Embeddings of Graphs with No Short Noncontractible Cycles

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We investigate embeddings of graphs on orientable 2-dimensional surfaces such that all face boundaries have fewer edges than every noncontractible cycle. We show that such embeddings are always minimum genus embeddings and that they share many properties with planar embeddings. For example, if the graph is 3-connected, then the embedding is unique. We use this to obtain a polynomially bounded algorithm for describing a minimum genus embedding with no short noncontractible cycles if such an embedding of the graph exists. We refine some of these results for triangulations.

1. INTRODUCTION

A fundamental graph result of Whitney [15] says that a 3-connected planar graph has a unique embedding on the sphere. This follows from his 2-switching theorem for planar embeddings of 2-connected graphs. Tutte [12] obtained Whitney's uniqueness theorem from a combinatorial description of the facial cycles in a 3-connected planar graphs: they are precisely the induced (i.e., chordless) nonseparating cycles. As a consequence, every automorphism of a 3-connected graph on the sphere can be extended to a homeomorphism of the sphere.

The difficulty in generalizing these results to higher surfaces can be illustrated by $K_7$ on the torus: Every $K_3$ in $K_7$ is a facial cycle in some embedding of $K_7$ on the torus and a nonfacial, noncontractible cycle in another embedding. Moreover, Lavrenchenko [6] proved that there are infinitely many 5-connected graphs on the torus which are not uniquely embeddable. Uniqueness results for embeddings have been established only

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for very special classes of graphs. Negami [7] proved that all 6-connected graphs (with three exceptions) on the torus are uniquely embeddable. (This class of graphs is very restricted as shown by Altschuler [1]). Negami [8] also showed that every 5-connected triangulation (other than \( K_6 \)) of the projective plane is uniquely embeddable.

In order to study automorphism properties of graphs on orientable surfaces Hutchinson [5] introduced embeddings which we shall here call large-edge-width embeddings (abbreviated LEW-embeddings). An LEW-embedding of a graph is an embedding such that every noncontractible cycle has more edges than every face boundary. Hutchinson asked if every 3-connected graph which has an LEW-embedding also has a Whitney embedding, i.e., an embedding such that every automorphism of the graph takes face boundaries to face boundaries.

In this paper we show that LEW-embeddings are always minimum genus embeddings and that they share many important properties with planar embeddings. First we establish some basic properties of embeddings in general using a purely combinatorial description. Then we observe that all face boundaries in an LEW-embedding of a 2-connected graph are cycles and we extend Tutte's characterization of the facial cycles: If a 3-connected graph \( G \) has an LEW-embedding, then there exists a natural number \( m \) such that the facial cycles are precisely the induced nonseparating cycles of length \( \leq m \). We generalize another result of Tutte by showing that, for every contractible cycle in an LEW-embedding of a 3-connected graph, the overlap graph (to be defined later) is connected and bipartite. We also generalize Whitney’s 2-switching theorem for planar graphs by showing that, if a 2-connected graph has an LEW-embedding, then all other minimum genus embeddings (also those that are not LEW-embeddings) can be obtained from the LEW-embedding by sequences of 2-switchings. In particular, every 3-connected graph which has an LEW-embedding is uniquely embeddable. The LEW-embedding is a Whitney embedding which answers Hutchinson’s question in the affirmative. The uniqueness result also answers the question of Lavrenchenko [6] if every triangulation with no noncontractible short cycle is uniquely embeddable. In fact, we give an affirmative answer for every triangulation on every orientable surface provided there are no 3-cycles which are both noncontractible and nonseparating. We also present a simple combinational description of such triangulations thus providing a partial solution to the problem of Ringel [9] of characterizing the graphs which triangulate some surface.

We apply these structural results to describe embedding algorithms. We present a general polynomial algorithm for finding a shortest cycle of a given type, for example a shortest noncontractible cycle. This implies a polynomial algorithm for deciding if a given embedding of a graph is an LEW-embedding. Then we obtain a polynomial algorithm which
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describes an LEW-embedding if the graph under consideration has an LEW-embedding. In particular, this is a polynomially bounded algorithm for determining the genus of a large class of graphs of unbounded genus. The problem of finding the genus of a general graph is NP-complete [11].

Another approach to studying minimum genus embeddings was suggested by Vitrany (see [13]). He asked if a graph of genus \( g \) embedded in a surface of genus \( \geq g \) must have representativity (in this paper called facewidth) \( \leq 2 \). While this is true for 3-connected planar graphs we show that it is false for 3-connected toroidal graphs.

2. NOTATION AND TERMINOLOGY

We use the same terminology as in [10]. We repeat some of the definitions here. A graph has no loops or multiple edges. The vertex set and edge set of a graph \( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. If \( A \subseteq V(G) \cup E(G) \), then \( G - A \) is obtained from \( G \) by deleting all vertices of \( A \) and all edges which are in \( A \) or which are incident with a vertex of \( A \). If \( G \) is connected and \( G - A \) is disconnected we say that \( A \) is separating and that \( A \) separates \( G \). If \( A \subseteq V(G) \), then \( G(A) = G - (V(G) \setminus A) \). If \( v \) is a vertex and \( A \) is the set of neighbours of \( v \), then we put \( N(v, G) = G(A) \) and \( N(v, G) = G(A \cup \{ v \}) \). A walk in \( G \) is a sequence \( u_1 e_1 u_2 e_2 \cdots u_p \) (which we also denote by \( u_1 u_2 \cdots u_p \)) where \( u_1, u_2, \ldots, u_p \) are vertices in \( G \) and \( e_i = u_i u_{i+1} \) is an edge of \( G \) for \( i = 1, 2, \ldots, p - 1 \). If \( v_1 = v_p \), then the walk is closed. In that case we regard the walks \( u_1 u_2 \cdots u_{p-1} v_1 \) and \( v_2 v_3 \cdots v_2 \), etc., to be the same walk. If a walk has no repetition of vertices, then it is a path. If \( v_p = v_1 \) and there is no other repetition of vertices, then the walk is a cycle.

It is convenient to think of embeddings topologically but we shall here treat them purely combinatorially in order to avoid intuitively clear but tricky topological arguments. The embeddings considered in this paper are those which in the literature (see, e.g. [3, 13]) are called 2-cell embeddings on orientable 2-dimensional compact surfaces. Thus an embedding of a connected graph \( G \) with vertex set \( \{ v_1, v_2, \ldots, v_n \} \) is a collection \( \Pi = \{ \pi_1, \pi_2, \ldots, \pi_n \} \) such that \( \pi_i \) is a cyclic permutation of the edges incident with \( v_i \) for \( i = 1, 2, \ldots, n \). We also refer to this embedding as the \( \Pi \)-embedding, and we say that \( G \) is a \( \Pi \)-embedded (or just embedded) graph. If \( e \) is an edge incident with \( v_i \), then the cyclic sequence \( e, \pi_i(e), \pi_i^2(e), \ldots \) is called the \( \Pi \)-clockwise ordering around \( v_i \). We say that the embedding \( \Pi' = \{ \pi'_1, \pi'_2, \ldots, \pi'_n \} \) is the same as \( \Pi \) if \( \pi'_i = \pi_i \) for \( i = 1, 2, \ldots, n \) or \( \pi'_i = \pi_i^{-1} \) for \( i = 1, 2, \ldots, n \).

If \( e = v_i v_j \), then we consider the closed walk \( v_i e v_j e' v_k e'' \cdots v_i \) where \( \pi_j(e) = e' = v_j v_k, \pi_k(e') = e'' \), etc. This is called the \( \Pi \)-facial (or just facial)
walk containing the edge \( e \) in the direction from \( v_i \) to \( v_j \). \( G \) also has a facial walk containing \( e \) in the direction from \( v_j \) to \( v_i \). These two facial walks containing \( e \) may or may not coincide. In order to describe an embedding \( \Pi \) it is sufficient to describe the \( \Pi \)-facial walks. It is even sufficient to describe the \( \Pi \)-facial walks without the orientation. Using the connectedness of \( G \) it is easy to see that the orientations can be assigned in only two ways. By definition, the two resulting embeddings are the same. A \( \Pi \)-facial cycle is a \( \Pi \)-facial walk which is a cycle.

If \( n = |V(G)|, q = |E(G)| \) and \( f \) is the number of \( \Pi \)-facial walks (which we shall also refer to as \( \Pi \)-faces), then we define the \( \Pi \)-genus \( g \) of \( G \) by Euler's formula
\[
n - q + f = 2 - 2g.
\]
The genus \( g(G) \) of \( G \) is defined as the minimum \( \Pi \)-genus taken over all embeddings of \( G \).

