotonic when $\theta_0 \leq 3.7$. The monotonicity fails as either the gain from the auction becomes small or the value of information advantage in the product market becomes large. Namely, the retailers are unwilling to reveal their signals as the procurement market price is small, as the capacity is small, and as the potential demand is large. Moreover, in all our numerical experiments, as long as the bidding function under the first-price auction is monotonic, then the monotonicity for the bids under the second-price auction holds, which suggests that the separating equilibrium is relatively harder to sustain in the first-price auction. The numerical experiments support our interpretation of Theorems 1 and 2.

5. Conclusions

In this paper, we analyze a two-stage supply chain setting, where two symmetric retailers bid for the capacity of a supplier and then compete in the consumer market. The supplier’s capacity is sold as a bundle, and right after the auction closes, both retailers can get additional items in a procurement market. We study this model under first- and second-price auctions, and analyze the impact of the information about the auction price disclosed by the supplier in the first stage of the game.

We characterize sufficient conditions for the existence of monotonic equilibrium bidding functions. These conditions are more restrictive for the first-price auction case. When an increasing equilibrium bidding function exists, the threat of revealing private information induces lower bids than the ones that would be submitted under conventional auctions (i.e., auctions with no resale). The first-price auction makes consumers better off if retailers receive very different demand signals. Under a second-price auction, the supplier collects higher expected revenue, and retailers receive higher expected profits as well. The consumers are also better off if the retailers receive similar signals.

Our model shows that traditional auctions may have a significant impact when put to work in the context of a supply chain because of the information asymmetry introduced when announcing their results, and despite some restrictive assumptions, it constitutes a starting point to understand the underlying economics in supply chain negotiations using auctions under interdependent-values.

As a possible extension, we can think of a similar model with budget constraints on the retailers’ side. Note that the equilibrium of a Cournot game as ours is independent of the retailers’ cash endowments. Che and Gale (1998) consider a single-period auction where bidders have a budget constraint and show that bids are distorted in that scenario. In our model, a more intriguing trade-off prevails. If a retailer spends too much in acquiring the capacity $C$ from the auction, she may not have enough money to trade in the procurement market after observing some information about her rival’s signal. This phenomenon deserves further investigation.

Another plausible extension would be to allow the supplier to split the capacity sold through the
auction. However, in case both retailers get part of it, the supplier should announce the prices paid by they both, which would break the information asymmetry introduced in the current setting in the late stages of the supply chain.

Acknowledgements

We would like to thank Ke-Wei Huang, Anshul Sheopuri (NYU), and the participants of the Seminar Series at the IOMS-Department, Stern School of Business (NYU), for many stimulating discussions. We would also like to thank David Reiley for generously sharing his data of online auctions.

References


A. Appendix

In this Appendix we extensively use the regions found in Figure 3 and divide our analysis by cases. To avoid redundancy, we present only three of them and refer the other cases to their analogous counterparts. Usually the analogy is made when the terms we consider differ merely in the marginal cost $c$, e.g., regions I and V.

Proof of Proposition 1

We first disregard the capacity constraint of the winner, and derive her first-order conditions given $s_l$.

Let $\Pi_{II}^w \equiv (\theta - q_w - q_l)q_w - c(q_w - C)^+$ denote the winner’s payoff function. By differentiating $\Pi_{II}^w$ with respect to $q_w$ where $q_w \leq C$, we obtain the first-order condition

$$q_{w}^*(s_w, s_l) = \frac{1}{2} (\theta - q_l(s_l) - c\{q_w^*(s_w, s_l) \geq C\}) ,$$  \hspace{1cm} (A.1)

where $\mathbb{I}\{\cdot\}$ is the indicator function. Similarly, we derive the first-order condition for the loser:

$$q_{l}^*(s_l) = \frac{1}{2} (E_{s_w}[\theta|s_w > s_l] - E_{s_w}[q_w(s_l)|s_w > s_l] - c) .$$

Observing that the differentiation $\theta - q_l^* - 2q_w - c\{q_w \geq C\}$ is decreasing in $q_w$, the marginal change of the winner payoff will look like Figure A.1. Now the optimal quantity $q_{II}^w(s_w, s_l)$ follows immediately from the comparison between marginal revenue and marginal cost. Note that the discontinuity occurs when capacity $C$ is hit. In Figure A.1, $s_0$ falls into the first case of equation (2) since the marginal change becomes negative before hitting $C$. $s_1$ there equals $S_{II}^1(s_l)$ because the marginal

![Figure A.1: Marginal change of the winner’s payoff vs quantity.](image)
change at $C$ from the right just turns negative. Likewise, $s_2 = S_2^I(s_j)$ because $s_2$ is the largest point whose marginal change is positive for all $q_w \leq C$. By continuity of $q_w^I(s_w, s_l)$, the values of $S_1^I(s_l)$ and $S_2^I(s_l)$ can be obtained by equating the capacity $C$ and $q_w^I(s_w, s_l)$ at the boundary points. □

**Proof of Proposition 2**

The following lemma is needed to show this proposition.

**Lemma 1.** In the equilibrium quantities described in equation (2),

$$0 < q_w^I(s_2) - q_w^I(s_1) < s_2 - s_1,$$

for signals $s_1 < s_2$.

**Proof:** The proof is by contradiction. Suppose that there exists $s_1^I$ such that $(q_w^I)'(s_1^I) = \rho_1^I > 1$. We can rewrite equation (2) as

$$q_w^I(s_l) = \frac{1}{2}[\theta_0 + \frac{3}{2}s_l - E_{s_w}[q_w^I(s_w, s_l)|s_w > s_l] - \frac{1}{2} - c].$$

(A.2)

The coefficient $s_w$ term in $q_w^I(s_w, s_l)$ is lower bounded by 0 according to equation (2), and hence $(q_w^I)'(s_1^I) = \rho_1^I > 1$ implies that there exists a signal $s_1^I$ such that $\frac{\partial}{\partial s_0}(q_w^I)(s_w, s_l)|s_i = s_1^I < \frac{3}{2} - 2\rho_1^I < 0$. Since $q_w^I$ will not change if the capacity is binding, this can happen only in either the first or the third case. But then this implies that $\frac{1}{2}(1 - \rho_1^I) \leq \frac{3}{2} - 2\rho_1^I$, which leads to $\rho_1^I \leq \frac{3}{4}$, a contradiction.

