Practical Wireless Network Coding and Decoding Methods for Multiple Unicast Transmissions

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Abstract

We propose a simple yet effective wireless network coding and decoding technique. It utilizes spatial diversity through cooperation between nodes which carry out distributed encoding operations dictated by generator matrices of linear block codes. For this purpose, we make use of greedy codes over the binary field and show that desired diversity orders can be flexibly assigned to nodes in a multiple unicast network, contrary to the previous findings in the literature. Furthermore, we present the optimal detection rule for the given model that accounts for intermediate node errors and suggest a network decoder using the sum-product algorithm. The proposed sum-product detector exhibits near optimal performance. We also show asymptotic superiority of network coding over a method that utilizes the wireless channel in a repetitive manner and give related rate-diversity trade-off curves. Finally, we extend the given encoding method through selective encoding in order to obtain extra coding gains.

Index Terms

Wireless network coding, cooperative communication, linear block code, sum-product decoding, unequal error protection

I. INTRODUCTION

In order to counteract the effects of fading in wireless communication networks, many ways of creating diversity for transmitted data have been proposed. Utilizing the spatial diversity inherent in wireless channels, cooperative communication [1] has been of great interest in recent years. In [2], [3] three methods to be used by relay nodes are described: amplify-forward (AF), decode-forward (DF) and detect-forward (DetF). The AF method attains full diversity, whereas other two cannot, unless the propagation of errors resulting from the decoding operation is avoided. One of the various ways to handle this problem is using

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CRC-based methods, which results in loss of spectral efficiency due to drop of a packet with only a few bit errors. An on/off weighting based on relay signal-to-noise power ratio (SNR) is given in [4]. Weighting of the signals either at the relay or at the receiver using the relay error probability is proposed in [5], [6]. Yet another idea is transmitting the log-likelihood ratios (LLR) of bits [7]. However, the soft information relaying methods in [5]–[7] suffer from quantization errors and high peak to average ratio problems. In addition, the AF method requires non-standard analog signal processing operations and the DF method leads to high complexity decoding operations especially for the relays. As an alternative, relays may use the simple DetF method, which is shown to avoid error propagation in [3], if the error probabilities at relays are known and the maximum a posteriori probability (MAP) detection is employed at the receiver. In this paper, we will concentrate on MAP-based detection rules at the receiver for achieving high diversity orders and DetF at the intermediate (relay) nodes due to ease of implementation and compatibility with currently established standards.

Network coding (NC) was initially proposed to enhance network throughput in wired systems with error-free links of unit capacity [8]. Later studies exhibited the good performance of random linear NC [9]. In wireless networks with nodes naturally overhearing transmissions, NC can be utilized to create diversity, reduce routing overhead, and introduce MAC layer gains as discussed for practical systems in [10]. Although most of the work in the literature concentrate on multicast transmission [11]; we deal with a network involving multiple unicast transmissions, which is inherent in real-life scenarios. Hence we formulate a multiple unicast transmission problem such that for each unicast transmission, there is a distinct diversity that is to be improved via spatial opportunities. The major goal in this paper is to introduce practical NC/decoding methods for improving the diversity order of a network through cooperation with the overall rate of transmission in mind.

We consider a simple NC scheme based on DetF. Given a relay combining strategy, which we represent by a generator matrix and a vector of transmit schedule, we investigate the diversity order of each source, which can be unequal. We propose a novel method for designing the generator matrix based on greedy codes over the binary field. The proposed method is very flexible in that any set of desired diversity levels for the sources can be achieved with the highest NC rate possible. The diversity analysis relies on the fact that an optimal MAP detector which employs the reliability information of the relays, avoids loss of diversity due to error propagation [3]. The numerical complexity of the MAP detector can be impractical. Thus we propose a practical approximation of the MAP detector: the sum-product detector.
A study based on flexible network codes in a two-source two-relay system with emphasis on unequal error protection is [12], where authors propose a suboptimal detection rule (distributed minimum distance detector) that is known to result in diversity order loss. Note that our model is more general, and captures full diversity due to the use of the sum-product detector with intermediate node reliability information. One of the studies closest to ours is [13], where the NC operation is fixed in construction yielding very large Galois Field (GF) sizes for increasing network size and relay nodes carry out complex DF operation for each transmission they overhear. Similarly in [14], DF is used in a fixed single-relay two-user scenario in order to provide diversity-multiplexing trade-off for NC. However, our generalized results indicate that any diversity order can be achieved for any unicast transmission even with the GF of size 2 by using greedy codes and simple DetF operation. Also independent from our work, in [15], similar results concerning diversity analysis for a system model resembling ours have been obtained using different methods.

The contributions of the current paper can be listed as follows. (1) Representation of wireless NC with generator matrix and transmission vector. (2) Unequal diversity analysis utilizing this representation. (3) Design of novel network codes based on greedy codes. (4) Investigation of greedy codes and maximum code rates for desired diversity levels. (5) Investigation of sum-product algorithm for decoding network codes with relay reliability information. The rest of the paper is organized as follows. In Section II, we present the wireless network model, the corresponding detection rule that is individually optimal for each user, and a practical enhancement on the proposed network encoding method. We investigate the details of network codes based on linear block codes with emphasis on greedy NC through some representative networks in Section III. We also give some asymptotic results based on the rate-diversity order trade-off curves for the proposed greedy NC method and a repetition coding method that represents a non-cooperative technique with no NC. In Section IV, we introduce the sum-product network decoder that has linear complexity order in terms of number users and intermediate node events and yields performance figures very close to that of the optimal decoder. Section V includes the numerical results for the mentioned network encoding and decoding methods and Section VI concludes the paper.