3. Basic Properties of Embeddings

If \( \Pi \) is an embedding of a connected graph \( G \) and \( H \) is a connected subgraph of \( G \), then the induced embedding of \( H \) (which we also refer to as \( \Pi \)) is obtained from that of \( G \) by ignoring all edges in \( E(G) \setminus E(H) \). More precisely, if \( e \) is an edge of \( H \) incident with the vertex \( v_i \), then the successor of \( e \) in the clockwise ordering around \( v_i \) in \( H \) is the first edge in the sequence \( \pi_i(e), \pi_i^2(e), \ldots \) which is in \( H \). If \( e \in E(G) \) and \( G - e \) is connected, then \( G - e \) has either one more or one less facial walk than \( G \) has. Hence the \( \Pi \)-genus of \( G - e \) is either equal to or is one less than the \( \Pi \)-genus of \( G \). The two \( \Pi \)-genera are equal if and only if \( e \) partitions a facial walk of \( G - e \) into two. In this case we shall use the expression that \( e \) is added to a face of \( G - e \) or that \( e \) is in a face if \( G - e \). If \( v \) has degree 1 in \( G \), then \( G \) and \( G - v \) have the same \( \Pi \)-genus. Since every connected subgraph of \( G \) can be obtained from \( G \) by successively deleting edges or vertices of degree 1, it follows that the \( \Pi \)-genus of every connected subgraph of \( G \) is smaller than or equal to the \( \Pi \)-genus of \( G \). Since a subgraph with only one edge has genus 0 it follows that every embedding has genus \( \geq 0 \). If the \( \Pi \)-genus of \( G \) is zero, then \( G \) can be obtained from a graph with one edge by successively adding vertices of degree one and edges in faces. This can also be done in the Euclidean plane such that a new edge is added as a polygonal arc in the face bounded by the appropriate facial walk and such that no crossings of arcs occur. Hence we have:

Proposition 3.1. If the \( \Pi \)-genus of a connected graph \( G \) is zero, then \( G \) can be drawn in the plane such that the edges are polygonal arcs no two of which cross and such that the \( \Pi \)-clockwise ordering is the same as the geometric clockwise ordering for every vertex.
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If \( G \) is a \( \Pi \)-embedded graph and \( C \) is a cycle of \( G \), then we define the left graph and right graph of \( C \) as follows: If \( e \) is an edge of \( C \) followed by \( e' = \pi_1(e) \), then all edges \( \pi_1(e), \pi_2(e), \ldots, \pi_{k-1}(e) \) are said to be on the left side of \( C \). An edge \( e'' \) which is not incident with \( C \) and which is connected by a path in \( G - C \) to an end of an edge of the left side of \( C \) is also said to be on the left side. Now the left graph of \( C \) denoted by \( G_L(C, \Pi, G) \) (or just \( G_L(C, \Pi) \), or \( G_L(C) \)) is defined as the edges on the left side of \( C \) together with all their ends. The right graph \( G_R(C, \Pi, G) \) is defined analogously.

**Proposition 3.2.** If \( C \) is a cycle in a \( \Pi \)-embedded graph \( G \) such that \( G_L(C) \) and \( G_R(C) \) have no edge in common, then the \( \Pi \)-genus of \( G \) equals the sum of the \( \Pi \)-genera of \( G_L(C) \cup C \) and \( G_R(C) \cup C \).

**Proof.** All \( \Pi \)-facial walks of \( G \) are either in \( G_L(C) \cup C \) or \( G_R(C) \cup C \). In addition, \( C \) is a \( \Pi \)-facial cycle in both \( G_L(C) \cup C \) and \( G_R(C) \cup C \). Now an easy count proves the proposition.

If one of \( G_L(C) \cup C, G_R(C) \cup C \) (say the former) in Proposition 3.2 has \( \Pi \)-genus zero, then we say that \( C \) is \( \Pi \)-contractible and we call \( G_L(C) \) the \( \Pi \)-interior of \( C \) and denote it by \( \text{int}(C, \Pi, G) \) (or just \( \text{int}(C, \Pi) \) or \( \text{int}(C) \)). We put \( \text{Int}(C, \Pi, G) = \text{int}(C, \Pi, G) \cup C \). We define the \( \Pi \)-exterior \( \text{ext}(C) \) and \( \text{Ext}(C) \) analogously.

**Proposition 3.3.** If \( H \) is a connected subgraph of a \( \Pi \)-embedded graph \( G \) such that \( G \) and \( H \) have the same \( \Pi \)-genus and such that each \( \Pi \)-facial walk of \( H \) is a cycle, then each \( \Pi \)-facial cycle in \( H \) is \( \Pi \)-contractible in \( G \). In other words, \( G \) is obtained from \( H \) by adding planar subgraphs to facial cycles in \( H \).

**Proof.** Let \( C \) be a facial cycle in \( H \). If \( G_L(C) \) and \( G_R(C) \) have an edge in common, then there exists a path \( P \) (or cycle) in \( G \) starting with an edge \( e \) in \( G_L(C) \) and ending with an edge in \( G_R(C) \) such that no intermediate vertex of \( P \) is on \( C \). Let \( P' \) be the subpath of \( P \) starting with \( e \) and ending with a vertex in \( H \) such that no intermediate vertex of \( P' \) is in \( H \). Now \( H \cup P' \) has fewer \( \Pi \)-facial walks than \( H \) and hence \( H \cup P' \) has larger \( \Pi \)-genus than \( H \), contradicting the assumption that \( H \) and \( G \) (and hence also every connected subgraph of \( G \) containing \( H \)) have the same \( \Pi \)-genus. So we have proved that \( G_L(C) \) and \( G_R(C) \) are edge-disjoint.

By Proposition 3.2, the \( \Pi \)-genus of \( G \) equals the sum of the \( \Pi \)-genera of \( G_L(C) \cup C \) and \( G_R(C) \cup C \). Since the former contains \( H \) (because \( C \) is facial in \( H \)) we conclude that \( C \cup C \) has \( \Pi \)-genus zero and hence \( C \) is \( \Pi \)-contractible in \( G \).
Corollary 3.4. If \( H \) is a connected subgraph of the \( \Pi \)-embedded graph \( G \) such that all \( \Pi \)-facial walks of \( H \) are cycles, then a cycle \( C \) in \( H \) is \( \Pi \)-contractible in \( H \) if and only if it is \( \Pi \)-contractible in \( G \).

Proof. A cycle which is \( \Pi \)-contractible in \( G \) is clearly \( \Pi \)-contractible in every connected subgraph which contains the cycle. Suppose now that \( C \) is \( \Pi \)-contractible in \( H \). By Proposition 3.3, \( G \) is obtained from \( H \) by adding planar subgraphs in faces of \( H \). Hence \( \text{Int}(C) \) remains planar when we extend \( H \) to \( G \).

Proposition 3.5. Let \( P_1, P_2, P_3 \) be three internally disjoint paths from a vertex \( x \) to a vertex \( y \) in a \( \Pi \)-embedded graph \( G \). If two of the three cycles \( C_{i,j} = P_i \cup P_j \ (1 \leq i < j \leq 3) \) are contractible, then all three of the cycles \( C_{i,j} \) are contractible, and one of them contains the third path in its interior.

Proof. Suppose that \( C_{1,2} \) and \( C_{2,3} \) are contractible. If \( P_3 \subseteq \text{Int}(C_{1,2}) \) or \( P_1 \subseteq \text{Int}(C_{2,3}) \) we have finished. So assume that \( P_3 \subseteq \text{Ext}(C_{1,2}) \) and \( P_1 \subseteq \text{Ext}(C_{2,3}) \). Now choose the orientation of \( P_i \cup P_j \ (1 \leq i < j \leq 3) \) such that we first walk from \( x \) to \( y \) along \( P_i \) and then from \( y \) to \( x \) along \( P_j \). Without loss of generality we can assume that \( \text{Int}(C_{1,2}) = G_1(C_{1,2}) \). Since \( P_3 \subseteq \text{Ext}(C_{1,2}) = G_1(C_{1,2}) \) we have \( P_1 \subseteq G_1(C_{2,3}) \). Since \( P_1 \subseteq \text{Ext}(C_{2,3}) \), we have \( G_1(C_{2,3}) = \text{Ext}(C_{2,3}) \). Now it follows that \( G_1(C_{1,3}) = G_1(C_{1,2}) \cup G_1(C_{2,3}) \cup P_3 \). Since \( C_{1,2} \) and \( C_{2,3} \) are \( \Pi \)-contractible an easy count shows that \( G_1(C_{1,3}) \cup C_{1,3} \) has \( \Pi \)-genus zero. This proves the proposition.

We now define the edge-width \( \text{ew}(G, \Pi) \) of a \( \Pi \)-embedded graph \( G \) as the length of a shortest \( \Pi \)-noncontractible cycle. We define the face-width \( \text{fw}(G, \Pi) \) as the smallest \( k \) such that \( G \) has a \( \Pi \)-noncontractible cycle which is a union of \( k \) paths each of which belongs to a \( \Pi \)-facial walk. (Topologically, the face-width is the smallest \( k \) such that the surface on which \( G \) is embedded has a noncontractible curve intersecting the graph in at most \( k \) points). If \( G \) has no \( \Pi \)-noncontractible cycle, we put \( \text{ew}(G, \Pi) = \text{fw}(G, \Pi) = \infty \). Clearly, \( \text{fw}(G, \Pi) \leq \text{ew}(G, \Pi) \). We say that \( \Pi \) is a large-edge-width embedding (abbreviated LEW-embedding) if every facial walk has length less than \( \text{ew}(G, \Pi) \). (If an edge is traversed twice in a facial walk, it contributes 2 to the length).