Similarly, suppose that there exists $s_2^I$ such that $(q_w^I)'(s_2^I) = \rho_2^I < 0$. Since from equation (2) the coefficient $s_w$ term in $q_w^I$ is upper bounded by $\frac{1}{2}$, the term associated to $s_w$ in $E_{s_w}[q_w^I(s_w, s_l)|s_w > s_l]$ contributes at most $\frac{1}{2} + \frac{1}{s_2}$. We can show that for some $s_2^I$ whose $q_w^I(s_2^I) \leq s_2^I$ implies $\frac{1}{2}(1 - \rho_2^I) \geq 1 - 2\rho_2^I \Rightarrow \rho_2^I \geq \frac{1}{3}$. Hence we conclude that $0 \leq q_w^I(s_2) - q_w^I(s_1) \leq s_2 - s_1, \forall s_1 \leq s_2$. □

Now we prove Proposition 2. We will focus on $S_1^I(s)$; the proof for $S_2^I(s)$ goes along the same argument. The first part of equation (5) follows from the fact that $S_2^I(s) \geq S_1^I(s), \forall s$. In the sequel, $s_1$ and $s_2$ are two distinct signals and $s_1 < s_2$.

**Case a): $S_1^I(s_2) = 1$**

We would like to prove that $S_1^I(s_1) = 1$ as well, i.e., the capacity constraint is never binding when $s_1 = s_1$.

Note that $S_1^I(s_2) = 1$ means that

$$\frac{1}{3}\theta_0 + \frac{1}{2} + \frac{1}{4}s_2 + \frac{1}{3}c - \frac{1}{12} \leq C,$$

and therefore

$$\frac{1}{3}\theta_0 + \frac{1}{2} + \frac{1}{4}s_1 + \frac{1}{3}c - \frac{1}{12} < \frac{1}{3}\theta_0 + \frac{1}{2} + \frac{1}{4}s_2 + \frac{1}{3}c - \frac{1}{12} \leq C,$$

which implies $S_1^I(s_1) = 1$. 31
Case b): $s_1 < S_1^{II}(s_1) < 1$ and $s_2 < S_1^{II}(s_2) < 1$

Suppose $s_1 < S_1^{II}(s_1) < 1$ and $s_2 < S_1^{II}(s_2) < 1$. Our goal here is to prove that $S_1^{II}(s_2) < S_1^{II}(s_1)$. Since in both cases the capacity constraint is not binding for some $s_w$ and binding for others,

$$S_1^{II}(s_2) = 2C - \theta_0 - s_2 + q_1^{II}(s_2)$$
$$= S_1^{II}(s_1) - (s_2 - s_1) + (q_1^{II}(s_2) - q_1^{II}(s_1)),$$

and therefore by Lemma 1,

$$0 < S_1^{II}(s_1) - S_1^{II}(s_2) < s_2 - s_1.$$

Case c): $S_1^{II}(s_1) = s_1$.

In this case, we would like to prove that $S_1^{II}(s_2) = s_2$ as well. The proof is by contradiction.

First we claim that $S_1^{II}(s_2) \neq s_2$. If this is not the case, the capacity constraint is never binding when $s_l = s_2$. By the argument in Case a), $S_1^{II}(s_1) = 1$ as well since $s_1 < s_2$, which contradicts the fact $S_1^{II}(s_1) = s_1$.

Therefore, the only possibility is that $s_2 < S_1^{II}(s_2) < 1$. In this case, $S_1^{II}(s_2) = 2C - \theta_0 - s_2 + q_1^{II}(s_2)$, and $S_1^{II}(s_1) = s_1$ implies $\frac{1}{2}(\theta_0 + s_1 + s_1 - q_1^{II}(s_1)) \geq C$. Then we can establish the following inequality:

$$S_1^{II}(s_2) \leq 2s_1 + \theta_0 - q_1^{II}(s_1) - \theta_0 - s_2 + q_1^{II}(s_2)$$

Thus,

$$s_2 < S_1^{II}(s_2) \leq 2s_1 + \theta_0 - q_1^{II}(s_1) - \theta_0 - s_2 + q_1^{II}(s_2),$$
$$\Rightarrow 2(s_2 - s_1) < q_1^{II}(s_2) - q_1^{II}(s_1),$$
$$\Rightarrow q_1^{II}(s_2) - q_1^{II}(s_1) > 2(s_2 - s_1),$$

which contradicts Lemma 1. Hence we conclude that $S_1^{II}(s_2) = s_2$. 

Proof of Theorem 1

Preliminaries

We start discussing several technical lemmas that lead to Theorem 1 and provide their economic intuition.

If the losing bid $\beta^{II}(z)$ is higher, the winner should expect the demand to be higher, and therefore the quantity she puts in the consumer market $q_1^{II}(y, z)$ should also be larger, and the magnitude by which the winning quantity increases should be reasonably bounded. Hence, we have
Lemma 2.

$$0 \leq q_{II}^{w}(y, z_2) - q_{II}^{w}(y, z_1) < \frac{1}{2}(z_2 - z_1), \forall z_1 < z_2.$$ (A.3)

**Proof:** We show the monotonicity by dividing the shape of $S_{II}^{I}(z)$ into cases. We write $z_i \in I$ if $z_i$ belongs to region I; $z_i \in II$ and $z_i \in III$ are defined analogously.

**Case 1:** $y \leq S_{II}^{I}(z_1), y \leq S_{I}^{II}(z_2)$

In this case, the capacity constraint is not binding, and therefore

$$q_{II}^{w}(y, z) = \frac{1}{3}\theta_0 + \frac{1}{2}y + \frac{1}{3}z + \frac{1}{3}c - \frac{1}{12},$$

which satisfies equation (A.3).