II. WIRELESS NETWORK MODEL

A. Detect and forward wireless network

In this work, we analyze a wireless network in which unicast transmission of data symbols, each belonging to a different source, is to be carried out utilizing NC at the intermediate nodes. Under the
general operation scheme, every node may act both as a member (source or destination) of a unicast
communication pair and as an intermediator (relay) node for other unicast pairs. Consider a subset of
nodes in which there are $k$ nodes transmitting data to a single receiver node, called node 0 henceforth,
and every transmission is heard by every other node. We are going to develop a network model based on
multiple unicast transmissions for this subset of nodes. Let the symbol transmitted by node $i$ be denoted
by $u_i$, for $i \in \{1, \ldots, k\}$. We assume $u_i$ to be statistically independent. The receiver is the destination
for one or more of the source symbols, and may act as a relay for the other sources’ symbols. Therefore,
the receiver may try to detect the data symbols for which it is the destination with a higher priority and
others with lower priority.

The transport of $k$ symbols is realized over $n$ transmissions, which will in fact form a round of network
coded communication (see Fig. 1 for a simple network with $k = 3$ and $n = 4$). We assume that these
transmissions are done in orthogonal channels. Moreover, the channel is assumed to be shared by a time
division multiple access technique for the sake of simplicity in model description and due to causality
requirements forcing the intermediate nodes to listen to a symbol before combining it through NC.

Let $u = [u_1 \ u_2 \ldots \ u_k]$ be the combined data vector for $k$ source nodes in the network, where $u_i$
is an element from the Galois field of size $q$, GF($q$). In time slot $j \in \{1, \ldots, n\}$, a transmitting node
$v_j \in \{1, \ldots, k\}$ forms a linear combination of its own and other nodes’ data. If $v_j$ detects all data to
be encoded correctly, it simply forms $c_j = u g_j$, where $g_j$ is a $k \times 1$ network encoding vector whose
entries are elements of GF($q$). Let $\hat{u}_i$ denote the estimate of the symbol of node $i$ at node $v_j$. Using these
estimates, node $v_j$ forms the noisy network coded symbol $\hat{c}_j = \hat{u} g_j$ that is also an element of GF($q$).
Then $v_j$ modulates and transmits this symbol to receiver node 0 as:

$$s_j = \mu(\hat{c}_j),$$  

where $\mu(.)$ denotes the mapping of a coded symbol to a constellation point. Although symbols may come
from any alphabet and non-binary constellations may be used, we will focus here on GF(2) and binary
phase-shift keying (BPSK) with $s_j = 1 - 2c_j$. Our assumption is that each vector $g_j$, source address $v_j$
and probability of error $p_e_j$ for the transmitted symbol are simply appended to the corresponding packet
with a negligible rate loss for packets of sufficient sizes and hence all of them are known at the receiving
nodes. We consider transmissions with no channel coding and deal with single network coded data symbol
$c_j$ which is a representative of all symbols within a packet transmitted by node $v_j$. At the end of a round
of transmissions, if no errors occur at the intermediate nodes, the overall vector of \( n \) symbols coded cooperatively in the network can be written as

\[
c = [c_1 \ c_2 \ \ldots \ c_n] = u [g_1 \ g_2 \ \ldots \ g_n] = uG,
\]

(2)

where \( G \) is the generator matrix (called the transfer matrix in [13]). The generator matrix characterizes the network code. However the generator matrix alone is not enough for a complete characterization, as it does not specify the order or transmissions. For a complete characterization we also need the vector of transmitting nodes, which is denoted by

\[
v = [v_1 \ v_2 \ \ldots \ v_n].
\]

(3)

The choices \( u, G, k, n \) for the parameters defining the operation of network are not arbitrary. They are used intentionally to point out the analogy to regular linear block codes. However, reliable detection of all data symbols, i.e., whole block \( u \), originating from a single error-free source is of interest for a regular decoder; whereas node 0 may desire to reliably detect, as an example, only \( u_1 \) using \( c \) which is cooperatively encoded. This difference and the diversity order for a symbol are clarified in Section III-A.