A subwalk of a walk is defined in the obvious way. If \( W \) is a closed walk and \( W' \) is a shortest closed subwalk of \( W \) of length at least one, then either \( W \) is of the form \( uvu \) (where \( u \) and \( v \) are distinct vertices) or else \( W' \) is a cycle. If \( W \) is a \( \Pi \)-facial walk and \( W' \) is of the form \( uvu \), then \( v \) has degree 1. If \( W' \) is a cycle, then a \( \Pi \)-facial walk distinct from \( W \) may contain the converse of \( W' \) as a subwalk. This will, however, only occur in special cases as the next lemma shows.
LEMMA 3.6. If $C$ is a cycle of a $\Pi$-embedded graph $G$ such that both $C$ and its converse are subwalks of $\Pi$-facial walks, then $C$ has at most two vertices which in $G$ have degree $>2$.

Proof. Let $C: v_0v_1\cdots v_kv_0$ be a subwalk of a $\Pi$-facial walk and assume that also $v_mv_{m-1}\cdots v_0v_k\cdots v_m$ is a subwalk of a $\Pi$-facial walk. Then for each $i$, where $1\leq i \leq k$ and $i \neq m$, the edge $v_iv_{i+1}$ is both the successor and predecessor of $v_iv_{i-1}$ in the $\Pi$-clockwise ordering around $v_i$. Therefore $v_i$ has degree 2 in $G$.

4. LEW-EMBEDDINGS

Our first result in this section justifies the concept of LEW-embeddings.

THEOREM 4.1. If $\Pi$ is an LEW-embedding of the graph $G$, then the $\Pi$-genus of $G$ equals the genus of $G$.

Proof. The proof is by induction on $q = |E(G)|$. If the $\Pi$-genus $g$ of $G$ is zero, there is nothing to prove. So we can assume that $g > 0$ and hence $q > 1$. Let $W_1, W_2, \ldots, W_f$ be the $\Pi$-facial walks of $G$.

Suppose now (reductio ad absurdum) that $G$ has an embedding $\Pi'$ of genus $g' < g$ such that the $\Pi'$-facial walks are $W'_1, W'_2, \ldots, W'_{f'}$, where $f' > f$. For each $i$, where $1 \leq i \leq f'$, let $C_i$ be a shortest closed subwalk of $W'_i$ of length at least 1. If some $C_i, 1 \leq i \leq f'$, is of the form $vuw$, then $v$ has degree 1 in $G$. The $\Pi$-embedding of $G - v$ is an LEW-embedding of genus $g$ and the $\Pi'$-embedding of $G - v$ has genus $g' < g$, a contradiction to the induction hypothesis. So we can assume that each $C_i (1 \leq i \leq f')$ is a cycle.

Assume first that $G$ has a $\Pi$-facial cycle $C$ such that $C = C_i = C_j$ for some $i, j (1 \leq i < j \leq f')$. By Lemma 3.6, $C$ has at most two vertices $x, y$ of degree 2. If $x \neq y$, then we delete the longest path in $C$ from $x$ to $y$ (except $x$ and $y$), and if $x = y$, then we delete $C - x$ from $G$. In either case $\Pi$ induces an LEW-embedding of genus $g$ of the resulting subgraph $H$. The $\Pi'$-embedding of $H$ is at most $g' < g$, a contradiction to the induction hypothesis.

Assume next that $G$ has no $\Pi$-facial cycle $C$ which is equal to two of the cycles $C_i$. Assume that the notation has been chosen such that $C_i = W_i$ for $i = 1, 2, \ldots, m$ and $C_i$ is distinct from each $W_1, \ldots, W_f$ for each $i \geq m + 1$. Let $q_i$ (respectively $q'_i$) be the length of $W_i$ (respectively $W'_i$) for $1 \leq i \leq f$ (respectively $1 \leq i \leq f'$). Then $q_1 + q_2 + \cdots + q_f = q'_1 + q'_2 + \cdots + q'_{f'} = 2q$. Also, $f < f'$, and $q_i \leq q'_i$ for $i = 1, 2, \ldots, m$, and $q_i < \text{ew}(G, \Pi)$ for $i = 1, 2, \ldots, f$. Hence for some $j, m + 1 \leq j \leq f'$, we have $q'_j < \text{ew}(G, \Pi)$. It follows that $C_j$ is a cycle which is $\Pi$-contractible but not $\Pi$-facial. Hence $\text{Ext}(C_j, \Pi)$ is a proper subgraph of $G$. As in the proof of Corollary 3.4
we conclude that a $\mathcal{I}$-noncontractible cycle in $\text{Ext}(C_j, \mathcal{I})$ is also $\mathcal{I}$-noncontractible in $G$. It follows that $\mathcal{I}$ is an LEW-embedding of $\text{Ext}(C_j, \mathcal{I})$. By Proposition 3.2 it has genus $g$. On the other hand, the $\mathcal{I}'$-embedding of $\text{Ext}(C_j, \mathcal{I})$ has genus $\leq g' < g$, a contradiction to the induction hypothesis.

The following results show that LEW-embeddings share many properties with planar embeddings.

**Proposition 4.2.** If $G$ is a 2-connected LEW-embedded graph, then every facial walk is a cycle.

**Proof.** Suppose (reductio ad absurdum) that $W'$ is a facial walk which is not a cycle. Let $W$ be a closed subwalk of length at least 1 and with no repetition of vertices. Since $G$ is 2-connected, $G$ has no vertex of degree 1, and hence $W$ is a cycle. Since $G$ is LEW-embedded, $W'$ has length $< \text{ew}(G)$ ad hence $W$ is contractible. Since $\text{Int}(W)$ is planar and 2-connected, every facial walk of $G$ in $\text{Int}(W)$ is a cycle. Hence $W'$ must be a facial walk in $\text{Ext}(W)$. $W$ is of the form $u_1u_2\ldots u_ku_1$ where only $u_i$ is incident with an edge in $\text{ext}(W)$. Since $W \neq W'$, there must be such an edge. Hence $u_1$ is a cutvertex, contrary to the assumption that $G$ is 2-connected.

The following result generalizes a result of Tutte [12] on planar graphs:

**Theorem 4.3.** If $\mathcal{I}$ is an LEW-embedding of a 3-connected graph $G$, then the $\mathcal{I}$-facial cycles are precisely those induced nonseparating cycles of $G$ which have length $< \text{ew}(G, \mathcal{I})$.

**Proof.** If $G$ is planar, then the $\mathcal{I}$-genus of $G$ is also zero, by Theorem 4.1, and the theorem reduces to that of Tutte. So assume that $g(G) > 0$.

If $C$ is an induced nonseparating cycle of length $< \text{ew}(G, \mathcal{I})$, then $C$ is $\mathcal{I}$-contractible because $\mathcal{I}$ is an LEW-embedding. In particular, $\text{Int}(C)$ and $\text{Ext}(C)$ have only $C$ in common. Since $\text{Ext}(C) \neq C$ and $C$ is induced and nonseparating we must have $\text{Int}(C) = C$. Hence $C$ is $\mathcal{I}$-facial.

Assume conversely that $C$ is a $\mathcal{I}$-facial cycle. Since $\mathcal{I}$ is an LEW-embedding, $C$ has length $< \text{ew}(G, \mathcal{I})$. We shall prove that $C$ is induced and nonseparating. If this were not the case, then some $\mathcal{I}$-facial cycle $C'$ which intersects $C$ would contain a segment of the form $v_1uv_2$ or $v_1uu'v_2$ where $u$, $u'$ are consecutive vertices of $C$ and the first edge $v_1u$ (or $v_1u'$) or the last edge $uv_2$ is a chord of $C$ or they are both edges joining $C$ to distinct components of $G - V(C)$. Let $P$ be the shortest path of $C'$ starting with the edge $uv_2$ and terminating at a vertex $z$ of $C$. Without loss of generality we can assume that $P$ has length at most $\frac{1}{2}|E(C')| < \frac{1}{2}\text{ew}(G, \mathcal{I})$. Hence two of the cycles in $C \cup P$ have length less than $\text{ew}(G, \mathcal{I})$ and are therefore $\mathcal{I}$-contractible. By Proposition 3.5, all three cycles of $C \cup P$ are
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Let \( \Pi \)-contractible and one of them, say \( C" \) satisfies \( \text{Int}(C") \supseteq C \cup P \). Since \( P \subseteq \text{Ext}(C) \) we conclude that \( C" \supseteq P \). Since \( P \) is a segment of a facial cycle, every path from a vertex in \( \text{Int}(C") - C" \) to \( \text{Ext}(C") - P \) must contain \( u \) or \( z \). But \( G \) is 3-connected and so there is no vertex in \( \text{Int}(C") - C" \). This implies that \( u \) and \( z \) are consecutive on \( C \) and that \( v_1 \) is in \( \text{Ext}(C") \). In particular \( C' \subseteq \text{Ext}(C") \). Since \( G \) has no multiple edges, \( P \) has length at least 2. Now \( G - \{u, z\} \) has no path from \( v_2 \) to \( v_1 \). This contradicts the assumption that \( G \) is 3-connected. Hence \( C \) is induced and nonseparating.