**Case 2:** $(z_1, z_2) \in (II, II)$

Since when $z < S_{II}^{I}(z) < 1$, $S_{II}^{I}(z)$ is decreasing in $z$, the critical point where $q_{II}^{w}(y, z)$ hits the capacity $C$ comes earlier. It suffices to show that while $q_{II}^{w}(y, z)$ is not hitting the capacity, it is increasing in $z$.

Following Proposition 1, we first observe that given $z$, $\frac{\partial}{\partial y} q_{II}^{w}(y, z) = \frac{1}{2}$, which means $q_{II}^{w}(y, z)$ is increasing in $y$ with a fixed rate. Therefore, for all $y$ such that $z_2 \leq y \leq S_{II}^{I}(z_2)$,

$$q_{II}^{w}(y, z_2) = q_{II}^{w}(S_{II}^{I}(z_2), z_2) - \frac{1}{2}[S_{II}^{I}(z_2) - y]$$

$$= C - \frac{1}{2}[S_{II}^{I}(z_2) - y]$$

$$\geq C - \frac{1}{2}[S_{II}^{I}(z_1) - y], \quad \text{since} \quad S_{II}^{I}(z_1) > S_{II}^{I}(z_2)$$

$$= q_{II}^{w}(S_{II}^{I}(z_1), z_1) - \frac{1}{2}[S_{II}^{I}(z_1) - y], \quad \text{by definition of} \quad S_{II}^{I}(z_1)$$

$$= q_{II}^{w}(y, z_1).$$

Moreover,

$$q_{II}^{w}(y, z_2) - q_{II}^{w}(y, z_1) = [C - \frac{1}{2}(S_{II}^{I}(z_2) - y)] - [C - \frac{1}{2}(S_{II}^{I}(z_1) - y)]$$

$$= \frac{1}{2}(S_{II}^{I}(z_1) - S_{II}^{I}(z_2)) \leq \frac{1}{2}(z_2 - z_1),$$

where the last inequality comes from Lemma 1 and equation (3).

**Remark:** Figure A.2 shows the relationship between $z_1, z_2, S_{II}^{I}(z_2)$, and $S_{II}^{I}(z_1)$. Referring to Figure 3, we fix the loser’s signal $s_l$ inside region II. Now we increase $s_w$ along the vertical line that passes $(s_l, 0)$, and draw the winner’s quantity $q_w$ as a function of $s_w$ in Figure A.2.
Figure A.2: An example to show the relative positions of \( z_1, z_2, S^{II}_1(z_2), \) and \( S^{II}_1(z_1). \)

Case 3: \( S^{II}_1(z_1) \leq y \leq S^{II}_2(z_1), S^{II}_1(z_2) \leq y \leq S^{II}_2(z_2) \)

In this case, the capacity constraint of the winner is binding for all \( z, \) i.e., \( q^{II}_w(y, z) = C, \forall z. \) Thus, the result holds.

Other Cases: The cases \( y \geq S^{II}_2(z_1), S^{II}_1(z_2) \leq y \leq S^{II}_2(z_2) \) and \( y \geq S^{II}_2(z_1), y \geq S^{II}_2(z_2) \) are very similar to Cases 2 and 3 respectively, and the results follow from analogous derivations. In other cases where \( z_1 \) and \( z_2 \) belong to different regions in Figure 3, we can use the triangle inequalities by inserting the boundaries of two regions \( s^*_t \) and \( s^{**}_t \) respectively.

Lemma 2 confirms the correctness of our intuition. When the losing bid increases, the winner’s expectation of the demand in the consumer market also increases, and therefore she puts more equilibrium quantity. This can be interpreted as the winner’s overestimation of the demand if the loser submits a bid higher than \( \beta^{II}(s) \). However, the increment of \( q^{II}_w(y, z) \) is bounded above.

Furthermore, if the loser’s signal increases, her optimal quantity should also increase at a reasonable rate:

Lemma 3.

\[
\frac{1}{2}(z_2 - z_1) \leq q^{II}_1(z_2) - q^{II}_1(z_1) < \frac{3}{4}(z_2 - z_1), \quad \forall z_1 < z_2.
\] (A.4)

Proof: Recall from equation (2) that

\[
q^{II}_1(z) = \frac{1}{2}\theta_0 + \frac{3}{4}z - \frac{1}{2}c - \frac{1}{2}\mathbb{E}_y[q^{II}_w(y, z)|y \geq z].
\]

If we increase \( z \) by \( \Delta z \), the term \( \frac{3}{4}z \) will increase by \( \frac{3}{4}\Delta z \). The term \( \mathbb{E}_y[q^{II}_w(y, z)|y \geq z] \) is bounded by 0 and \( \frac{1}{2} \) according to Lemma 2. Therefore, equation (A.4) is valid. □
Note that this lemma provides tighter upper and lower bounds for the first-order difference of \( q^H_t(z) \) than Lemma 1. The next lemma says that if a type-\( s \) retailer loses in the auction, her expected partial payoff will be decreasing in her reported type.