B. Optimal Network Decoding Using Relay Reliability Information

The intermediate nodes are assumed to use the DetF technique (hard decision with no decoding operation) due to its simplicity. In a wireless network, an intermediate node \( v_j \) has a noisy detection result \( \hat{u} \) of \( u \). Let us express the resulting noisy network coded symbol as

\[
\hat{c}_j = c_j + e_j,
\]

(4)

where \( e_j \) denotes this propagated error and we observe that a possible error in \( \hat{u} \) propagates to \( \hat{c}_j \) after the network encoding operation dictated by \( g_j \) is realized. We assume that node \( v_j \) knows the probability mass function of \( e_j \), or equivalently the relay reliability information. This assumption is not unrealistic as it can be determined by the estimation of the channel gains of the links connected to node \( v_j \), along with the reliability information forwarded to node \( v_j \). The received signal by node 0 at time slot \( j \) is then \( y_j = h_j s_j + w_j \), where \( h_j \) is the channel gain coefficient resulting from fading during the \( j \)th slot for the link between \( v_j \) and node 0 and \( w_j \) is the noise term for the same link. The gain coefficient is circularly symmetric complex Gaussian (CSCG), zero-mean with variance \( E_s \), i.e., it has distribution \( \mathcal{CN}(0, E_s) \).
The noise term is CSCG with \( \mathbb{CN}(0, N_0) \). The usual independence relations between related variables representing fading and noise terms exist. The overall observation vector of length \( n \) at node 0 is

\[
y = Hs + w, \quad (5)
\]

where \( y = [y_1 \ldots y_n]^T, s = [s_1 \ldots s_n]^T = \mu(\hat{c}^T), w = [w_1 \ldots w_n]^T \) and \( H \) is a diagonal matrix whose diagonal elements are independent channel gains \( h_1, h_2, \ldots, h_n \) for the links connected to node 0. It is assumed that \( H \) is perfectly known at the receiver. Combining the coded symbols in a network code vector, we obtain

\[
\hat{c} = c + e = uG + e, \quad (6)
\]

where \( e = [e_1 \ldots e_n] \) is the error vector. We assume that \( e \) is independent of \( c \) although dependence can be incorporated in the sum-product network decoder developed in Section II. This independence assumption is valid directly for BPSK modulation, whereas in a general modulation scheme the Euclidean distances between various constellation point pairs differ and an error term \( e_i \) depends on the symbol being transmitted. As a result, using (4), (5), and (6), the observation vector at node 0 is

\[
y = H \mu(uG + e)^T + w. \quad (7)
\]

Thus node 0 has access to the likelihood \( p(y|u, e) \) and \( p(e) = \prod_{j=1}^{n} p(e_j) \), assuming the errors are independent. As shown in [3], in order to avoid the propagation of errors occurring at intermediate nodes, node 0 has to utilize the reliability information \( p(e) \). Then, the a posteriori probability of the source bit of interest, say \( u_1 \), can be calculated by using the Bayes’ rule:

\[
p(u_1|y) = \sum_{u_2, \ldots, u_k} \sum_{e_1, \ldots, e_n} \alpha \, p(y|u, e) \prod_{j=1}^{n} p(e_j), \quad (8)
\]

where \( \alpha \) is a normalizing constant that does not depend on \( u_1 \). The MAP estimate of \( u_1 \) at node 0 is denoted by \( \hat{u}_1 \) and obtained as

\[
\hat{u}_1 = \arg \max_{u_1} p(u_1|y) = \arg \max_{u_1} \sum_{u_2, \ldots, u_k} \sum_{e_1, \ldots, e_n} p(y|u, e) \prod_{j=1}^{n} p(e_j), \quad (9)
\]

which is the individually optimum detector for \( u_1 \). As a result, for the optimal detection of \( u_1 \), the receiver node needs the relay reliability information vector: \( p_e = [p_{e_1} \ldots p_{e_n}] \), where \( p_{e_j} \) depends on the probability mass function of \( e_j \). We will observe the performance of this detection rule in Section V-A.
with the assumption that instantaneous reliability information for each bit of codeword \( \hat{c} \) is appended to the packet by the intermediate node.

The main problem related to the MAP-based detection rule of (9) is the complexity of required operations, which grows exponentially both in the number of nodes \( k \) and the number of possible error events \((n - k)\). This is handled in Section IV where we suggest a practical network decoding technique.

C. Selective Network Coding

The NC described in Section II-A is a static method in the sense that the generator matrix \( G \) is fixed. In static NC, node \( v_j \) always combines the symbols of a pre-determined set of users, even when it knows that the reliability for one of those users is low. When a symbol estimate with low reliability is combined with a symbol with high reliability, the reliability of the resulting network code symbol is low. Thus, it is intuitive to expect some gains in performance by forcing the intermediate nodes not to combine (network encode) the symbols that have very low instantaneous reliability. In [16] and [4], various forms of channel state information are used to determine thresholds for relaying decisions. In [13] and [17], for relays assuming DF operation, successful decoding of channel code for a source is the required condition for combining its data in the network encoded data. Here, we propose a method called Selective Network Coding (SNC) that imposes a certain probability threshold value on the reliability of the candidate symbols to be encoded at intermediate nodes that adapt DetF. In this way, any symbol that is sufficiently reliable is included in network encoding and the resulting encoding vector \( g_j \) is appended to the transmitted packet so that node 0 still has the instantaneous generator matrix \( G \) at the end of \( n \) transmissions. Clearly, SNC inherently includes usage of random generator matrices although the utilized generator matrix may assume in average a form dictated by some predetermined (and optimal if possible) linear block code structure like the greedy code generator matrices that are to be discussed in Section III-B. The improvement in BER performance for sample network coded systems using SNC is shown in Section V-D. The optimization of the reliability thresholds is not handled in this work. Rather, expected error probabilities for each transmission without SNC are taken as the thresholds on the instantaneous error probability values.