**Corollary 4.4.** A 3-connected graph has at most one LEW-embedding.

**Proof.** Suppose that \( \Pi \) and \( \Pi' \) are LEW-embeddings of the 3-connected graph \( G \). Suppose that \( \text{ew}(G, \Pi) < \text{ew}(G, \Pi') \). By Theorem 4.3, every \( \Pi \)-facial cycle is \( \Pi' \)-facial. Since every edge is in two \( \Pi \)-facial cycles and two \( \Pi' \)-facial cycles, we conclude that the \( \Pi \)-facial cycles are the same as the \( \Pi' \)-facial cycles and hence \( \Pi = \Pi' \).

We shall later show the stronger statement that a 3-connected graph \( G \) which has an LEW-embedding only has one embedding of genus \( g(G) \). For that we shall make use of a result on overlap graphs which we now define. If \( C \) is a cycle in a graph \( G \), then a \( C \)-component in \( G \) is either a \( K_2 \) whose ends (but not its edge) are in \( C \) or it consists of a connected component \( H \) of \( G - C \) together with all edges from \( H \) to \( C \) and the ends of these edges. Two \( C \)-components \( H, H' \) are said to avoid one another if \( C \) has two vertices \( x, y \) such that all vertices of \( H \cap C \) are in one of the segments of \( C \) from \( x \) to \( y \), and all vertices of \( H' \cap C \) are in the other segment of \( C \) from \( x \) to \( y \). If \( H \) and \( H' \) do not avoid one another, then they overlap. Now the overlap graph \( O(G, C) \) for a cycle \( C \) in a graph \( G \) is the graph whose vertices are the \( C \)-components such that two vertices of \( O(G, C) \) are adjacent if and only if they overlap. It is easy to see that \( O(G, C) \) is connected if \( G \) is 3-connected (or, more generally, a subdivision of a 3-connected graph) and that \( O(G, C) \) is bipartite if \( G \) is planar. Tutte [12] proved that a graph \( G \) is planar if and only if \( O(G, C) \) is bipartite for every cycle \( C \). An extension and a short proof was given in [10]. Here we extend part of Tutte's result to LEW-embeddings.

**Theorem 4.5.** Let \( C \) be a cycle of length \(< \text{ew}(G)\) in a 2-connected nonplanar LEW-embedded graph \( G \). Then \( O(G, C) \) is bipartite. \( G \) has precisely one \( C \)-component \( H \) such that \( C \cup H \) is nonplanar. If \( G \) is a subdivision of a 3-connected graph, then \( O(G, C) \) is connected and the partite class of \( O(G, C) \) containing \( H \) is precisely the set of \( C \)-components in \( \text{Ext}(C) \).

**Proof.** Since \( C \) has length \(< \text{ew}(G) \), \( C \) is contractible and hence each
C-component is either in \( \text{int}(C) \) or in \( \text{ext}(C) \). Moreover, two C-components in \( \text{int}(C) \) cannot overlap because \( \text{Int}(C) \) is planar.

We shall show that no two C-components in \( \text{ext}(C) \) overlap. Let \( C' \) be a cycle of \( G \) with the following properties:

(i) \( C' \) is contractible and \( \text{Int}(C') \supseteq C \).

(ii) Every vertex of \( C' \) which is incident with an edge in \( \text{ext}(C') \) is in \( C \).

(iii) \( \text{int}(C') \) has as many edges as possible subject to (i) and (ii).

Since \( C \) satisfies (i) and (ii), \( C' \) exists. Consider a path \( P \) on \( C' \) such that \( P \) has only its two ends in common with \( C \). Then \( P \) is contained in a C-component \( H' \). Since no intermediate vertex of \( P \) is incident with an edge of \( \text{ext}(C) \) it follows that \( H' \subseteq \text{Int}(C') \). Since \( \text{Int}(C') \) is planar it follows that \( H' \) does not overlap any C-component of \( G \) in \( \text{ext}(C) \). More generally, no two C-components in \( \text{ext}(C') \cap \text{Int}(C') \) overlap.

We claim that there is only one C-component \( H \) of \( G \) in \( \text{ext}(C') \). For otherwise, \( G \) would have a facial cycle \( C'' \) in \( \text{Ext}(C') \) containing edges from at least two distinct \( C' \)-components. As in the proof of Theorem 4.3, we conclude that \( C'' \) has a segment \( P' \) of length \( < \frac{1}{2} \text{ew}(G) \) such that \( P' \) has its ends but no edge in common with \( C' \). As in the proof of Theorem 4.3, we conclude that the three cycles of \( C \cup P' \) are contractible and that one of them, say \( S \), satisfies \( S \supseteq P' \), \( \text{Int}(S) \supseteq C \). We can assume that no intermediate vertex of \( P' \) is incident with an edge in \( \text{ext}(S) \). (For otherwise \( C'' \) is in \( \text{Int}(S) \) and then we consider instead of \( P' \) any other segment of \( C'' \) which connects two vertices of \( C \) and which has no edge in common with \( C' \).) Since \( S \) is contractible, each segment of \( C' \) in \( \text{ext}(C) \) is either in \( \text{ext}(S) \) or \( \text{int}(S) \). We then obtain a contradiction so the maximality property of \( C' \) by letting \( P' \) replace the segment of \( C' \) in \( \text{int}(S) \) connecting the ends of \( P' \).

We have shown that there is at most one C-component \( H \) in \( \text{ext}(C') \). Clearly, \( H \) does not overlap any C-component in \( \text{Int}(C') \cap \text{ext}(C) \). Hence \( O(G, C) \) is bipartite. Since \( G \) is nonplanar, \( H \) exists and \( H \cup C \) is nonplanar. This proves the last part of Theorem 4.5.

5. POLYNOMIALLY BOUNDED ALGORITHMS FOR FINDING SHORTEST NONCONTRACTIBLE CYCLES AND FOR DESCRIBING LEW-EMBEDDINGS

Before we describe an algorithm that produces an LEW-embedding we shall first investigate a given embedding. We shall describe an algorithm for finding a shortest noncontractible cycle. Thus we can determine the edge-width and decide if the embedding is an LEW-embedding.

If \( \Pi \) is an embedding of a connected graph \( G \) and \( C \) is a cycle of \( G \), then
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clearly we can find \( G_r(C) \) and \( G_I(C) \) in polynomial time and we can determine the \( \Pi \)-genus of \( G_r(C) \cup C \) and \( G_I(C) \cup C \) in polynomial time. In particular, we can decide in polynomial time if \( C \) is \( \Pi \)-contractible.

We shall now describe a polynomially bounded algorithm \( A_1 \) which finds, in polynomial time, a shortest \( \Pi \)-noncontractible cycle if the \( \Pi \)-genus of \( G \) is positive. The algorithm \( A_1 \) is a special case of a general algorithm which we call the \textit{fundamental cycle method} described below.

Consider a family \( F \) of cycles in a connected graph \( G \). For each vertex \( x \) in \( G \) we let \( T_x \) be a breadth-first tree rooted at \( x \), i.e., a tree such that for each vertex \( y \), the distance between \( x \) and \( y \) in \( T_x \) equals the distance between \( x \) and \( y \) in \( G \). For each edge \( e \) in \( E(G) \setminus E(T_x) \) we let \( C(e, T_x) \) be the unique cycle in \( T_x \cup \{ e \} \). Among all cycles \( C(e, T_x) \) we let \( C_r \) denote a shortest one in \( F \). The cycle \( C_r \) is a shortest cycle in \( F \) provided \( F \) satisfies the following condition which we call the \( 3 \)-path-condition: If \( P_1, P_2, P_3 \) are internally disjoint paths connecting two vertices \( x \) and \( y \) and if two of the cycles in \( P_1 \cup P_2 \cup P_3 \) are not in \( F \), then also the third cycle in \( P_1 \cup P_2 \cup P_3 \) is not in \( F \).

**Theorem 5.1.** If \( F \) is a collection of cycles in a graph \( G \) satisfying the \( 3 \)-path condition, then the fundamental cycle method results in a shortest cycle in \( F \).

**Proof.** Let \( x \) be a vertex in \( G \) such that some shortest cycle in \( F \) contains \( x \). Let \( T_x \) be any breadth-first tree rooted at \( x \). Let \( C \) be a shortest cycle in \( F \) such that \( C \) contains \( x \) and such that \( |E(C) \setminus E(T_x)| \) is minimum subject to these conditions. Let \( P_1 : x x_1 x_2 \ldots x_k \) and \( P_2 : x y_1 y_2 \ldots y_k \) be the two paths in \( C \) starting from \( x \) where \( |E(C)| \) equals \( 2k \) or \( 2k + 1 \).

We first claim that \( x_i \) and \( y_i \) have distance \( i \) to \( x \) in \( G \) for \( i = 1, 2, \ldots, k \). For otherwise, \( T_x \) would have a path \( P \) such that \( P \cap C = \{ u_i, u_j \} \) where \( u_i \in \{ x_i, y_i \}, u_j \in \{ x_j, y_j \} \) and \( P \) has length \( < |i - j| \). By the minimality of \( C \), two of the cycles in \( C \cup P \) are not in \( F \). But, \( C \) is in \( F \) contradicting the assumption that \( F \) satisfies the \( 3 \)-path-condition.