**Lemma 4.** \( \int_s^1 \frac{\partial}{\partial z} \pi^H_l(z, s, y) \big|_{z=s} dy \leq 0. \)

**Proof:** Define \( g(s) = \int_s^1 \frac{\partial}{\partial z} \pi^H_l(z, s, y) \big|_{z=s} dy. \)

\[
g(s) = \int_s^1 \frac{\partial}{\partial z} (q^H_l(z, s)[\theta_0 + y + s - q^H_w(y, z) - q^H_l(z, s) - c]) \big|_{z=s} dy \\
= \int_s^1 \left( \frac{\partial q^H_l(z, s)}{\partial z} \right) [\theta_0 + y + s - q^H_w(y, z) - q^H_l(z, s) - c] \\
+ q^H_l(z, s) \left[ -\frac{\partial q^H_w(y, z)}{\partial z} - \frac{\partial q^H_l(z, s)}{\partial z} \right] \big|_{z=s} dy. \tag{A.5}
\]

Recalling that \( q^H_l(z, s) = q^H_l(z) + \frac{1}{2}(s - z) \) and Proposition 1, we have

\[
\frac{\partial q^H_l(z, s)}{\partial z} \big|_{z=s} = (q^H_l)'(s) - \frac{1}{2}, \\
\frac{\partial q^H_w(y, z)}{\partial z} \big|_{z=s} = \begin{cases} 0, & S^H_1(z) \leq y \leq S^H_2(z), \\ \frac{1}{2}[1 - (q^H_l)'(s)], & \text{otherwise}. \end{cases}
\]

Thus, equation (A.5) can be rewritten as

\[
g(s) = \left( (q^H_l)'(s) - \frac{1}{2} \right) \left( E_y[\theta | s, y \geq s] - E_y[q^H_w(y, s) | s, y \geq s] - q^H_l(s) - c \right) \\
+ q^H_l(s) \left( 1 - \frac{1}{2} (q^H_l)'(s) \right) \left[ 1 - (S^H_2(s) - S^H_1(s)) \right] - q^H_l(s) \left[ (q^H_l)'(s) - \frac{1}{2} \right] (S^H_2(s) - S^H_1(s)) \\
= \frac{1}{2} \left( 1 - (S^H_2(s) - S^H_1(s)) \right) q^H_l(s) \left[ (q^H_l)'(s) - 1 \right]. \tag{A.6}
\]

where the last equality follows from the fact \( q^H_l(s) = \frac{1}{2} \left( E_y[\theta | s, y \geq s] - E_y[q^H_w(y, s) | s, y \geq s] - c \right). \)

Since \( 1 - (S^H_2(s) - S^H_1(s)) \) and \( q^H_l(s) \) are positive, and \( (q^H_l)'(s) - 1 \) is negative by Lemma 3, we conclude that \( g(s) < 0 \), which completes the proof. \( \square \)

The average marginal change of a type-\( s \) loser’s partial payoff is obtained by integrating over her opponent’s signal \( y \), with \( y \geq s \). For notational convenience, we assume that \( \pi^H_l(z, s, y) \) is differentiable with respect to \( z \) at boundary points. If it is not differentiable, then we should look for subgradients rather than the gradient from the first-order condition.

Now, we complete the proof of Theorem 1. We will first derive a necessary condition that an equilibrium bidding function must satisfy, and then provide the verification of the proposed bidding function.
1. Necessary Condition

The first-order condition with respect to $z$ is as follows:

$$\frac{\partial \Pi^H(z|s)}{\partial z} = [\pi^H_w(s, z) - \beta^H(z) - \pi^I_I(z, s, z)] + \int_z^1 \frac{\partial}{\partial v} \pi^H_I(v, s, y)|_{v=z} dy.$$

(A.7)

The truth-telling equilibrium requires that the bidder’s optimal strategy is to reveal her own type. Thus, we conjecture the equilibrium bidding function $\beta^H(s)$ as follows:

$$\beta^H(s) = \pi^H_w(s, s) - \pi^I_I(s, s, s) + \int_s^1 \frac{\partial}{\partial z} \pi^H_I(z, s, y)|_{z=s} dy$$

(A.8)

where the last inequality is given by Lemma 4. Note that by Proposition 1, the equilibrium quantities $q^H_w(z, s)$ and $q^H_I(y)$ are all uniquely determined for any values $z, s, y$, and hence the terms $\pi^H_I(s, s, s)$, $\pi^I_I(s, s, s)$, and $\frac{\partial}{\partial z} \pi^H_I(z, s, y)|_{z=s}$ are all unique since these are generated from $q^H_w(z, s)$ and $q^H_I(y)$. Thus, within the class of symmetric equilibria, if there exists an equilibrium bidding function that is strictly increasing in the signal, then it must be uniquely determined by equation (A.8).

2. Verification: Monotonicity of $\beta^H(s)$

For the second-price auction, our goal is to show that $\beta^H(s)$ is strictly increasing in $s$, i.e., $\lim_{z \to s} \frac{\beta^H(z) - \beta^H(s)}{z - s} > 0$, $\forall s \in [0, 1]$. Except region IV where the capacity constraint is binding for $q^H_w(y, y)$, we obtain

$$\pi^H_w(z, z) - \pi^H_I(z, z, z) - [\pi^H_w(s, s) - \pi^H_I(s, s, s)] = (2 - 2\rho_l)q^H_w(s, s) - (1 - \frac{1}{2}\rho_l)q^H_I(s, s) + \rho_l c I\{s \in V\},$$

where we have ignored the second-order terms and $\rho_l$ is such that $q^H_I(z, z) = q^H_I(s, s) + \rho_l(z - s)$. We will divide the analysis according to Figure 3. Note that $q^H_w(s, s)$ takes values along the 45-degree line.

**Case 1:** $s \in I$

In this case, $\rho_l = \frac{1}{2}$, $S^I_1(s) = S^I_2(s) = 1$. From equation (A.6),

$$g(s) = \int_s^1 \frac{\partial}{\partial z} \pi^H_I(z, s, y)|_{z=s} dy = \frac{1}{2} (1 - (S^I_2(s) - S^I_1(s))) q^H_I(s) \left((\beta^H)'(s) - 1\right),$$

and hence $\beta^H(z) - \beta^H(s) = (z - s)(q^H_w(s, s) - \frac{3}{4} q^H_I(s, s) + \frac{1}{2} c - \frac{1}{8}) + o(z - s)$. In other words, $(\beta^H)'(z) = q^H_w(s, s) - \frac{3}{4} q^H_I(s, s) + \frac{1}{2} c - \frac{1}{8} > 0$ by Assumption 2.

