Legendre wavelets direct method for variational problems

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Abstract

A direct method for solving variational problems using Legendre wavelets is presented. An operational matrix of integration is first introduced and is utilized to reduce a variational problem to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique. © 2000 IMACS. Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Orthogonal functions (OFs) and polynomial series have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. The approach is based on converting the underlying differential equations into an integral equation through integration, approximating various signals involved in the equation by truncated orthogonal series $\Phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_{n-1}(t)]^T$ and using the operational matrix of integration $P$, to eliminate the integral operations. The elements $\phi_0(t), \phi_1(t), \ldots, \phi_{n-1}(t)$ are the basis functions, orthogonal on a certain interval $[a, b]$. The matrix $P$ can be uniquely determined based on the particular OFs. Special attention has been given to applications of Walsh functions \[1\], block pulse functions \[2\], Laguerre polynomials \[3\], Legendre polynomials \[4\], Chebyshev polynomials \[5\] and Fourier series \[6\]. All previously mentioned OFs are supported on the whole interval $a \leq t \leq b$. This kind of global support is evidently a drawback for certain analysis work, especially systems involving abrupt variations or local functions vanishing outside a short interval of time or space \[7\].

The direct method of Ritz and Galerkin in solving variational problems has been of considerable concern and is well covered in many textbooks \[8,9\]. Chen and Hsiao \[1\] introduced the Walsh series...
method to variational problems. Due to the nature of the Walsh functions, the solution obtained were piecewise constant. References [3–5] used Laguerre polynomials, Legendre polynomials and Chebyshev series approaches, respectively, to derive continuous solutions for the first example in [1]. Furthermore, Razzaghi and Razzaghi [6,10] applied the Fourier series and Taylor series, respectively, to derive a continuous solution for the second example in [1] which is an application to the heat conduction problem. In [10], it is shown that to obtain the Taylor series coefficient, an ill-conditioned matrix commonly known as Hilbert matrix is used. Hence, the Taylor series is not suitable for the solution of the second example in [1].

In the present paper, we introduce a new direct computational method to solve variational problems. The method consists of reducing the variational problem into a set of algebraic equations by first expanding the candidate function as Legendre wavelets with unknown coefficients. The Legendre wavelets are introduced. The operational matrix of integration and integration of the product of two Legendre wavelet vectors are given. These matrices are then used to evaluate the coefficients of the Legendre wavelets in such a way that the necessary conditions for extremization is imposed. Here, we will demonstrate the results by considering the illustrative examples discussed in [3] and the second example in [1]. It is shown that while the same results are obtained in the first example, the Legendre wavelets approach produces an exact solution for the heat conduction problem.

2. Properties of Legendre wavelets

2.1. Wavelets and Legendre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter \( a \) and the translation parameter \( b \) vary continuously we have the following family of continuous wavelets as [11]

\[
\psi_{a,b}(t) = |a|^{-1/2} \psi \left( \frac{t-b}{a} \right), \quad a, b \in \mathbb{R}, \quad a \neq 0.
\]

If we restrict the parameters \( a \) and \( b \) to discrete values as \( a = a_0^{-k}, b = nb_0a_0^{-k} \), \( a_0 > 1, b_0 > 0 \) and \( n \), and \( k \) positive integers, we have the following family of discrete wavelets:

\[
\psi_{k,n}(t) = |a_0|^{k/2} \psi \left( a_0^{-k}t - nb_0 \right),
\]

where \( \psi_{k,n}(t) \) forms a wavelet basis for \( L^2(\mathbb{R}) \). In particular, when \( a_0 = 2 \) and \( b_0 = 1 \), then \( \psi_{k,n}(t) \) forms an orthonormal basis [11].

Legendre wavelets \( \psi_{n,m}(t) = \psi(k, \hat{n}, m, t) \) have four arguments; \( \hat{n} = 2n - 1, n = 1, 2, 3, \ldots, 2^{k-1}, \) \( k \) can assume any positive integer, \( m \) is the order for Legendre polynomials and \( t \) is the normalized time. They are defined on the interval \([0, 1]\) as

\[
\psi_{n,m}(t) = \begin{cases} 
\sqrt{m + \frac{1}{2}} 2^{k/2} P_m(2^k t - \hat{n}), & \text{for } \frac{\hat{n} - 1}{2^k} \leq t \leq \frac{\hat{n} + 1}{2^k}, \\
0, & \text{otherwise}
\end{cases}
\]