III. LINEAR BLOCK CODES UTILIZED AS NETWORK CODES

Our goal is now to explore the error performance metrics for network coding/decoding described in Section II. When the conventional block coding is considered, the average error performance over all
data symbols is of interest. Therefore, for a linear block code, the metric utilized for comparison is the minimum distance. However, there is a vector of distinct minimum distances (defined as separation vector in [19]) for data symbols, whenever we are interested in performance of individual symbols that originate from different source nodes as with NC. This idea is exemplified in [12] in the context of NC for simple networks. We will generalize and use this idea for investigating diversity orders assigned to source symbols in a network. For demonstration, let us start with a simple $q$-ary symmetric channel model for the transmission of each one of the $n$ symbols. Then we denote the received vector at node 0 as

$$r = uG + t,$$  \hspace{1cm} (10)

where $t$ is the $1 \times n$ error vector of independent terms from GF$(q)$. For the case of conventional coding, the joint MAP decoding

$$\hat{u} = \arg \max_u p(u|r) = \arg \max_u p(r|u)$$  \hspace{1cm} (11)

is used, where all vectors of data symbols (all possible $u$ vectors) are assumed to be equally likely. Therefore, an error is the event that at least one of the detected symbols $\hat{u}_i$ is different than the original symbol $u_i$, i.e., $\hat{u} \neq u$. In contrast, in NC, the priority of individual sources may happen to be different from the point of view of receiver and erroneous detection of high-priority symbols may determine the performance figure. Here, the optimal way of detecting distinct symbols follows individual MAP decoding:

$$\hat{u}_i = \arg \max_{u_i} \sum_{\{u_1, \ldots, u_k\}\backslash u_i} p(r|u_1, u_2, \ldots, u_k)$$  \hspace{1cm} (12)

and we are interested in errors $\hat{u}_i \neq u_i$, where priority may depend on $i$. In this work, we assume instantaneous intermediate node error probability knowledge and MAP detection at the receiver. Hence we may directly make use of the minimum distance values for a given generator matrix $G$ in determining the error performance through diversity orders without considering error propagation due to transmissions at the intermediate nodes. In the following sections, we identify the minimum distances for each source symbol uniquely in order to characterize the error performance of a network coded system.

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1 Minimum distance is equal to the diversity order in the case that independent channels are used for transmission of coded symbols [18].
A. A Network Code Example

Let us consider an example network code with \( n = 4 \) transmission slots, \( k = 3 \) sources and transmissions over \( \text{GF}(2) \) with data rate \( r = \frac{k}{n} = \frac{3}{4} \) bits/transmission:

\[
G = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}, \quad \mathbf{v} = [1 \ 2 \ 3 \ 2].
\]

(13)

According to the generator matrix \( G \) and the vector of transmitting nodes \( \mathbf{v} \), in the first two time slots (corresponding to the first two columns of \( G \) and the first two entries of \( \mathbf{v} \)), node 1 and node 2 transmit \( u_1 \) and \( u_2 \) respectively. In the third time slot, node 3 encodes its own data symbol \( u_3 \) together with the detection result at the first time slot \( \hat{u}_1 \) through a simple XOR operation over \( \text{GF}(2) \): \( \hat{c}_3 = \hat{u}_1 + u_3 \). In the last slot, once again node 2 uses the channel to transmit the network encoded data \( \hat{c}_4 = \hat{u}_1 + u_2 \) with its own estimate of \( u_1 \). This single round of network coded transmissions is summarized in Fig. 1. One can show that the minimum distance for \( G \) is 1. However, we will see that an error event requires at least 2 bit errors for detection of \( u_1 \) at receiver node 0.

Let all the data bits be equal to 0 without loss of generality, i.e., \( \mathbf{u} = [0 \ 0 \ 0] \). Hence the transmitted codeword is expected to be \( \mathbf{c} = [0 \ 0 \ 0 \ 0] \) in case of no intermediate node errors. The error event for \( u_1 \) corresponds to its detection as 1. This erroneous detection can occur for sequence detections \( \hat{\mathbf{u}} \in \{[100], [101], [110], [111]\} \). The incorrect codewords \( \hat{\mathbf{c}} \) corresponding to these detected vectors are \([1011], [1001], [1110], [1100] \), respectively. When these codewords are compared to the codeword \([0000] \), it is clear that at least 2 bit errors are needed to cause an error event. Hence the diversity order for \( u_1 \) in this setting is said to be 2. The erroneous detection for other bits can be investigated in a similar fashion. Focusing on \( u_3 \) and hypothesizing \( \mathbf{u} = [000] \), \( u_3 \) is incorrectly detected when \( \hat{\mathbf{u}} \in \{[001], [011], [101], [111]\} \). The corresponding codewords are \([0010], [0111], [1001], [1100] \). Therefore, a single bit error can cause erroneous detection of \( u_3 \). As seen in the example, the error performance for a particular data symbol may differ from that for another. This claim is verified through simulations in Section V.