**Case 2:** $s \in II$

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When \( s \in II \), \( S^I_2(s) = 1 \) and \( S^I_1(s) = 2C - \theta_0 - s + q^I_1(s) \). Therefore,
\[
\beta^I(z) - \beta^I(s) = (2 - 2\rho_t)q^I_1(s, s) - \frac{1}{2}(\rho^2_t - 3\rho_t + 3)q^I_1(s, s) - \frac{1}{2}\rho_t(1 - \rho_t)S^I_1(s) + o(z - s).
\]
Note that from Lemma 3, \( \frac{1}{3} \leq \rho_t \leq \frac{3}{4} \), and \( S^I_1(s) \leq 1 \). A sufficient condition for monotonicity under these parameters is that \( q^I_1(s, s) > \frac{13}{16}q^I_1(s, s) + \frac{3}{16} \).

Case 3: \( s \in III \)

Here we have \( S^I_2(s) - S^I_1(s) = 1 - c \), and therefore
\[
\beta^I(z) - \beta^I(s) = (2 - 2\rho_t)q^I_1(s, s) - (1 - \frac{1}{2}\rho_t)q^I_1(s, s) - \frac{1}{2}\rho_t(1 - \rho_t) + c(1 - \frac{1}{2}(1 - \rho_t)) + o(z - s).
\]
Note that when \( \frac{1}{2} \leq \rho_t \leq \frac{3}{4} \), \( c(1 - \frac{1}{2}(1 - \rho_t)) \) is always positive. A sufficient condition for \( \beta^I(s) \) being increasing is that \( q^I_1(s, s) > \frac{5}{4}q^I_1(s, s) + \frac{3}{16} \).

Case 4: \( s \in IV \)

In this case, the capacity constraint is binding and \( g(s) = -\frac{1}{2}(1 - \rho_t)(1 - q^I_1(s, s)) \).
\[
\beta^I(z) - \beta^I(s) = (2 - 2\rho_t)q^I_1(s, s) - 2q^I_1(s, s) - \frac{1}{2}\rho_t(1 - \rho_t) + o(z - s).
\]
\( \frac{1}{2}\rho_t(1 - \rho_t) \) achieves its maximum at \( \rho_t = 2 - \sqrt{2} \), and hence if \( q^I_1(s, s) \geq \frac{8}{5}q^I_1(s, s) + \frac{(\sqrt{2} - 1)(2 - \sqrt{2})}{2\sqrt{2}} \), the monotonicity holds. The constant term is roughly 0.086.

Case 5: \( s \in V \)

In region V, \( \rho = \frac{1}{2} \), \( S^I_1 = S^I_2 = s \), and hence \( g(s) = -\frac{1}{2}q^I_1(s, s) \). It can be verified that \( \beta^I(z) - \beta^I(s) = (z - s)(q^I_1(s, s) - \frac{3}{4}q^I_1(s, s) - \frac{1}{3}) + o(z - s) \), and therefore monotonicity holds.

We conclude that the bidding function \( \beta^I(s) \) is strictly increasing since Assumption 2 is sufficient for all cases.

3. Verification: Incentive compatibility

We will follow Milgrom and Weber (1982) to verify that the proposed bidding function is indeed an equilibrium. The idea is to show that a type-\( s \) bidder’s payoff is unimodal in her reported type, and it achieves the maximum at the truth-telling value \( s \).

Suppose the other player adopts that bidding function. Differentiating the expected payoff with respect to \( z \), we obtain
\[
\frac{\partial \Pi^I(z|s)}{\partial z} = (\pi^I_w(s, z) - \beta^I(z) - \pi^I_1(z, s, z)) + \int_z^1 \frac{\partial}{\partial v} \pi^I_1(v, s, y)|_{v=z} dy
\]
\[
= (\pi^I_w(s, z) - \pi^I_1(z, s, z)) - (\pi^I_1(z, z) - \pi^I_1(z, z, z)).
\]
Our goal is to show that if \( z < s \), then
\[
\frac{\partial \Pi^H(z|s)}{\partial z} < 0,
\]
and if \( z > s \), then
\[
\frac{\partial \Pi^H(z|s)}{\partial z} > 0.
\]

We divide the proof into four cases in the sequel.

**Case 1:** \( z, s \leq S^H_1(z) \).

Recall that
\[
\pi^H_w(s, z) - \pi^H_l(z, s, z) = q^H_w(s, z) \left( (\theta_0 + s + z - q^H_w(s, z) - q^H_l(z)) \right) \\
- q^H_l(z, s) \left( (\theta_0 + z + s - q^H_w(z, z) - q^H_l(z, s) - c) \right),
\]
and \( q^H_l(z, s) = q^H_l(z, z) + \frac{1}{2}(s - z) \). In this case, the capacity constraint is never binding, and thus
\[
q^H_w(s, z) = q^H_w(z, z) + \frac{1}{2}(s - z).
\]
After simple manipulations, we can rewrite equation (A.9) as follows
\[
\frac{\partial \Pi^H(z|s)}{\partial z} = \frac{1}{2}(s - z)[q^H_w(z, z) - q^H_l(z, z) + c].
\]

The multiplicative term \( q^H_w(z, z) - q^H_l(z, z) + c \) is always positive by Assumption 2. Hence \( \partial \Pi^H(z|s)/\partial z \) is positive if \( s - z > 0 \) and negative if \( s - z < 0 \).

**Case 2:** \( S^H_1(z) \leq z, s \leq S^H_2(z) \).

In this case, \( q^H_w(s, z) = q^H_w(z, z) = C \).
\[
\frac{\partial \Pi^H(z|s)}{\partial z} = \frac{1}{2}(s - z) \left( C - \frac{1}{2}q^H_w(z, z) - \frac{1}{2}[\theta_0 + z + s - C - q^H_l(z, s) - c] \right) \\
= \frac{1}{2}(s - z) \left( C - \frac{1}{2}[\theta_0 + z + \frac{s + z}{2} - c - C] \right).
\]

Note that \( C = q^H_w(z, z) \) here and \( \frac{1}{2}[\theta_0 + z + \frac{s + z}{2} - c - C] \leq \frac{1}{2}[\theta_0 + z + \frac{s + z}{2} - c - C] = q^H_l(z) \) since \( s \leq 1 \). Thus, by Assumption 2, the multiplicative term is positive and the payoff is unimodal.