(1)

where \( m = 0, 1, \ldots, M-1, n = 1, 2, 3, \ldots, 2^{k-1} \). In Eq. (1), the coefficient \( \sqrt{m + \frac{1}{2}} \) is for orthonormality, the dilation parameter is \( a = 2^{-k} \) and translation parameter is \( b = \hat{n}2^{-k} \). Here, \( P_m(t) \) are the well-known
Legendre polynomials of order \( m \) which are orthogonal with respect to the weight function \( w(t) = 1 \) on the interval \([-1, 1]\), and satisfy the following recursive formula [12]:

\[
P_0(t) = 1, \quad P_1(t) = t, \quad P_{m+1}(t) = \left( \frac{2m+1}{m+1} \right) tP_m(t) - \left( \frac{m}{m+1} \right) P_{m-1}(t), \quad m = 1, 2, 3, \ldots
\]

### 2.2. Function approximation

A function \( f(t) \) defined over \([0, 1)\) may be expanded as

\[
f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t),
\]

where \( c_{n,m} = (f(t), \psi_{n,m}(t)) \) in which \((, )\) denotes the inner product.

If the infinite series in Eq. (2) is truncated, then Eq. (2) can be written as

\[
f(t) \simeq \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t),
\]

where \( C \) and \( \Psi(t) \) are \( 2^{k-1}M \times 1 \) matrices given by

\[
C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}]^T
\]

\[
\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \psi_{20}(t), \ldots, \psi_{2M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M-1}(t)]^T.
\]

The integration of the vector \( \Psi(t) \) defined in Eq. (4) can be obtained as

\[
\int_0^t \Psi(t') \, dt' = P \Psi(t), \tag{5}
\]

where \( P \) is the \((2^{k-1}M)\times(2^{k-1}M)\) operational matrix for integration and is given in [13] as

\[
P = \frac{1}{2^k} L F F \ldots F
\]

\[
\begin{bmatrix}
L & F & F & \ldots & F \\
O & L & F & \ldots & F \\
\vdots & O & \ddots & \ddots & \vdots \\
O & O & \cdots & O & L
\end{bmatrix}
\]

In Eq. (6) \( F \) and \( L \) are \( M \times M \) matrices given by

\[
F = \begin{bmatrix}
2 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix},
\]

\[
L = \begin{bmatrix}
& \ddots & \ddots & \cdots & \ddots \\
\vdots & \ddots & \ddots & \cdots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \\
0 & 0 & \cdots & 0 & \\
0 & 0 & \cdots & 0 & \\
\end{bmatrix}
\]

\[
F = \begin{bmatrix}
2 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
& \ddots & \ddots & \cdots & \ddots \\
\vdots & \ddots & \ddots & \cdots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \\
0 & 0 & \cdots & 0 & \\
0 & 0 & \cdots & 0 & \\
\end{bmatrix}
\]
The integration of the product of two Legendre wavelet function vectors is obtained as

\[ I = \int_0^1 \Psi(t)\Psi^T(t) \, dt \]  

(7)

where \( I \) is an identity matrix.

3. Legendre wavelets direct method

Consider the problem of finding the extremum of the functional

\[ J(x) = \int_0^1 F[t, x(t), \dot{x}(t)] \, dt. \]  

(8)

The necessary condition for \( x(t) \) to extremize \( J(x) \) is that it should satisfy the Euler–Lagrange equation

\[ \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}} \right) = 0 \]

with appropriate boundary conditions. However, the above differential equation can be integrated easily only for simple cases. Thus, numerical and direct methods such as the well-known Ritz and Galerkin methods have been developed to solve variational problems. Here we consider a Ritz direct method for solving Eq. (7) using the Legendre wavelets.

Suppose the rate variable \( \dot{x}(t) \) can be expressed as

\[ \dot{x}(t) = C^T \Psi(t). \]  

(9)

Using Eq. (5), \( x(t) \) can be represented as

\[ x(t) = \int_0^t \dot{x}(t') \, dt' + x(0) \]

\[ = C^T P \Psi(t) + [x(0), 0, \ldots, 0, x(0), 0, \ldots, 0, \ldots, x(0), 0, \ldots, 0]^T \Psi(t). \]  