Generalizing the method for obtaining the diversity order corresponding to \( u_i \), one can make use of Algorithm 1. In Algorithm 1, the function \( \text{dec}2\text{GF}q(.) \) returns a q-ary pattern corresponding to the input decimal number and the function \( \text{numberofnonzero}(.) \) returns the number of non-zero entries in the input vector. The algorithm searches within all possible codewords leading to erroneous detection of \( u_i \) and


Algorithm 1 Algorithm for finding the diversity order corresponding to symbol $u_i$ for a $(n, k, d)$ code with given generator matrix $G$

\[
\begin{align*}
\text{minimumdistance} & \leftarrow n \\
\text{indexvector} & \leftarrow [1 \ 2 \ \cdots \ k] \setminus i \\
\text{for } j = 1 \text{ to } 2^{k-1} \text{ do} \\
\text{errorpattern} & \leftarrow \text{dec2GF}q(j - 1) \\
\text{errdatavector}[i] & \leftarrow 1 \\
\text{errdatavector}[\text{indexvector}] & \leftarrow \text{errorpattern} \\
\text{errcodevector} & \leftarrow \text{errdatavector} \ast G \\
\text{errcodedistance} & \leftarrow \text{numberofnonzero(errcodevector)} \\
\text{minimumdistance} & \leftarrow \text{min}(\text{minimumdistance}, \text{errcodedistance}) \\
\text{end for}
\end{align*}
\]

returns the minimum distance as the diversity order of it. Since diversity order is an asymptotic quantity that is defined for SNR values tending to infinity, the intermediate node errors and the exact form of $v$ are irrelevant to the procedure used for obtaining a diversity order value. On the other hand, it is wiser that each column $g_j$ of $G$ should be used as the encoding function of a $v_j$ such that $g_j(v_j) \neq 0$, otherwise possibly an extra relaying error is also included in the encoded data symbol. Therefore, $v$ clearly affects the coding gain corresponding to the bit error rate (BER) versus SNR curve of $u_i$.

B. Greedy Codes

In this study, we make use of some well-known linear block codes while constructing network codes that are to be used for the analysis of data rate and diversity orders for distinct symbols in Section III-C and simulation of BER in Section V. However, the cooperative network coded operation described in this work and the resulting performance figures for a unicast pair are more general and applicable to any linear block code.

In comparison with the network coded operation, we consider the repetition coding scheme which is in fact a degenerate NC scheme. In repetition coding, the nodes transmit in the order dictated by the same $v$ vector and each source node simply transmits its own data repeatedly with no combining operation over GF($q$). Following the $n$ transmissions of $k$ source/intermediate nodes, the receiver node combines the data received for each source symbol optimally to generate the detection results. On the other hand, we rely on the family of block codes known as greedy codes with network coded operation. These $(n, k, d)$ codes are selected with the following parameters: blocklength (number of transmission slots) $n$, dimension (number of unicast pairs) $k$, and minimum distance (minimum diversity order) $d$. Greedy codes are known to satisfy or be very close to the optimal dimensions for all blocklength-minimum distance pairs [20].
Moreover, they are readily available for all dimensions (number of nodes) and minimum distances unlike some other optimal codes. Hence, even in an ad hoc wireless network with time-varying size (number of nodes), any desired diversity order can be satisfied by simply broadcasting the new greedy code generator matrix $G$ to be utilized in subsequent rounds of communication.

As an example, let us consider a network that consists of $k = 3$ nodes transmitting their data symbols over GF(2). If a round of communication is composed of $n = 6$ transmission slots, we deal with codes of type $(6, 3, d)$, which have a code rate of $\frac{1}{2}$. Starting with the generator matrix and transmitting node vector corresponding to the repetition coding, we have

$$G = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}, \quad v = [1 \ 2 \ 3 \ 1 \ 2 \ 3].$$  \hspace{1cm} (14)

It is easily observed that, since each data bit is transmitted twice over independent channels, this method satisfies only a diversity order of 2 for all bits $u_1$, $u_2$, and $u_3$. In contrast, a diversity order of 3 for all sources can be achieved using NC, with the same code rate. As an example, the NC that achieves this performance can be obtained using the $(6, 3, 3)$ greedy code, as follows:

$$G_1 = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}, \quad v_1 = [1 \ 2 \ 3 \ 1 \ 2 \ 3].$$  \hspace{1cm} (15)

It should also be noted that greedy codes accommodate each unicast pair with equal diversity order due to the greedy algorithm utilized in their construction. Moreover, contrary to the findings in [13], it is easy to obtain any required diversity order for any data bit even using GF(2). There is no limitation due to the number of unicast pairs in terms of the desired diversity order as long as each one of the $n$ symbol transmissions to each node is realized over independent channels, which is a natural assumption for many wireless communication scenarios. If we need an increase in data rate, through a trade-off mechanism, we can assign decreased diversity orders to the lower-priority unicast pairs. This is accomplished by omitting some columns of a greedy code generator matrix in order to decrease number of transmissions. The columns to be excluded can be decided by running Algorithm [1] in Section [III-A] on candidate punctured generator matrices. As an example, the following punctured $(5, 3, 2)$ code is obtained by omitting the last
column of $G_1$ and has a data rate $\frac{3}{5}$ that is higher than those of above two codes:

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad v_2 = [1 \ 2 \ 3 \ 1 \ 2].$$  