**Case 3:** \( z, s \geq S^H_2(z) \)

Mimicking the treatment of Case 1, we get
\[
\frac{\partial \Pi^H(z|s)}{\partial z} = \frac{1}{2}(s - z) \left( q^H_w(z, z) - q^H_l(z, z) \right),
\]

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where the term in brackets is positive by Assumption 2.

**Case 4: General case.**

Now we consider the case where the capacity constraint is binding only for one of \( q^I_w(s, z) \) and \( q^I_w(z, z) \), This includes both \( z \leq S^I_1(z) \leq s \leq S^I_2(z) \) and \( S^I_1(z) \leq z \leq S^I_2(z) \leq s \). In this case, \( q^I_w(s, z) - q^I_w(z, z) = \frac{1}{2}\rho(s - z) \), where \( 0 \leq \rho \leq 1 \). We can rewrite \( \pi^I_w(s, z) - \pi^I(z, s, z) \) as follows:

\[
\pi^I_w(s, z) - \pi^I(z, s, z) = (q^I_w(z, z) + \frac{1}{2}(S^I_1(z) - z)) (q^I_w(z, z) + (s - z) - \frac{1}{2}(S^I_1(z) - z)) - (q^I(z, z) + \frac{1}{2}(s - z))^2 \\
= \pi^I_w(z, z) - \pi^I(z, z, z) + (s - z) (q^I_w(z, z) - q^I(z) + \frac{1}{2}\rho (1 - \frac{1}{2}\rho)(s - z) - \frac{1}{4}(s - z)),
\]

and hence

\[
\frac{\partial \Pi^I(z|s)}{\partial z} = (s - z) \left( q^I_w(z, z) - q^I(z) + \frac{1}{2}\rho (1 - \frac{1}{2}\rho)(s - z) - \frac{1}{4}(s - z) \right).
\]

Note that Cases 1 and 2 can be regarded as two special cases of this general case, with \( \rho = 1 \) and \( \rho = 0 \) respectively, which bounds \( \frac{1}{2}\rho (1 - \frac{1}{2}\rho)(s - z) \) from below and above. Thus the multiplicative term is positive according to Cases 1 and 2, and the payoff is unimodal in this case as well.

**Remark:**

It can be verified that the assumption needed to guarantee the monotonicity of bidding function \( \beta^I(s) \) is stronger than that for the unimodality of retailers’ expected payoffs (incentive compatibility). This phenomenon does not occur in the conventional auctions (e.g., Milgrom and Weber (1982)).

Similar results apply to the first-price auction as well. See the proof of Theorem 2.

**Proof of Theorem 2**

1. **Necessary condition**

The first-order condition of \( \Pi^I(z|s) \) is

\[
\pi^I_w(z, s, z) - \beta^I(z) - \pi^I_l(s, z) + \int_0^z \pi^I_w(z, s, y)dy - \int_0^z (\beta^I)'(z)dy = 0.
\]

Rewriting the above equation, we find an expression of \( (\beta^I)'(z) \) as follows:

\[
(\beta^I)'(z) = \frac{1}{z} \left( \pi^I_w(z, s, z) - \pi^I_l(s, z) + \int_0^z \pi^I_w(z, s, y)dy - \beta^I(z) \right),
\]

The closed-form solution of \( \beta^I(z) \) can be obtained from this differential equation by the same procedure in Krishna (2002, Proposition 6.3). Note that the proof of Proposition 6.3 in Krishna requires the signal affiliation to guarantee the unimodality for the last steps in his derivation. In our case signals are non-affiliated, but unimodality is guaranteed from Assumption 3.
Together with the incentive compatibility condition, the bidding function is

\[
\beta^I(s) = \int_0^s \left( \pi_w^I(y, y, y) - \pi_l^I(y, y) + \int_0^y \pi_{w, 1}^I(y, y, v) dv \right) dL(y|s),
\]

(A.10)

where

\[
L(y|s) = \exp \left( - \int_y^s \frac{1}{v} dv \right) = \frac{y}{s}.
\]

Plugging \(L(y|s)\) in equation (A.10), we obtain the bidding function \(\beta^I(s)\). Since Proposition 4 guarantees that \(q_w^I(z, s)\) and \(q_l^I(y, z)\) are unique, all terms in equation (A.10) are known, i.e., there is only one bidding function that satisfies the first-order condition. Thus, if there exists a symmetric, strictly increasing equilibrium, it must be uniquely determined by equation (A.10).

Next, we will show that \(\pi_w^I(z, s, y)\) is decreasing in \(z\), for all pair \((s, y)\). This leads to the conclusion \(\beta^I(s) < \frac{1}{s} \int_0^s \left( \pi_w^I(y, y, y) - \pi_l^I(y, y) \right) dy\). Observing equation (9), the differentiation can be expressed as follows:

\[
\frac{\partial}{\partial z} \pi_w^I(z, s, y) = \frac{\partial}{\partial z} q_w^I(z, s) (\theta_0 + s + y - q_w^I(z, s) - q_l^I(y, z)) - c \frac{\partial}{\partial z} (q_w^I(z, s) - C) + q_l^I(z, s) \left( - \frac{\partial}{\partial z} q_w^I(z, s) - \frac{\partial}{\partial z} q_l^I(y, z) \right)
\]

(A.11)

Note that by Proposition 4, \(-\frac{1}{8} \leq \frac{\partial}{\partial z} q_w^I(z, s) \leq 0\) and \(\frac{1}{4} \leq \frac{\partial}{\partial z} q_l^I(y, z) \leq \frac{1}{2}\). Thus, the last term of equation (A.11) is negative. Observe that the first two terms can be combined into \(q_w^I(z, s) \frac{\partial}{\partial z} q_w^I(z, s)\) regardless of whether \(q_w^I(z, s) > C\) or not. Hence it is negative as well because \(\frac{\partial}{\partial z} q_w^I(z, s) \leq 0\), and we conclude that \(\pi_w^I(z, s, y)\) is decreasing in \(z\).