We next claim that \( T_x \) contains both paths \( P_1, P_2 \) if \( x_k \neq y_k \) and that \( T_x \) contains one of \( P_1, P_2 \) (say \( P_1 \)) and \( P_2 - y_k \) if \( x_k = y_k \). For otherwise, \( T_x \) would contain a path \( P \) as in the previous paragraph except that now \( P \) has length \( |i - j| \). We obtain a contradiction as in the previous paragraph unless the ends \( u_i \) and \( u_j \) are on the same path \( P_1, P_2 \) (say on \( P_1 \)). Since \( F \) satisfies the \( 3 \)-path-condition, the cycle \( C' \) obtained from \( C \) by replacing the segment of \( P_i \) from \( x_i \) to \( x_j \) by \( P \) is in \( F \). Now \( |E(C') \setminus E(T_x)| < |E(C) \setminus E(T_x)| \). This contradiction shows that \( E(C) \setminus E(T_x) \) consists of an edge \( e \) incident with \( y_k \). Hence \( C = C(e, T_x) \).

Theorem 5.1 shows that there is a polynomially bounded algorithm for the following problems:
(1) Find a shortest cycle which has odd intersection with a prescribed edge set in an undirected graph.

(2) Find a shortest nonbalanced cycle in a directed graph. (A cycle is balanced if it has the same number of directed edges in both directions).

By Proposition 3.5, the family of noncontractible cycles in an embedded graph satisfies the 3-path-condition. So we get:

**Theorem 5.2.** There exists a polynomially bounded algorithm $A_1$ which finds a shortest noncontractible cycle for any graph embedding of positive genus.

J. Hutchinson (private communication) raised the question of finding a shortest noncontractible cycle which does not separate the surface. In our combinatorial framework this question is reformulated as follows: We say that a cycle $C$ in a $\Pi$-embedded graph is $\Pi$-separating if $G_1(C, \Pi)$ and $G_1(C, \Pi)$ are edge-disjoint. (We have previously defined a separating cycle. It should be noted that a separating cycle need not be $\Pi$-separating and a $\Pi$-separating cycle need not be separating). Now Hutchinson's question is that of finding a shortest non-$\Pi$-separating cycle. An easy extension of Proposition 3.5 shows that the set of non-$\Pi$-separating cycles satisfies the 3-path-condition. Hence Theorem 5.1 gives an affirmative answer to Hutchinson's question.

In this paper we do not treat embeddings on nonorientable surfaces. However, it is easy to describe such embeddings combinatorially. A graph embedding is then nonorientable if and only if the graph has a cycle such that the left side and the right side interchange when we traverse the cycle. We call a cycle with this property a Möbius cycle. It is easy to show that the set of Möbius cycles satisfies the 3-path-condition. Hence a shortest one can be found in polynomial time by Theorem 5.1. Other interesting types of (shortest) noncontractible cycles can be found as follows: Let $F'$ be the collection of contractible cycles and let $F''$ be a (small) collection of non-contractible cycles. Let $F$ be those cycles which are not in the subspace of the cycle space generated by $F' \cup F''$. Then $F$ satisfies the 3-path-condition and thus Theorem 5.1 applies to $F$. More generally, Theorem 5.1 applies to a family $F$ of cycles whenever the cycles not in $F$ generate a cycle space that contains no cycle of $F$. In other words, we can find, in polynomial time, a shortest cycle outside a given subspace of the cycle space.

We now describe an algorithm for finding LEW-embeddings.

**Theorem 5.3.** There exists a polynomially bounded algorithm $A_2$ which, for any 3-connected graph $G$, describes an LEW-embedding of $G$ or tells that $G$ has no LEW-embedding.
Proof. If $G$ has an LEW-embedding and $e_1, e_2$ are two edges incident with the same vertex $v$, then Theorem 4.3 implies that $G$ has an induced nonseparating cycle $C(e_1, e_2)$ of length $<\text{ew}(G)$ through $e_1$ and $e_2$ if and only if $e_1$ and $e_2$ are consecutive in the clockwise ordering around $v$. Moreover, the cycle $C(e_1, e_2)$ is unique if it exists. By Corollary 4.4, $G$ has at most one LEW-embedding. That LEW-embedding is uniquely described by the facial cycles which are the cycles of the form $C(e_1, e_2)$. We shall describe a polynomially bounded algorithm which finds the cycles $C(e_1, e_2)$ or decides that $G$ has no LEW-embedding.

First we find a shortest cycle $C'(e_1, e_2)$ through $e_1$ and $e_2$. If $C'(e_1, e_2)$ is induced and nonseparating, then we write $C^*(e_1, e_2) = C'(e_1, e_2)$ by which we mean that either $C(e_1, e_2)$ does not exist or $C(e_1, e_2) = C'(e_1, e_2)$. (As we do not know the edge-width we do not at this time say that $C'(e_1, e_2)$ is a face boundary. However, as $C'(e_1, e_2)$ is a shortest cycle through $e_1, e_2$ and it is induced and nonseparating, there cannot be another cycle through $e_1, e_2$ which is the face boundary in an LEW-embedding, by Theorem 4.3. So listing $C^*(e_1, e_2)$ means that we have found $C(e_1, e_2)$ if it exists, but we may erase $C^*(e_1, e_2)$ later). If $G$ has at least two $C'(e_1, e_2)$-components, then we form the overlap graph $O(G, C'(e_1, e_2))$. We can assume that it is bipartite and connected since otherwise Theorem 4.5 implies that $C(e_1, e_2)$ does not exist. We obtain the same conclusion if there is more than one $C'(e_1, e_2)$-component which together with $C'(e_1, e_2)$ forms a nonplanar graph. This can be checked in polynomial (even linear) time [4]. On the other hand, we can assume that there is a $C'(e_1, e_2)$-component which together with $C'(e_1, e_2)$ forms a nonplanar graph since otherwise $G$ is planar in which case we have finished. Now we draw $C'(e_1, e_2)$ as a convex polygon and we draw all $C'(e_1, e_2)$-components in the "planar partite class" of $O(G, C'(e_1, e_2))$ inside $C'(e_1, e_2)$. (By Corollary 4.4 and Theorem 4.5, all the faces inside $C'(e_1, e_2)$ must be faces in any LEW-embedding). If the resulting graph has a facial cycle inside $C'(e_1, e_2)$ containing $e_1$ and $e_2$, then we let $C(e_1, e_2)$ be that facial cycle. If there is no facial cycle inside $C'(e_1, e_2)$ that contains $e_1$ and $e_2$ and, furthermore, the common end $v$ of $e_1$ and $e_2$ is incident with an edge $e'$ not on or inside $C'(e_1, e_2)$, then $e_1$ and $e_2$ cannot be consecutive in the clockwise ordering around $v$ in an LEW-embedding. Therefore, we say that $C(e_1, e_2)$ does not exist. If $v$ is not incident with such an edge $e'$, then we know all facial cycles (except one) which contain $v$ and hence also the clockwise or anticlockwise ordering around $v$ in the unique LEW-embedding of $G$ if it exists. We list these cycles and terminate our investigation at $v$. If this situation does not occur, then we perform the algorithm for each pair of edges incident with $v$. If, at the end, we have listed three cycles $C(e, e_1), C(e, e_2), C(e, e_3)$, then the LEW-embedding does not exist. If we have listed two cycles $C(e, e_1), C(e, e_2)$, then we omit
all cycles of the form $C^*(e, e')$. If we do not have two cycles containing a particular edge $e$, then the other face boundary (or boundaries) containing this edge must be induced and nonseparating. Recall that we have previously found all short induced nonseparating cycles $C'(e, e_1)$. The required face boundary (or boundaries) must be the shortest (or two shortest) cycles from this set. We now add this cycle (or cycles) to our collection of face boundaries, so that every edge is in two potential face boundaries. (If there is no cycle (or cycles) to add, there is no LEW-embedding.) We then check if the resulting cycles of the form $C(e, e, e_j)$ define a cyclic permutation around $v$. If so, then that is the clockwise or anticlockwise ordering around $v$ in the unique LEW-embedding of $G$ if it exists. If not, then the LEW-embedding does not exist.

We repeat this algorithm for all vertices of $G$. Then either we conclude that $G$ has no LEW-embedding or we find the clockwise or anticlockwise orientation around every vertex. Defining one of these to be the clockwise orientation and using the connectedness of $G$ we obtain the clockwise ordering around each other vertex. We can use the clockwise ordering around a vertex $v$ to decide if the cyclic ordering around a neighbour $u$ is clockwise or anticlockwise because we know at least one cycle through the edge $uv$ which must be facial.

The embedding $\Pi$ which we have found is the unique LEW-embedding of $G$ if $G$ has such an embedding. We now use the algorithm $A_1$ of Theorem 5.2 to decide if $\Pi$ is an LEW-embedding.

Note that the algorithm $A_2$ in Theorem 5.3 always describes a minimum genus embedding for all 3-connected graphs that have LEW-embeddings. However, $A_2$ describes an embedding which is a minimum genus embedding for many 3-connected graphs that do not have LEW-embeddings, namely those which, roughly speaking, have "locally large edge-width."

6. 2-SWITCHINGS AND UNIQUENESS OF EMBEDDINGS

We have previously proved that an LEW-embedding is a minimum genus embedding (Theorem 4.1) and that a 3-connected graph has at most one LEW-embedding (Corollary 4.4). In this section we shall prove that a 3-connected graph which has an LEW-embedding has only one minimum genus embedding. We shall derive this from an extension of Whitney's 2-switching theorem to LEW-embeddings.