(10)
We can also express $t$ in terms of $\Psi(t)$ as

$$
t = 2^{1-3k/2} \left[ 1, \frac{1}{\sqrt{3}}, 0, \ldots, 0 | 3, \frac{1}{\sqrt{3}}, \ldots, 0, \ldots, 0 | 2^k - 1, \frac{1}{\sqrt{3}}, 0, \ldots, 0 \right]^T \Psi(t)
$$

$$= d^T \Psi(t). \quad (11)$$

Substituting Eqs. (9)–(11) in Eq. (8), the functional $J(x)$ becomes a function of $c_{n,m}$, where $n = 1, 2, \ldots, 2^{k-1}$, and $m = 0, 1, 2, \ldots, M - 1$. Hence, to find the extremum of $J(x)$, we solve

$$\frac{\partial J}{\partial c_{n,m}} = 0, \quad n = 1, 2, \ldots, 2^{k-1}, \quad m = 0, 1, \ldots, M - 1. \quad (12)$$

The above procedure is now used to solve the following variational problems.

4. Illustrative examples

4.1. Example 1

Consider the problem of finding the minimum of the functional [3]

$$J(x) = \int_0^1 [\dot{x}^2 + t \dot{x} + x^2] \, dt \quad (13)$$

with boundary conditions

$$x(0) = 0, \quad x(1) = \frac{1}{4} \quad (14)$$

Using Eqs. (9)–(11) and (13), we get

$$J(x) = \int_0^1 [C^T \Psi(t) \Psi^T(t) C + C^T \Psi(t) \Psi^T(t) d + C^T P \Psi(t) \Psi^T(t) P^T C] \, dt$$

using Eq. (6), we obtain

$$J(x) = C^T C + C^T d + C^T P P^T C. \quad (15)$$

Eq. (9) and the boundary conditions in Eq. (14) imply

$$x(1) = C^T \int_0^1 \Psi(t) \, dt = \frac{1}{4}.$$ 

Let

$$v = \int_0^1 \Psi(t) \, dt,$$

then we have

$$C^T v = \frac{1}{4}. \quad (16)$$

We now minimize Eq. (15) subject to Eq. (16) using the Lagrange multiplier technique. Suppose

$$\tilde{J}(x) = J(x) + \lambda \left( C^T v - \frac{1}{4} \right)$$
Table 1

<table>
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<th>Exact</th>
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</thead>
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<td>0.000000</td>
</tr>
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</tr>
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</tr>
<tr>
<td>1</td>
<td>0.249999</td>
<td>0.250000</td>
</tr>
</tbody>
</table>

where $\lambda$ is the Lagrange multiplier. Using Eq. (12), we solve
\[
\frac{\partial \hat{J}}{\partial C} = 0, \quad \frac{\partial \hat{J}}{\partial \lambda} = 0,
\]
or
\[
2C + d + 2PP^T C + \lambda v = 0, \quad C^T v = \frac{1}{4}.
\]
(17)

By choosing $M = 3$ and $k = 3$, Eq. (17) is solved from which the coefficient vector $C$ and the Lagrange multiplier $\lambda$ can be found. Table 1 gives the approximate values of $x(t)$ using the present approach together with the exact solutions.

4.2. Example 2. Application to the heat conduction problem

Consider the extremization of
\[
J = \int_0^1 \left( \frac{1}{2} \dot{x}^2 - xg(t) \right) dt = \int_0^1 F(t, x, \dot{x}) dt,
\]
where $g(t)$ is a known function satisfying
\[
\int_0^1 g(t) dt = -1,
\]
with the boundary conditions
\[
\dot{x}(0) = 0, \quad \dot{x}(1) = 0.
\]
(19)

Schechter [14] gave a physical interpretation for this problem by noting an application in heat conduction and Chen and Hsiao [1] considered the case where $g(t)$ is given by
\[
g(t) = \begin{cases} 
-1, & 0 \leq t \leq \frac{1}{4}, \quad \frac{1}{2} \leq t \leq 1 \\
3, & \frac{1}{4} \leq t \leq \frac{1}{2}
\end{cases}
\]
(20)
and gave an approximate solution using Walsh functions. The exact solution is

\[
x(t) = \begin{cases} 
\frac{1}{2}t^2, & 0 \leq t \leq \frac{1}{4} \\
-\frac{3}{2}t^2 + t - \frac{1}{8}, & \frac{1}{4} \leq t \leq \frac{1}{2} \\
\frac{1}{2}t^2 - t + \frac{3}{8}, & \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

Here, we solve the same problem using Legendre wavelets. First we assume

\[
\dot{x}(t) = C^T \Psi(t). 
\]

(21)

