SOME HADAMARD LIKE INEQUALITIES VIA CONVEX AND s-CONVEX FUNCTIONS AND THEIR APPLICATIONS FOR SPECIAL MEANS

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Abstract. In this study, the author establish some inequalities of Hadamard like based on convex and s-convexity in the second sense. Some applications to special means of positive real numbers are also given.

1. Preliminaries

1.1. Definitions.

Definition 1. [15] *A function $f : I \to \mathbb{R}$ is said to be convex on I if inequality*

$$f\left(tx+(1-t)\,y\right)\leq tf\left(x\right)+(1-t)\,f\left(y\right) \tag{1.1}$$

holds for all $x,y\in I$ and $t\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Geometrically, this means that if P,Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below the chord PR.

Definition 2. [11] *Let $s\in(0,1]$. A function $f:(0,\infty]\to[0,\infty]$ is said to be s-convex in the second sense if*

$$f\left(tx+(1-t)\,y\right)\leq t^{s}f\left(x\right)+(1-t)^{s}\,f\left(y\right), \tag{1.2}$$

for all $x,y\in(0,b]$ and $t\in[0,1]$. This class of s-convex functions is usually denoted by K_s^2.

Certainly, s-convexity means just ordinary convexity when $s=1$.

1.2. Theorems.

Theorem 1. ***The Hermite-Hadamard inequality:*** *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and $u,v\in I$ with $u<v$. The following double inequality:*

$$f\left(\frac{u+v}{2}\right)\leq\frac{1}{v-u}\int_{u}^{v}f\left(x\right)dx\leq\frac{f\left(u\right)+f\left(v\right)}{2} \tag{1.3}$$

is known in the literature as Hadamard's inequality (or Hermite-Hadamard inequality) for convex functions. If f is a positive concave function, then the inequality is reversed.

2000 *Mathematics Subject Classification.* Primary 26A15, 26D10; Secondary 26D07, 26D15.
Key words and phrases. Hadamard' inequality, convexity, s-convexity.
This paper is in final form and no version of it will be submitted for publication elsewhere.

Theorem 2. [8] *Suppose that $f:[0,\infty)\rightarrow[0,\infty)$ is an $s-$convex function in the second sense, where $s\in(0,1]$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L_1([0,1])$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{u+v}{2}\right)\leq\frac{1}{v-u}\int_u^v f(x)\,dx\leq\frac{f(u)+f(v)}{s+1}. \tag{1.4}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.4). The above inequalities are sharp. If f is an s-concave function in the second sense, then the inequality is reversed.

For recent results and generalizations concerning Hadamard's inequality and concepts of convexity and s-convexity see [1]-[20] and the references therein.

Throughout this paper we will use the following notations and conventions. Let $J=[0,\infty)\subset\mathbb{R}=(-\infty,+\infty)$, and $u,v\in J$ with $0<u<v$ and $f'\in L[u,v]$ and

$$\begin{aligned}A(u,v)&=\frac{u+v}{2},\ G(u,v)=\sqrt{uv},\ I(u,v)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},\\ L_p(u,v)&=\left(\frac{v^{p+1}-u^{p+1}}{(p+1)(v-u)}\right)^{1/p},\ u\neq v,\ p\in\mathbb{R},\ p\neq-1,0\end{aligned}$$

be the arithmetic mean, geometric mean, identric mean, generalized logarithmic mean for $u,v>0$ respectively.

2. Some new Hadamard like inequalities

In order to establish our main results, we first establish the following lemma.

Lemma 1. *Let $f:J\rightarrow\mathbb{R}$ be a differentiable function on J°. If $f'\in L[u,v]$, then*

$$\begin{aligned}&\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\\ =&\frac{(v-x)^2}{(v-u)^2}\int_0^1(tu+(1-t)v)f'(tx+(1-t)v)\,dt\\ &+\frac{(x-u)^2}{(v-u)^2}\int_0^1(tv+(1-t)u)f'(tx+(1-t)u)\,dt\end{aligned}$$

for each $t\in[0,1]$ and $x\in[u,v]$.