Suppose that $C$ is a $\Pi$-contractible cycle in a $\Pi$-embedded 2-connected graph $G$. Suppose further that only two vertices $x$ and $y$ of $C$ are incident with edges in $\text{ext}(C)$. Then we define a new embedding $\Pi'$ as follows: For each vertex $z$ in $\text{Ext}(C, \Pi) - \{x, y\}$, the $\Pi'$-clockwise ordering around $z$ is the same as the $\Pi$-clockwise ordering. For each $z$ in $\text{Int}(C, \Pi) - \{x, y\}$, the
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III'-clockwise ordering around z is the III-anticlockwise ordering around z. If \( e_1, e_2, ..., e_k, e_{k+1}, ..., e_p \) is the clockwise ordering around \( x \) (or \( y \)) in \( III \) such that \( e_1, e_2, ..., e_k \) are in \( \text{Int}(C, III) \) and \( e_{k+1}, ..., e_p \) are in \( \text{ext}(C, III) \), then the clockwise ordering around \( x \) (or \( y \)) in \( III' \) is: \( e_k, e_{k-1}, ..., e_1, e_{k+1}, ..., e_p \). We say that \( III' \) is obtained from \( III \) by a 2-switching (of \( C \)). We say that the edges in \( \text{Int}(C, III) \) are involved in that 2-switching. If all \( III \)-facial walks are cycles, then also all \( III' \)-facial walks are cycles. Moreover, only two facial cycles are affected by the 2-switching. Whitney [15] proved that if \( III \) and \( III' \) are planar (i.e., genus zero) embeddings of a 2-connected graph \( G \), then \( III' \) can be obtained from \( III \) by a sequence of 2-switchings. We now generalize this to LEW-embeddings.

**Theorem 6.1.** If \( III \) is an LEW-embedding of a 2-connected graph \( G \), then any other embedding \( III' \) of \( G \) of genus \( g(G) \) is obtained from \( III \) by a sequence of 2-switchings.

Before we prove Theorem 6.1 we derive the following corollary.

**Corollary 6.2.** Let \( G \) be a 2-connected graph that has an LEW-embedding. Then a cycle \( C \) in \( G \) is contractible in every embedding of genus \( g(G) \) if and only if \( C \) is contractible in some embedding of genus \( g(G) \). If, in addition, \( G \) is a subdivision of a 3-connected graph, then \( G \) has only one embedding of genus \( g(G) \).

**Proof.** Let \( III \) be an LEW-embedding of \( G \). By Theorem 4.1, \( III \) is of genus \( g(G) \). By Theorem 6.1, every other embedding of \( G \) of genus \( g(G) \) is obtained from \( III \) by a sequence of 2-switchings. So, it is sufficient to show that a 2-switching of a contractible cycle \( C \) does not make a contractible cycle \( C' \) noncontractible. This is clear if \( C \subseteq \text{Int}(C') \) or \( C \subseteq \text{Ext}(C') \). So we can assume that \( C' \) is the union of two paths \( P, P' \) such that \( P \subseteq \text{Int}(C) \) and \( P' \subseteq \text{ext}(C) \). Using the assumption that \( \text{Int}(C) \) has genus zero an easy count shows that \( \text{Int}(C') \) has the same genus before and after the 2-switching.

We now prove Theorem 6.1 (and hence also Corollary 6.2) by induction on \( |E(G)| \). If \( g(G) = 0 \), then Theorem 6.1 follows from Whitney's theorem. So assume that \( g(G) > 0 \) and hence \( |E(G)| > |V(G)| \geq 4 \).

Suppose first that \( G \) is not a subdivision of a 3-connected graph (without multiple edges). Then \( G \) is the union of two proper subgraphs \( G_1, G_2 \) such that \( G_1 \cap G_2 \) consists of two vertices \( x, y \) such that none of \( G_1, G_2 \) is just a path from \( x \) to \( y \). Now \( G \) has two \( III \)-facial walks (which are cycles by Proposition 4.2) \( C_1, C_2 \) such that both \( C_1 \) and \( C_2 \) contain edges from both \( G_1 \) and \( G_2 \) and such that all four paths of the form \( G_i \cap C_j \) \( (1 \leq i \leq 2, 1 \leq j \leq 2) \) are distinct. (To verify this is an easy exercise which we leave for the reader). Let \( P_2 \) be a shortest path in \( C_2 \) such that \( P_2 \) has its ends but
no intermediate vertex in common with $C_1$. Now $C_1$ has length $< \text{ew}(G, \Pi)$ and $P_2$ has length $< \frac{1}{2} \text{ew}(G, \Pi)$. Hence two cycles in $C_1 \cup P_2$ have length $< \text{ew}(G, \Pi)$ and by Proposition 3.5, $C_1 \cup P_2$ has a $\Pi$-contractible cycle $C'$ such that $\text{Int}(C', \Pi) \supseteq C_1 \cup P_2$. Since $g(G) > 0$, $\text{Ext}(C', \Pi)$ has $\Pi$-genus $> 0$ and hence $C' \neq C_1$. Hence $C' \supseteq P_2$.

We claim that $G$ has a $\Pi$-contractible cycle $C$ such that only two vertices of $C$ are incident with edges in $\text{ext}(C, \Pi)$. If $C_2 \subseteq \text{Ext}(C', \Pi)$, then the union of $P_2$ and the segment of $C_1$, in $\text{int}(C', \Pi)$ can play the role of $C$. On the other hand, if $C_2 \subseteq \text{Int}(C', \Pi)$, then the union of the segments of $C_1$ and $C_2$ in $\text{int}(C', \Pi)$ contains a cycle (because of the intersection properties of $C_1$ and $C_2$) which can play the role of $C$. This shows that $C$ exists.

We choose $C$ such that $\text{Int}(C, \Pi)$ has as few edges as possible. Let $x$ and $y$ be the two vertices incident with edges in $\text{ext}(C, \Pi)$. Let $Q_1, Q_2$ be the two segments of $C$ from $x$ to $y$ such that $|E(Q_1)| \leq |E(Q_2)|$. The minimality of $C$ implies that $Q_1$ has no chord and that $H - \text{Int}(C, \Pi) - Q_1$ has only one component. Now we consider the embeddings of $G - H$ induced by $\Pi$ and $\Pi'$. Since $C$ is $\Pi$-contractible, the $\Pi$-embedding of $G - \text{int}(C, \Pi)$ has genus $g(G)$, and then it is also easy to see that the $\Pi$-embedding of $G - H$ induced by $\Pi$ and $\Pi'$ has genus $g(G)$. Clearly, the $\Pi$-embedding of $G - H$ is an $\text{LEW}$-embedding and is therefore of genus $g(G - H)$, by Theorem 4.1. In particular, $g(G - H) = g(G)$. Since $\Pi'$ is a minimum genus embedding of $G$ and $g(G) = g(G - H)$, it follows that the $\Pi'$-genus of $G - H$ is $g(G - H)$.

By the induction hypothesis, the $\Pi'$-embedding of $G - H$ can be obtained from the $\Pi$-embedding of $G - H$ by a sequence of 2-switchings. Since $H$ is attached only to $Q_1$, which is part of a facial cycle in each of the embeddings of $G - H$ we can modify the sequence of 2-switchings so that it becomes a sequence of 2-switchings of $G$. This transforms $\Pi$ into an embedding $\Pi''$ of $G$ such that $\Pi''$ and $\Pi'$ agree on $G - H$. Since all $\Pi$-facial walks in $G - H$ are cycles the same holds for the $\Pi''$-facial walks (and hence also the $\Pi'$-facial walks) of $G - H$. By the last sentence of Proposition 3.3, the embedding $\Pi'$ is obtained from that of $G - H$ by adding $H$ to a facial cycle $S$. If $S$ contains $Q_1$, then clearly $\Pi'$ is obtained from $\Pi''$ by at most one 2-switching. (Note that the minimality of $\text{Int}(C, \Pi)$ ensures that there is only one planar embedding of $\text{Int}(C)$ with $Q_1$ being on a facial cycle, by Whitney's 2-switching theorem for planar graphs). On the other hand, if $S$ does not contain $Q_1$, then all intermediate vertices of $Q_1$ have degree 2 in $G$. By the minimality of $\text{Int}(C, \Pi)$, all intermediate vertices of $Q_2$ have degree 2 in $G$. Since $Q_1$ has length $< \frac{1}{2} \text{ew}(G, \Pi)$, $Q_1$ has the following property in the $\Pi$-embedding of $G$ (by Proposition 3.5):

If $S'$ is a $\Pi$-facial cycle containing $x$ and $y$, then the three cycles of $Q_1 \cup S'$ are $\Pi$-contractible. The proof of Corollary 6.2 shows that $Q_1$ still has that property after a sequence of 2-switchings. In particular, the three cycles in $Q_1 \cup S$ are $\Pi''$-contractible in $G - H$ and hence also
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Now we can perform a sequence of 2-switchings which transform \( \Pi' \) into an embedding which is obtained from \( \Pi' \) by first deleting \( Q_2 - \{x, y\} \) and then adding \( Q_2 \) such that \( Q_2 \) together with \( Q_1 \) form a facial cycle. The resulting embedding of \( G \) is the same as \( \Pi'' \) (possibly after switching \( Q_1 \cup Q_2 \)).