2. Verification: Monotonicity of \(\beta^I(s)\)

Similar to Milgrom and Weber (1982), a sufficient condition for the monotonicity of bidding function in the first-price auction is that the integrand of \(\beta^I(s)\) is increasing, i.e.,

\[
\pi_w^I(y, y, y) - \pi_l^I(y, y) + \int_0^y \frac{\partial}{\partial u} \pi_{w, 1}^I(u, y, v)|_{u=y} dv
\]

is increasing in \(y\). Recall that \(\frac{\partial}{\partial u} \pi_w^I(u, y, v)|_{u=y} = -q_w^I(y, y) \frac{\partial}{\partial u} q_l^I(v, u)|_{u=y}\) by equation (A.11), the integral is \(-\frac{1}{2} \int_0^y q_w^I(u, u) du\) if \(S_1^L \leq y \leq S_2^L\), and is \(-\frac{1}{2} \int_0^y q_w^I(u, u) du\) otherwise. Note also that

\[
\pi_w^I(y, y, y) - \pi_l^I(y, y) = q_w^I(y, y)[\theta_0 + y + y - q_w^I(y, y) - q_l^I(y, y) - c(q_w^I(y, y) - C)^+] - q_l^I(y, y)[\theta_0 + y + y - q_w^I(y, y) - q_l^I(y, y) - c].
\]

Now we will divide our analysis into three cases, depending on the regions to which \(y\) belongs.

Case 1: \(y \leq S_1^L\)
We first consider the case $y \in S^I_1$. Let
\[
\beta^I(z) - \beta^I(y) = \pi^I_w(z, z, z) - \pi^I_w(s, z) + \int_0^z \frac{\partial}{\partial v} \pi^I_w(v, z, v) |_{u=z} dv \]
\[
- \left[ \pi^I_w(y, y, y) - \pi^I_w(y, y) + \int_0^y \frac{\partial}{\partial u} \pi^I_w(u, y, y) |_{u=y} du \right].
\]
Our goal is to show that $\lim_{z \to y} \frac{\beta^I(z) - \beta^I(y)}{z - y} > 0$, $\forall y$. Note that in this case $q^I_w(z, z) = q^I_w(y, y) + \frac{1}{2}(z - y)$ and $q^I_w(z, z) = q^I_w(y, y) + \frac{3}{2}(z - y)$. Therefore after some algebra, $\beta^I(z) - \beta^I(y)$ can be rewritten as $(z - y)[\frac{2}{3} q^I_w(y, y) - q^I_w(y, y) + \frac{1}{2}c] + o(z - y)$, where $\lim_{x \to 0} o(x)/x = 0$. Hence $\lim_{z \to y} \frac{\beta^I(z) - \beta^I(y)}{z - y} > 0$ by Assumption 3 in this case.

Case 2: $y > S^I_2$

Similar to Case 1 except that $\beta^I(z) - \beta^I(y) = (z - y)[\frac{2}{3} q^I_w(y, y) - q^I_w(y, y)] + o(z - y)$, and hence the result holds.

Case 3: $S^I_1 \leq y \leq S^I_2$

In this case the capacity constraint is binding, i.e., $q^I_w(z, z) = q^I_w(y, y) = C$, and $q^I_w(z, z) = q^I_w(y, y) + (z - y)$. It can be verified that $\beta^I(z) - \beta^I(y) = (z - y)[\frac{2}{3} q^I_w(y, y) - 2q^I_w(y, y)] + o(z - y)$, which is strictly positive under Assumption 3. The proof of monotonicity is now complete.

3. Verification: Incentive compatibility

We will show that the expected payoff of a type-$s$ bidder is unimodal in the reported type $z$ with maximum achieved at $z = s$. Plugging the bidding function $\beta^I(z)$ into the expected payoff, we obtain
\[
\frac{\partial}{\partial z} \Pi^I(z|s) = \pi^I_w(z, s, z) - \beta^I(z) - \pi^I_l(s, z)
\]
\[
+ \int_0^z \frac{\partial}{\partial v} \pi^I_w(v, s, y) |_{u=z} dv - \int_0^z (\beta^I)'(z) dy
\]
\[
= (\pi^I_w(z, s, z) - \pi^I_l(s, s)) - (\pi^I_w(z, z, z) - \pi^I_l(z, z)).
\]

Recall that
\[
\pi^I_w(z, s, z) - \pi^I_l(s, s) = q^I_w(z, s)[\theta_0 + s + z - q^I_w(z, s) - q^I_l(z, z) - c(q^I_w(z, s) - C)^+]
\]
\[
- q^I_l(s, z)[\theta_0 + s + z - q^I_w(z, z) - q^I_l(s, z) - c], \tag{A.12}
\]
and
\[
q^I_l(s, z) = q^I_l(z, z) + \frac{1}{2}(s - z).
\]

In the following, we will divide the problem into cases to prove that the differentiation is positive when $z < s$ and negative if $z > s$.

Case 1: $q^I_w(z, z) < C$ and $q^I_w(z, z) < C$
In this case, the capacity constraint is not binding. Thus

\[ q^I_w(z, s) = q^I_w(z, z) + \frac{1}{2}(s - z). \]

Plugging it into equation (A.12), we obtain

\[ \frac{\partial}{\partial z} \Pi^I(z|s) = \frac{1}{2}(s - z) \left( q^I_w(z, z) - q^I(\frac{1}{2}(s - z)) + c \right), \]

where the multiplicative term \( q^I_w(z, z) - q^I(\frac{1}{2}(s - z)) + c \) is positive by Assumption 3.