In view of Eq. (20), we write Eq. (18) as

\[
J = \frac{1}{2} \int_0^1 \dot{x}(t) \, dt + 4 \int_0^{1/4} x(t) \, dt - 4 \int_0^{1/2} x(t) \, dt + \int_0^1 x(t) \, dt
\]

or

\[
J = \frac{1}{2} \int_0^1 C^T \Psi(t) \Psi(t) C \, dt + 4 C^T P \int_0^{1/4} \Psi(t) \, dt - 4 C^T P \int_0^{1/2} \Psi(t) \, dt - C^T P \int_0^1 \Psi(t) \, dt.
\]

Let

\[
w(t) = \int_t^1 \Psi(t') \, dt',
\]

then, using Eq. (6) we have

\[
J = \frac{1}{2} C^T C + C^T P [4w(\frac{1}{4}) - 4w(\frac{1}{2}) + w(1)].
\]

(22)

The boundary conditions in Eq. (19) can be expressed in terms of Legendre wavelets as

\[
C^T \Psi(0) = 0, \quad C^T \Psi(1) = 0.
\]

(23)

We now minimize Eq. (22) subject to Eq. (23) using the Lagrange multiplier technique. Suppose

\[
J^* = J + \lambda_1 C^T \Psi(0) + \lambda_2 C^T \Psi(1),
\]

where \(\lambda_1\) and \(\lambda_2\) are the two multipliers. Using Eq. (12), we obtain

\[
\frac{\partial J^*}{\partial C} = C + P \left[4w(\frac{1}{4}) - 4w(\frac{1}{2}) + w(1)\right] + \lambda_1 \Psi(0) + \lambda_2 \Psi(1) = 0.
\]

(24)

By choosing \(M = 3\) and \(k = 3\), Eqs. (23) and (24) define a set of 14 simultaneous linear algebraic equations from which the coefficient vector \(C\) and the multipliers \(\lambda_1\) and \(\lambda_2\) can be found. In Eq. (24), \(P\) is the \(12 \times 12\) operational matrix of integration given in Eq. (6), and

\[
4w(\frac{1}{4}) - 4w(\frac{1}{2}) + w(1) = [\frac{1}{16}, 0, 0, -\frac{3}{32}, 0, 0, \frac{1}{16}, 0, 0, 1, 0, 0]^T.
\]

Using Eq. (21), we get

\[
\dot{x}(t) = \begin{cases} 
\frac{1}{16} \psi_{10} + \frac{1}{48} \sqrt{3} \psi_{11}, & 0 \leq t \leq \frac{1}{4} \\
-\frac{1}{16} \psi_{20} - \frac{1}{16} \sqrt{3} \psi_{21}, & \frac{1}{4} \leq t \leq \frac{1}{2} \\
-\frac{3}{16} \psi_{30} + \frac{1}{48} \sqrt{3} \psi_{31}, & \frac{1}{2} \leq t \leq \frac{3}{4} \\
-\frac{1}{16} \psi_{40} + \frac{1}{48} \sqrt{3} \psi_{41}, & \frac{3}{4} \leq t \leq 1.
\end{cases}
\]
where

\[
\begin{align*}
\psi_{10} &= 2, & \psi_{11} &= \sqrt{12}(8t - 1), & 0 \leq t \leq \frac{1}{4} \\
\psi_{20} &= 2, & \psi_{21} &= \sqrt{12}(8t - 3), & \frac{1}{4} \leq t \leq \frac{1}{2} \\
\psi_{30} &= 2, & \psi_{31} &= \sqrt{12}(8t - 5), & \frac{1}{2} \leq t \leq \frac{3}{4} \\
\psi_{40} &= 2, & \psi_{41} &= \sqrt{12}(8t - 7), & \frac{3}{4} \leq t \leq 1.
\end{align*}
\]

hence we get

\[
x(t) = \begin{cases} 
\frac{1}{2}t^2, & 0 \leq t \leq \frac{1}{4} \\
-\frac{3}{2}t^2 - t - \frac{1}{8}, & \frac{1}{4} \leq t \leq \frac{1}{2} \\
\frac{1}{2}t^2 - t + \frac{3}{8}, & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

which is the exact solution.

5. Conclusion

The Legendre wavelets operational matrix \( P \), together with the integration of the product of two Legendre wavelets vectors, are used to solve the variational problems. The present method reduces a variational problem into a set of algebraic equations. The integration of the product of two Legendre wavelet function vectors is an identity matrix, hence making Legendre wavelets computationally very attractive. It is also shown that the Legendre wavelets provide an exact solution for the heat conduction problem in Section 4.2.

References