Suppose now that \( G \) is a subdivision of a 3-connected graph (without multiple edges). We shall prove that \( \Pi = \Pi' \) in this case. Suppose therefore (reductio ad absurdum) that \( \Pi \neq \Pi' \). Since \( G \) has no vertex of degree 1 and since \( G \) has no cycle satisfying the conclusion of Lemma 3.6 we can use the proof of Theorem 4.1 to conclude that \( G \) has a cycle \( C \) (corresponding to \( C_j \) in the proof of Theorem 4.1) such that \( C \) is a subwalk of a \( \Pi' \)-facial walk, \( C \) is not a \( \Pi \)-facial cycle, and \( C \) has length \( \leq \ell(G, \Pi) \) and is therefore \( \Pi \)-contractible. (In order to establish the existence of \( C_j \) in the proof of Theorem 4.1 we only need that \( \Pi' \neq \Pi \) and \( f' \geq f \).)

Clearly, \( \Pi \) is an LEW-embedding of \( \text{Ext}(C, \Pi) \). Moreover, as \( G \) is a subdivision of 3-connected graph, \( \text{Ext}(C, \Pi) \) is 2-connected, and any separating set of two vertices in \( \text{Ext}(C, \Pi) \) is either contained in \( C \) or separates a path from the rest of \( \text{Ext}(C, \Pi) \). In particular, every 2-switching of the \( \Pi \)-embedding of \( \text{Ext}(C, \Pi) \) must switch a cycle that contains a segment of \( C \).

Now \( \Pi \) is an LEW-embedding of \( \text{Ext}(C, \Pi) \) of genus \( g(G) \). By Theorem 4.1, it is also of genus \( g(\text{Ext}(C, \Pi)) \). Hence \( \Pi' \) is also a minimum genus embedding of \( \text{Ext}(C, \Pi) \). By the induction hypothesis, there exists a sequence of 2-switchings transforming the \( \Pi \)-embedding of \( \text{Ext}(C, \Pi) \) into an embedding \( \Pi'' \) such that \( \Pi' \) and \( \Pi'' \) are the same for \( \text{Ext}(C, \Pi) \). Since all \( \Pi \)-facial walks of \( \text{Ext}(C, \Pi) \) are cycles, the same holds for the \( \Pi'' \)-facial walks. By Proposition 3.3, the \( \Pi'' \)-embedding of \( G \) is obtained from the \( \Pi'' \)-embedding of \( \text{Ext}(C, \Pi) \) by adding planar subgraphs to \( \Pi'' \)-facial cycles.

Now consider a cycle \( C' \) satisfying conditions (i), (ii), (iii) in the proof of Theorem 4.5 (where int and ext refer to the \( \Pi \)-embedding of \( G \)). The proof of Theorem 4.5 shows that \( G \) has only one \( C \)-component \( H \) in \( \text{ext}(C') \). Moreover, \( H \cup C \) is nonplanar. In particular, \( H \) is not just a path. \( H \) has at least three vertices on \( C \). Hence \( H \cup C \) is a subdivision of a 3-connected graph and there is no 2-switching of \( \text{Ext}(C, \Pi) \) (or any graph obtained from \( \text{Ext}(C, \Pi) \) by a sequence of 2-switchings) which involves edges in \( H \).

Consider any \( \Pi \)-facial cycle \( C'' \) which contains edges from both \( H \) and \( C' \). Then \( C'' \) cannot contain two distinct segments in \( H \) connecting vertices in \( C \). For if that were the case then we would obtain a contradiction to the maximality of \( C' \) precisely as in the proof of Theorem 4.5 (where we considered a cycle containing edges from two distinct \( C \)-components in \( \text{Ext}(C') \)). So \( C'' \) consists of a path \( P' \subseteq C' \) and a path \( S \subseteq \text{ext}(C', \Pi) \) having
only the ends \( u, v \) in common with \( C \). If possible, we choose \( C'' \) such that \( P' \) is not a segment of \( C \). If this is not possible, then there is no 2-switching of the \( \Pi' \)-embedding of \( \text{Ext}(C, \Pi) \). In that case we let \( C'' \) be chosen such that it is one of the \( \Pi' \)-facial cycles that is used when we add \( C \)-components to \( \text{Ext}(C, \Pi) \) in order to obtain the \( \Pi'' \)-embedding of \( G \) from the \( \Pi'' \)-embedding of \( \text{Ext}(C, \Pi) \). In each case \( \{u, v\} \) is a separating set of \( G \) which shows that \( G \) is not a subdivision of a 3-connected graph. This is clear if \( \Pi = \Pi'' \). On the other hand, if \( \Pi'' \neq \Pi \), then \( P' \notin C \) and \( \{u, v\} \) separates \( \text{Ext}(C, \Pi) \) into two graphs none of which is a path. Since \( C' \) and \( C'' \) are \( \Pi \)-contractible the third cycle \( S \) in \( C' \cup C'' \) is also \( \Pi \)-contractible, and all 2-switchings of the \( \Pi \)-embedding of \( \text{Ext}(C, \Pi) \) involve only edges in the planar graph \( \text{Int}(S, \Pi) \). After the sequence of 2-switchings, \( C \) is \( \Pi'' \)-facial but, by the definition of \( C \), we cannot use \( C \) when we add \( C \)-components going from the \( \Pi'' \)-embedding of \( \text{Ext}(C, \Pi) \) to the \( \Pi' \)-embedding of \( G \). Hence \( \{u, v\} \) separates \( G \), contradicting the assumption that \( G \) is a subdivision of a 3-connected graph.

We can use the proof of Theorem 6.1 to obtain a polynomially bounded algorithm for describing minimum genus embeddings of a large class of graphs. Since the genus of a graph is the sum of the genera of its blocks [2], it is sufficient to consider 2-connected graph. If \( G \) is 2-connected but not a subdivision of a 3-connected graph, then we try to find a decomposition \( G = G_1 \cup G_2 \) such that \( G_1 \cap G_2 \) consists of two vertices \( x \) and \( y \) and such that \( G_2 \) is a 2-connected graph which can be drawn in the plane with \( x \) and \( y \) on the boundary of the unbounded face. It can be checked in polynomial time if such a decomposition exists. If it does not exist, then \( G \) has no LEW-embedding (as the proof of Theorem 6.1 shows). If the decomposition does exist, we replace \( G_2 \) by a shortest path in \( G_2 \) from \( x \) to \( y \). Then we repeat the algorithm on the resulting graph which has an LEW-embedding if \( G \) has an LEW-embedding and which has the same genus as \( G \). Continuing like this we either conclude that \( G \) has no LEW-embedding or we end up with a subgraph \( H \) of \( G \) which is a subdivision of a 3-connected graph. A close inspection of the proof of Theorem 5.3 shows that the algorithm \( A_2 \) also works for subdivisions of 3-connected graph. Thus we get:

**Theorem 6.3.** There exists a polynomially bounded algorithm \( A_3 \) with the following property: If \( G \) is a 2-connected graph, then either \( A_3 \) describes an embedding of \( G \) of genus \( g(G) \) or else it tells that \( G \) has no LEW-embedding.

Note that \( A_3 \) may produce a minimum genus embedding even if \( G \) has no LEW-embedding. On the other hand, it may also happen that \( G \) has an LEW-embedding and that \( A_3 \) does not produce that. It is not clear how to obtain an LEW-embedding in that case.
7. Triangulations

A triangulation is an embedded graph in which every facial walk is a 3-cycle (i.e., a cycle of length 3). Ringel [9] raised the question of characterizing those graphs which are triangulations. He pointed out that a triangulation $G$ is locally Hamiltonian, i.e., every neighbourhood graph $N(v, G)$ has a Hamiltonian cycle. It must also satisfy Euler's formula which together with the equation $2q = 3f$ implies that $q - 3n + 6$ is a nonnegative integer divisible by 6. But these two conditions are not sufficient for a graph to be a triangulation. Ringel's question, when restricted to triangulations with no noncontractible triangles is solved (at least from an algorithmic point of view) by Theorem 5.3 since every triangulation is 3-connected as the following result shows.

**Proposition 7.1.** Let $G$ be a connected locally Hamiltonian graph. Then $G$ is 3-connected. Moreover, if $\{v, x, y\}$ is a separating set of three vertices in $G$, then $G(\{v, x, y\})$ is a 3-cycle and $xy$ is not an edge of any Hamiltonian cycle in $N(v, G)$. $G - \{x, y, v\}$ has precisely two components $H_1, H_2$ and each of the graphs $G(V(H_i) \cup \{x, y, v\}) (i = 1, 2)$ is locally Hamiltonian.