(16)

This punctured network code satisfies a diversity order of 3 for $u_1$ and an order of 2 for both $u_2$ and $u_3$. If $u_1$ is of higher priority, this unequal error protection would be preferable especially when the higher rate of the code is considered. In case of larger diversity order requirement, $d = 4$ as an example, we may simply utilize the $(7, 3, 4)$ greedy code with rate $\frac{3}{7}$.

$$G_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad v_3 = [1 \ 2 \ 3 \ 1 \ 2 \ 3 \ 1].$$  

(17)

A final problem is the selection of vector $v$. Our basic assumption is that $mathbf{v}$ satisfies causality so that no intermediate node $v_j$ tries to transmit another node’s symbol before hearing at least one copy of it. This causality problem can be solved trivially by using only systematic generator matrices. For the transmitting nodes corresponding to the non-systematic part of $G$, as described in Section III-A, one can select each entry $v_j$ such that $g_j(v_j) \neq 0$ for each column $g_j$. For the columns that have more than one non-zero entry, a random selection between candidate $v_j$’s will merely affect the coding gains assigned to these nodes. As a result, one can force the number of transmissions of each node within a round to be equalized as much as possible for similar coding gain improvements of nodes. In the way exemplified in this section, one can choose a network code satisfying desired error protection properties for a determined network size with adequate data rate quite flexibly.

C. Theoretical Gains in Rate and Diversity for NC

In this section, we investigate the asymptotic (high-SNR) rate and diversity gains of NC through use of the family of greedy network codes detailed in Section III-B, although the results are still valid for any other family of optimal or close-to-optimal codes. Fig. 2 shows the diversity gains attainable using greedy NC (with punctured codes in case no corresponding greedy code exists) with respect to the repetition coding scenario. The rate-diversity trade-off curves of both cases are plotted for a network of $k = 3$ nodes with increasing number of transmissions and hence decreasing rate. We are interested in three types of
network diversity orders; average, minimum and maximum, since the orders corresponding to each one of
the three nodes may be unequal in general. The curves with no markers represent the (average) network
diversity orders for both scenarios, which is defined as the arithmetic mean of orders for three nodes.
For a rate of 0.43 bits/transmission, with greedy code (7, 3, 4), the network diversity order for NC is 4.
The minimum, maximum, and average diversity orders are equal for this case. In contrast, the repetition
scheme results in an average order of nearly 2.33 with the worst node observing a minimum order of 2
and the best node a maximum order of 3, which would mean a high SNR loss asymptotically for all three
nodes in the network.

In Fig. 3, we now fix the desired network diversity order to \( d = 3 \) and observe the rate advantage of
the NC for increasing network size. Note that for all cases diversity orders for \( k \) users are equal to 3.
For a network of \( k = 25 \) nodes, the rate with NC is \( \frac{25}{30} \) (with greedy code \( (30, 25, 3) \)) and the rate of the
repetition scheme is \( \frac{15}{25} \) (always equal to \( \frac{1}{2} \) for a diversity order of 3). The rate advantage ratio is then 2.5.
In the asymptotic case, as \( k \to \infty \) and hence as \( n \to \infty \), NC using optimal codes in construction will
have a rate advantage converging to 3 since the rate for network coded case can be shown to tend to 1
using the Gilbert-Varshamov bound [21] for arbitrarily large \( n \). In general, the rate advantage of NC over
the repetition scenario becomes simply \( d \), the desired network diversity order. As a result, increasing the
network size improves the network coded system’s efficiency in comparison to repetition coding.

IV. SUM-PRODUCT NETWORK DECODER

It is clear that the complexity of the optimal rule for detection of any unicast transmission symbol \( u_i \)
grows exponentially, since the number of additions and multiplications in (9) increase exponentially in the
number of users \( k \) and the number of transmissions \( n \). Therefore, this rule becomes quickly inapplicable
even for moderate-size networks. Recently the sum-product iterative decoding, which is often utilized for
decoding of low-density parity-check (LDPC) codes, is suggested for decoding general linear block codes
as well [22].

Here, under the Rayleigh fading assumption detailed in Section II-B we make use of sum-product
decoding and compare its performance with that of the optimal detection rule. We add a variable node
for each coded symbol \( c_j \) and a check node for each data symbol \( u_i \) and for each observation \( y_j \) in order
to describe the underlying linear block code structure of the network code. The only difference with the
regular linear block code decoder is that the check nodes corresponding to the observations should include
the effect of relay error events. Hence for the NC system given in (13), we refer to the graph presented
in Fig. 4 for sum-product decoding at the receiver node, namely node 0. It is seen that this simple graph of Fig. 4 has no cycles in it.