**Case 2:** \( q^I_w(z, s) = q^I_w(z, z) = C \)

In this case, there is no difference between \( q^I_w(z, s) \) and \( q^I_w(z, z) \). After some algebra, the differentiation becomes

\[ \frac{\partial}{\partial z} \Pi^I(z|s) = \frac{1}{2}(s - z) \left( C - \frac{1}{2}(q^I(\frac{1}{2}(s - z)) + q^I(s, z)) \right). \]

Now the multiplicative term is

\[ 2[q^I_w(z, z) - q^I(\frac{1}{2}(s - z)) - \frac{1}{4}(s - z)] \geq 2 \left( q^I_w(z, z) - q^I(\frac{1}{2}(s - z)) - \frac{1}{4}(1 - z) \right), \]

and hence the differentiation is positive iff \( z < s \).

**Case 3:** General case.

We first rewrite the expression for \( \pi^I_w(z, s, z) - \pi^I_w(s, z) \). After some algebra, we obtain that if \( S^I_1 \leq z \leq S^I_2 \), \( \pi^I_w(z, s, z) = q^I_w(z, s) \left( q^I_w(z, s) + \frac{1}{2}z \right) \); otherwise, \( \pi^I_w(z, s, z) = q^I_w(z, s) \left( q^I_w(z, s) + \frac{3}{4}z \right) \). Thus

\[ \frac{\partial}{\partial z} \Pi^I(z|s) = \begin{cases} (s - z) \left( q^I_w(z, z) - q^I(\frac{1}{2}(s - z)) + (\frac{1}{2}\rho(1 - \frac{1}{2}\rho) - \frac{1}{4})(s - z) + \frac{3}{8}\rho z \right) & \text{if } z \leq S^I_1, \\ (s - z) \left( q^I_w(z, z) - q^I(\frac{1}{2}(s - z)) + (\frac{1}{2}\rho(1 - \frac{1}{2}\rho) - \frac{1}{4})(s - z) + \frac{1}{8}\rho z \right) & \text{otherwise}. \end{cases} \]

Let us consider the term \( (\frac{1}{2}\rho(1 - \frac{1}{2}\rho) - \frac{1}{4})(s - z) + \frac{1}{8}\rho z \). When \( s - z < 0 \), \( (\frac{1}{2}(1 - \frac{1}{2}\rho) - \frac{1}{4})(s - z) > 0 \) since \( \frac{1}{2}(1 - \frac{1}{2}\rho) \leq \frac{1}{4} \) for \( \rho \in [0, 1] \). With the last term being positive, we obtain by Assumption 3 that \( \frac{\partial}{\partial z} \Pi^I(z|s) \) is negative for \( z > s \).

It remains to show that this holds for \( z < s \) as well. When \( s - z > 0 \), both \( (\frac{1}{2}\rho(1 - \frac{1}{2}\rho) - \frac{1}{4})(s - z) \) and \( \frac{1}{8}\rho z \) are minimized at \( \rho = 0 \), which corresponds to Case 2. Thus, the multiplicative term is bounded below by Case 2 and hence is positive for all cases.

**Remark:**

From the proofs of Theorems 1 and 2, the monotonicity of bidding functions is relatively hard to guarantee when the capacity constraint is binding in both auctions, i.e., Case III for the first-price auction and region IV for the second-price auction. In these regions, as the signal \( s \) increases,
$\pi^I_w(s, s)$ and $\pi^{II}_w(s, s, s)$ are restricted by the capacity constraint, while $\pi^I_w(s, s, s)$ and $\pi^{II}(s, s)$ keep growing without any constraint. This phenomenon makes it harder to ensure that bidding functions are monotonic.

**Proof of Theorem 3**

To prove Theorem 3, it suffices to show the following monotonicity result:

Suppose that $q^I_w(s_w) + q^I_l(s_w, s_l) \geq q^{II}_w(s_w, s_l) + q^{II}_l(s_l)$. Then for all $y \geq s_w$,

$$q^I_w(y) + q^I_l(y, s_l) \geq q^{II}_w(y, s_l) + q^{II}_l(s_l), \quad \forall y \geq s_w, \quad \text{(A.13)}$$

and

$$q^I_w(s_w) + q^I_l(s_w, z) \geq q^{II}_w(s_w, z) + q^{II}_l(s_l), \quad \forall z \leq s_l. \quad \text{(A.14)}$$

Given that $s_l$ is fixed and $y \geq s_w$ in inequality (A.13), let $\rho^I_w, \rho^{II}_w$ denote the coefficients of winner’s signals in the first-price and second-price auctions respectively. $\rho^I_w, \rho^{II}_w$ capture the marginal change of the total quantities resulting from one unit change in $s_w$. The monotonicity is equivalent to the fact $\rho^I_w \geq \rho^{II}_w$, and hence in the sequel we will verify it. According to Lemma 3, we know that $\frac{1}{2} \leq \rho^I_w \leq \frac{3}{4}$. Since $q^{II}_l(s_l)$ is independent of the winner’s signal, the bounds of $\rho^{II}_w$ are $0 \leq \rho^{II}_w \leq \frac{1}{2}$. Therefore, $\rho^I_w \geq \rho^{II}_w$ in all cases of inequality (A.13).

Now we focus on inequality (A.14). Given that $s_w$ is fixed and $s_l \geq z$, we compare the coefficients of loser’s signals between these two auctions (similarly to the treatment for inequality (A.13)). Let $\rho^I_l, \rho^{II}_l$ denote these coefficients. It suffices to show that $\rho^I_l \leq \rho^{II}_l$ in all cases. First we observe from Proposition 3 that $\rho^I_l = \frac{1}{2}$, $\forall s_w \in [0, 1]$. Combining Proposition 1 and Lemma 3, we obtain

$$0 + \frac{1}{2} \leq \rho^{II}_l \leq \frac{1}{2} + \frac{3}{4},$$

which implies $\frac{1}{2} \leq \rho^{II}_l \leq \frac{5}{4}$. Hence, $\rho^I_l \leq \rho^{II}_l$. In fact it can be shown that $\rho^{II}_l$ has tighter bounds, i.e., $\frac{1}{2} \leq \rho^I_l \leq \frac{3}{4}$. $\square$