**Proof.** Let $S$ be a smallest separating vertex set of $G$. Pick a vertex $v$ in $S$. Since $S \setminus \{v\}$ does not separate $G$, $v$ is joined to all components of $G - S$. Since $N(v, G)$ has a Hamiltonian cycle $C$, we conclude that $|S| \geq 3$. If $|S| = 3$, then $G - S$ has only two components $H_1, H_2$ (because $C - \{x, y\}$ has only two components), and $v$ is joined to both $x$ and $y$. The same argument with $x$ instead of $v$ shows that $xy$ is an edge in $G$. Clearly, $xy$ is a chord of $C$. Since $G$ is locally Hamiltonian and $G(S)$ is a complete graph, it is easy to see that $G(V(H_i) \cup \{x, y, v\})$ is locally Hamiltonian for $i = 1, 2$.

The next result sheds further light on Ringel's question.

**Theorem 7.2.** Let $G$ be a connected graph. If $G$ is a triangulation with no noncontractible 3-cycles, then $G$ satisfies (1) and (2) below.

1. $G$ is locally Hamiltonian;
2. Every edge of $G$ is in precisely two nonseparating 3-cycles.

Conversely, if $G$ satisfies (1) and (2), then $G$ triangulates some compact, connected, 2-dimensional manifold.

**Proof.** The first part of the theorem follows from Theorem 4.3.

To prove the second part we form a topological space as follows: For each nonseparating 3-cycle $xyzx$ in $G$ we consider an equilateral triangle of side lengths 1 and with corners $x', y', z'$ such that all the triangles are disjoint. Then identify a side in one triangle with a side in another whenever the two sides correspond to the same edge of $G$. We shall prove that the
resulting space is locally homeomorphic to a disc. It is sufficient to verify this at every corner of a triangle. Let $v$ be any vertex of $G$ and let $C': v_1 v_2 \cdots v_r v_1$ be a Hamiltonian cycle of $N(v, G)$. By Proposition 7.1, each 3-cycle which contains $v$ and an edge of $C'$ is nonseparating. By (2), these are the only nonseparating 3-cycles containing $v$. Hence the topological space we have constructed is locally homeomorphic to a disc.

One can show that conditions (1) and (2) are necessary and sufficient for $G$ to triangulate a compact 2-manifold without noncontractible 3-cycles (in the topological sense).

We can reduce the NP-complete problem of deciding if a cubic graph $G$ has a Hamiltonian path to the problem of deciding if a graph is locally Hamiltonian. Just take two disjoint copies of $G$ and add three new vertices each of which is joined to all other vertices. Then the resulting graph is locally Hamiltonian if and only if $G$ has a Hamiltonian path. Hence it is NP-complete to decide if condition (1) in Theorem 7.2 is satisfied. It is easy to check condition (2). Having verified that (2) holds (1) can be checked by a planarity algorithm as the next result shows.

**Proposition 7.3.** Let $G$ be a connected graph satisfying (2) in Theorem 7.2. The $G$ satisfies (1) if and only if every closed neighbourhood graph $\overline{N}(v, G)$ is planar and 3-connected.

**Proof.** If $\overline{N}(v, G)$ is 3-connected and planar, then $N(v, G)$ is 2-connected and can be drawn such that the facial cycle in $N(v, G)$ corresponding to the face containing $v$ is a Hamiltonian cycle of $N(v, G)$.

Suppose conversely that $G$ is locally Hamiltonian. Let $v$ be any vertex of $G$ and let $C: v_1 v_2 \cdots v_k v_1$ be a Hamiltonian cycle of $N(v, G)$. By Proposition 7.1, $G$ is a 3-connected and each 3-cycle through $v$ and an edge of $C$ is nonseparating. We draw $C$ as a convex polygon in the plane. We draw $v$ outside $C$ and all chords of $C$ as straight line segments. We claim that this gives a planar drawing of $N(v, G)$. For suppose that $v_i v_j$ and $v_j v_i$ are crossing chords. Then $C - \{v_i, v_j\}$ consists of two paths which belong to the same component of $G - \{v, v_i, v_j\}$. Each vertex $z$ is joined to $v$ by three internally disjoint paths (because $G$ is 3-connected). Hence also $z$ belongs to the same component of $G - \{v, v_i, v_j\}$ as $C - \{v_i, v_j\}$. This shows that $G - \{v, v_i, v_j\}$ is connected. But then $v v_i$ belongs to the three nonseparating 3-cycles $v v_{i-1} v v_{i+1} v$, $v v_i v v_i v v_i v$, contradicting (2).

Lavrenchenko [6] asked if every triangulation with no short noncontractible cycles on the torus is uniquely embeddable on the torus. The following result goes much further.

**Theorem 7.4.** If $G$ is a triangulation, then $G$ has only one embedding of genus $g(G)$ provided $G$ satisfies one of the conditions (a) or (b) below.
(a) $G$ has no noncontractible, nonseparating 3-cycles.

(b) The noncontractible 3-cycles and the induced noncontractible 4-cycles are pairwise disjoint.

**Proof.** Suppose first that $\Pi$ is a triangulation of $G$ satisfying (a). Let $\Pi'$ be any embedding of $G$ of genus $g(G)$. Since $\Pi'$ has as many facial walks as $\Pi$ and since the $\Pi'$-facial walks have length at least three and since the sum of the lengths of the facial walks equals $2|E(G)|$ for both $\Pi$ and $\Pi'$, we conclude that $\Pi$ and $\Pi'$ are both triangulations of genus $g(G)$. It is sufficient to prove that every $\Pi'$-facial cycle $C$ is a $\Pi$-facial cycle. It follows from Proposition 7.1 that $C$ is nonseparating. By (a), $C$ is $\Pi$-contractible. Since a nonseparating $\Pi$-contractible 3-cycle is $\Pi$-facial it follows that $\Pi = \Pi'$.

Suppose next that $\Pi$ satisfies (b). We insert a new vertex of degree 2 on each edge of $G$ which belongs to a noncontractible induced 4-cycle or 3-cycle. In the resulting graph $G'$ (which we may consider to be $\Pi$-embedded) every noncontractible cycle has length at least five but every facial cycle has length at most four. By Theorem 6.1, $G'$ has only one embedding of genus $g(G') = g(G)$. Hence $G$ has only one embedding of genus $g(G)$.

8. **Embeddings of Small Face-Width**

Theorem 4.1 shows that every embedding $\Pi$ of $G$ of genus $> g(G)$ must have small edge-width $\text{ew}(G, \Pi)$. Since the face-width $\text{fw}(G, \Pi)$ is not larger than $\text{ew}(G, \Pi)$, also the face-width must be small. R. P. Vitray (see [13]) asked if $\text{fw}(G, \Pi) < 10^{10}$ for every embedding $\Pi$ which is not of genus $g(G)$. He even asked if $\text{fw}(G, \Pi) \leq 2$ for every such embedding. Robertson (private communication) has verified this when $G$ is a 3-connected planar graph. We shall present a short proof of this result and then we show that it does not extend to 3-connected cubic toroidal graphs.

**THEOREM 8.1.** If $\Pi'$ is an embedding of genus $> 0$ of a planar 3-connected graph $G$, then

$$\text{fw}(G, \Pi') \leq 2.$$ 

**Proof.** Let $\Pi$ be the planar embedding of $G$. If $G$ has a $\Pi'$-facial walk $W$ which is not a cycle, then a shortest closed subwalk of $W$ of length $> 0$ is a cycle $C$. Since $G$ has no cutvertex, $C$ is not $\Pi'$-contractible and hence $\text{fw}(G, \Pi') = 1$ in this case. So assume that all $\Pi'$-facial walks are cycles.

If $G$ has a vertex $v$ such that the $\Pi'$-ordering around $v$ is neither the $\Pi$-ordering nor its inverse, then $G$ has two $\Pi'$-facial cycles $C$ and $C'$ that
cross (in the planar drawing $\Pi'$) at $v$. If the $\Pi'$-ordering around every vertex of $v$ is either the $\Pi$-ordering or its inverse, then we consider two adjacent vertices, $u, v$ such that $\Pi$ and $\Pi'$ are the same at $v$, and $\Pi$ is the inverse of $\Pi'$ at $u$. Then the two $\Pi'$-facial cycles $C, C'$ containing $uv$ "cross" at the edge $uv$. In any case $C$ and $C'$ will meet again at a vertex $z \neq v, u$ and now we let $C''$ be a cycle which is the union of a segment of $C$ and a segment of $C'$ such that none of these segments consists of $v$ or $e$ only. Since $G$ is 3-connected, $C''$ is not $\Pi'$-contractible. Hence

$$fw(G, \Pi') = 2.$$  

**Theorem 8.2.** For each natural number $g$ there exists a cubic 3-connected toroidal graph which has an embedding of genus $> g$ and face-width $> 3$.

**Proof.** Consider the graph $G_k$ on the torus indicated in Fig. 1 below. Suppose that each vertical or horizontal line meets $k$ (or no) 4-cycles. Then $G_k$ has $8k^2$ vertices, $12k^2$ edges and $4k^2$ faces (on the torus). Now consider the embedding $\Pi$ such that the ordering agrees with the toroidal embedding precisely at the black vertices in Fig. 1. Then $\Pi$ has $2k^2 + 4k$ facial cycles and hence $\Pi$ has genus $k^2 - 2k + 1$. It is easy to see that $fw(G_k, \Pi) > 3$.

**References**

5. J. P. Hutchinson, Automorphism properties of embedded graphs, *J. Graph Theory* 8 (1984), 35–49.