The iterative sum-product algorithm requires the log-likelihood ratios (LLRs) for the check nodes $u_1, u_2, u_3, y_1, y_2, y_3, y_4$ at the initialization step. The data bits ($u_i$'s), which are assumed to be 0 and 1 with equal probability, should be simply initialized to LLR values of 0. The LLR of an observation node is calculated using the corresponding relay reliability information in addition to the known channel gain based on (7). Since $y_j = h_j \mu(ug_j + e_j) + w_j$, defining $x_j \triangleq ug_j + e_j$, the LLR of $y_j$ is obtained as

$$LLR(y_j) = \ln \frac{p(y_j|ug_j = 0)}{p(y_j|ug_j = 1)} = \ln \frac{(1 - p_{e_j})p(y|x_j = 0) + p_{e_j}p(y|x_j = 1)}{(1 - p_{e_j})p(y|x_j = 1) + p_{e_j}p(y|x_j = 0)}$$

$$= \ln \frac{(1 - p_{e_j}) p(y|x_j = 0)}{p_{e_j} p(y|x_j = 1)} + 1$$

$$= \ln \frac{(1 - p_{e_j})}{p_{e_j}} + \frac{p(y|x_j = 0)}{p(y|x_j = 1)}, \quad (18)$$

where $p_{e_j}$ is the probability that the relaying node at time $j$ made error(s) during detection of an odd number of data bits that are used in its NC rule $g_j$. Next we define

$$LLR(e_j) \triangleq \ln \frac{1 - p_{e_j}}{p_{e_j}} \quad \text{and} \quad LLR(y'_j) \triangleq \ln \frac{p(y|x_j = 0)}{p(y|x_j = 1)} = 4\Re\{h^*_j y_j\}/N_0, \quad (19)$$

where $h^*_j$ is the conjugated gain of the channel over which the modulated symbol $s_j = \mu(\hat{c}_j)$ is transmitted by node $v_j$ and we use the fact that $w_j$ is Gaussian distributed (see Section II-B) in obtaining $LLR(y'_j)$. As a result the LLR for node $y_j$ is computed as

$$LLR(y_j) = \ln \frac{\exp(LLR(e_j)) \exp(LLR(y'_j)) + 1}{\exp(LLR(e_j)) + \exp(LLR(y'_j))}. \quad (20)$$

Following the initialization step, the sum-product decoder carries on iterations over the given Tanner graph to generate the estimated a posteriori LLRs for the data bits. If the number of iterations is fixed, the sum-product decoder utilized is known to have a complexity order of $O(n)$. In contrast, the optimal decoder has a computational load in the order of $O(2^n)$, which makes the sum-product network decoder a strong alternative for increasing network size and number of transmissions. The number of iterations used and other operational parameters for the sum-product decoder are given in Section V-C, where we show that performance figures comparable to that of the optimal decoder are possible.
V. Numerical Results

A. Sample Network-I: Simulation Results

The results in this subsection are based on Sample Network-I of (13), consisting of only 4 nodes in order to observe the fundamental issues. For BER results, at least 100 bit errors for each data bit \( u_1 \), \( u_2 \), and \( u_3 \) are collected through Monte Carlo simulations for each SNR value. In each run, data bits, intermediate node errors and complex channel gains are randomly generated with their corresponding probability distributions. The solid lines in Fig. 5 show the BER values for the optimal detector operating under the realistic scenario of intermediate node errors, whereas the dashed lines depict the performance of the genie-aided no-intermediate-error network with the same optimal detection. Finally, the dotted lines are for the detector that neglects possible intermediate errors.

It is observed in Fig. 5 that different diversity orders for bits of different nodes are apparent for optimal detection under intermediate errors. The diversity order for \( u_1 \) is observed to be 2 according to the slope of the corresponding BER curve. This is in agreement with the analytical results in Section III-A where it was shown that an error event corresponds to at least 2 bit errors for the detection of \( u_1 \) and \( u_2 \), so the diversity orders for these bits are 2 when there are no intermediate node errors. It is seen in Fig. 5 that the intermediate node errors cause no loss of diversity for \( u_1 \) and \( u_2 \), but an SNR loss of 1.5 dB. Hence the optimal detection rule of (9) is said to avoid the problem of error propagation. The loss for \( u_3 \), whose diversity order is 1, with respect to the hypothetical no-intermediate-error network is around 2.5 dB. The performance deteriorates significantly for especially \( u_1 \) and \( u_2 \) when intermediate errors are neglected in detection (dotted lines), i.e., \( p_{e_3} = p_{e_4} = 0 \) is assumed. Not only an SNR loss is endured but also the diversity gains for them disappear.

B. Sample Network-II: Simulation Results

Next, we verify the analytical results concerning the diversity orders for a set of three nodes operating under three different network codes constructed in Section III-B. Moreover, the unequal error protection performance of one of these codes is identified together with the rate advantage it provides.

The repetition method is represented by \( G \) and \( v \) in (14). To construct Code-1 and Code-2, we make use of the greedy code of (15) and the punctured greedy code of (16) respectively. Fig. 6 exhibits the BER curves for the repetition scenario with \( n = 6 \) transmissions (dashed lines), for NC scenarios with Code-1 with \( n = 6 \) (solid lines) and Code-2 with \( n = 5 \) (dotted lines). The optimal detector of (9) is utilized for
this simulation. Clearly, Code-1 has superior performance with an average network diversity order of 3. However, the lower rate of Code-1 (and also repetition coding) in comparison to Code-2 should also be noted. For Code-2, on the other hand, bits \( u_2 \) and \( u_3 \) observe a diversity order of 2 while \( u_1 \) observes an order of 3. With this unequal protection in mind, the average network diversity order for Code-2 is
\[
\frac{2+2+3}{3} \approx 2.33,
\]
which is higher than that of the repetition coding with order 2. In addition to improved diversity, Code-2 has also the advantage of increased overall rate and decreased decoding delay due to usage of 5 slots instead of 6. It is preferable especially for a network that puts higher priority on \( u_1 \) than on \( u_2 \) and \( u_3 \).

C. Performance of the Sum-Product Decoding for Network Coded Systems

In this section the performance figures for the sum-product iterative network decoder described in Section IV are presented in comparison with the optimal detection rule of (9), which has an exponential complexity order. The network coded communication system of interest is given in (15) with the corresponding transmitting node vector. The number of iterations for the sum-product type decoder is limited to 4 and no early termination is done over parity checks. Here, a minimum of 150 bit errors are collected for each data bit.

In Fig. 7 we identify the fact that the sum-product decoder maintains almost the same BER performance as the optimal decoding rule. The SNR loss due to usage of sum-product decoder is less than 0.1 dB for a BER value of \( 10^{-3} \) for all data bits. Achieving full-diversity with a linear complexity order, sum-product type decoding may serve as an ideal method for decoding in network coded wireless systems despite the fact that the corresponding Tanner graph contains cycles. Similar results were also reported previously in [22], [23] for loopy Tanner graphs.

D. Performance of Selective Network Coding (SNC)

The selective network encoding operation defined in Section II-C is applied in this section on the Sample Network-II of Section V-B. The performance improvement for the selective encoding over the static (using fixed \( G \) with no selection of symbols to be encoded) encoding method is again shown using the sum-product iterative decoder of Section IV. The instantaneous intermediate node error probabilities are compared with average error probabilities (dictated by \( G \)) and data of the nodes whose error probabilities are below the corresponding average values (thresholds) are combined by the intermediate node. In Fig. 8, we observe that SNC offers an SNR improvement of 0.6 dB for BER set to \( 10^{-3} \) over the static NC method.
E. Performance of NC under Slow-Fading Channel Model

All discussion and the results presented up until this section rely on the assumption that all channel gain coefficients related to the observations at node 0 are independent. Hence a block fading model over time slots is utilized. However, it is also possible under many communication scenarios that the variation of a channel gain coefficient is not rapid enough for such an assumption. Then it is also possible that all transmissions from a selected source node to node 0 observe the same fading condition leading to the degradation in BER performance due to loss in diversity. Therefore, we finalize the numerical results by providing the BER curves of NC and repetition coding under the assumption that within a round of \( n \) transmissions, only the transmissions from distinct source nodes observe independent fading, i.e., \( h_j \) and \( h_m \) are independent if \( v_j \neq v_m \) and otherwise \( h_j = h_m \). In Fig. 9 we investigate the BER curves for the Sample Network-II operating under this slow fading assumption. The repetition coding is represented by (14) and NC is realized by (15). It is seen that the repetition coding scheme merely results in a diversity order of 1 for each symbol as expected. On the other hand, NC yields an order of 2 via the cooperative diversity obtained due to intermediate nodes transmitting over independent channels. The SNR losses incurred by not utilizing NC are shown to further increase in great amounts for the slow-fading channel scenario.

VI. Conclusions

We formulated a NC problem for cooperative unicast transmissions. A generator matrix \( G \) and a vector of transmitting nodes \( v \) are used to represent the linear combinations performed at intermediate nodes. We derived a MAP-based detection rule utilizing \( G \), \( v \), and the error probabilities at the intermediate nodes. A method for obtaining the performance determining parameter as the diversity order for individual source nodes is proposed for any given \( G \). Although this method depends on the assumption that the intermediate node errors are negligible, through simulations we showed that our detection rule avoids error propagation in the system and hence yields no diversity loss by using reliability information for the network coded symbols. We presented design examples for network codes via greedy block codes, which may also provide unequal diversity orders to nodes with proper puncturing. Over given design examples, we obtained rate-diversity trade-off curves and the rate advantage realized by using NC instead of repeating source symbols. Moreover, the sum-product iterative network decoder with linear complexity order is proposed and shown to perform quite close to the optimal rule. Furthermore, the selective NC scheme combining only the
reliably detected data at cooperating nodes is shown to yield additional coding gains. Identifying gains of NC for purely random $G$ matrices in large networks, studying the effects of imperfect information on channel gains and relay error probabilities will be addressed in future work. Finally, it would be also interesting to operate suggested wireless NC methods under asymmetrical channel gains, which can be more realistic for ad hoc networks.

REFERENCES


Fig. 1. Sample network coded transmission scenario
Fig. 2. Network diversity orders for greedy NC and repetition coding for given data rates.
Fig. 3. Increasing rate advantage of greedy NC for increasing network size.

Fig. 4. Tanner graph for network coded system of [13]
Fig. 5. BER performance for data bits of different nodes for optimal detection.
Fig. 6. BER performance for repetition coding and NC with greedy codes.
Fig. 7. BER curves for the individual MAP decoder of (9) and the sum-product iterative decoder.
Fig. 8. Selective and static network encoding BER curves
Fig. 9. BER vs. SNR curves for slow fading channel