Proof. Integrating by parts, we get

$$
\begin{aligned}
& \frac{(v-x)^2}{(v-u)^2}\int_0^1 (tu+(1-t)\,v)\, f'(tx+(1-t)\,v)\,dt \\
& +\frac{(x-u)^2}{(v-u)^2}\int_0^1 (tv+(1-t)\,u)\, f'(tx+(1-t)\,u)\,dt \\
= & \frac{(v-x)^2}{(v-u)^2}\left[(tu+(1-t)\,v)\,\frac{f(tx+(1-t)\,v)}{x-v}\bigg|_0^1-(u-v)\int_0^1\frac{f(tx+(1-t)\,v)}{x-v}dt\right] \\
& +\frac{(x-u)^2}{(v-u)^2}\left[(tv+(1-t)\,u)\,\frac{f(tx+(1-t)\,u)}{x-u}\bigg|_0^1-(v-u)\int_0^1\frac{f(tx+(1-t)\,u)}{x-u}dt\right] \\
= & \frac{(v-x)^2}{(v-u)^2}\left[\frac{uf(x)-vf(v)}{x-v}-\frac{v-u}{(x-v)^2}\int_x^v f(\mu)\,d\mu\right] \\
& +\frac{(x-a)^2}{(v-u)^2}\left[\frac{vf(x)-uf(u)}{x-u}-\frac{v-u}{(x-u)^2}\int_u^x f(\mu)\,d\mu\right] \\
= & \frac{(v-x)\,(vf(v)-uf(x))+(x-u)\,(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu.
\end{aligned}
$$

□

Theorem 3. *Let $f : J \to \mathbb{R}$ be a differentiable function on J°. If $|f'|$ is convex on $[u,v]$, then*

$$
\begin{aligned}
& \left|\frac{(v-x)\,(vf(v)-uf(x))+(x-u)\,(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq & \frac{(v-x)^2}{6\,(v-u)^2}\left[(2u+v)\,|f'(x)|+(u+2v)\,|f'(v)|\right]+\frac{(x-u)^2}{6\,(v-u)^2}\left[(u+2v)\,|f'(x)|+(2u+v)\,|f'(u)|\right]
\end{aligned}
$$

for each $x \in [u,v]$.

Proof. Using Lemma 1 and from properties of modulus, and since $|f'|$ is convex on $[u,v]$, then we obtain

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\int_0^1 (tu+(1-t)v)\,|f'(tx+(1-t)v)|\,dt \\
&+\frac{(x-u)^2}{(v-u)^2}\int_0^1 (tv+(1-t)u)\,|f'(tx+(1-t)u)|\,dt \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\int_0^1 (tu+(1-t)v)\,(t|f'(x)|+(1-t)|f'(v)|)\,dt \\
&+\frac{(x-u)^2}{(v-u)^2}\int_0^1 (tv+(1-t)u)\,(t|f'(x)|+(1-t)|f'(u)|)\,dt \\
=\ & \frac{(v-x)^2}{6(v-u)^2}\left[(2u+v)|f'(x)|+(u+2v)|f'(v)|\right] \\
&+\frac{(x-u)^2}{6(v-u)^2}\left[(u+2v)|f'(x)|+(2u+v)|f'(u)|\right].
\end{aligned}$$

The proof is completed. □

Theorem 4. *Let $f:J\to\mathbb{R}$ be a differentiable function on J°. If $|f'|$ is s-convex on $[u,v]$ for some fixed $s\in(0,1]$, then*

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\left[\frac{((s+1)u+v)|f'(x)|+(u+(s+1)v)|f'(v)|}{(s+1)(s+2)}\right] \\
&+\frac{(x-u)^2}{(v-u)^2}\left[\frac{(u+(s+1)v)|f'(x)|+((s+1)v+v)|f'(u)|}{(s+1)(s+2)}\right]
\end{aligned}$$

for each $x\in[u,v]$.

Proof. The proof of Theorem 4 is similar to Theorem 3. □

Remark 1. *In Theorem 3, if we take $s=1$, then Theorem 4 reduces to Theorem 3.*

Corollary 1. *In Theorem 4, if we choose $x=\frac{u+v}{2}$, we get*

$$\begin{aligned}
&\left|\frac{vf(v)-uf(u)}{2(v-u)}+\frac{1}{2}f\left(\frac{u+v}{2}\right)-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{1}{4}\left[\frac{((s+1)u+v)\left|f'\left(\frac{u+v}{2}\right)\right|+(u+(s+1)v)|f'(v)|}{(s+1)(s+2)}\right] \\
&+\frac{1}{4}\left[\frac{(u+(s+1)v)\left|f'\left(\frac{u+v}{2}\right)\right|+((s+1)u+v)|f'(u)|}{(s+1)(s+2)}\right]
\end{aligned}$$

Proposition 1. *Let $u,v\in J^{\circ},\ 0<u<v$ and $s\in(0,1]$, then*

$$\begin{aligned}
&\left|\frac{s+1}{2}L_s^s(u,v)+\frac{1}{2}A^s(u,v)-L_s^s(u,v)\right| \\
=\ & \left|(s-1)L_s^s(u,v)+A^s(u,v)\right| \qquad (2.1)\\
\leq\ & \frac{s}{s+1}A(u,v)+\frac{s}{s+2}A(u^s,v^s)+\frac{sG^2(u,v)}{(s+1)(s+2)}A\left(u^{s-2},v^{s-2}\right)
\end{aligned}$$

Proof. The proof follows from Corollary 1 applied to the s-convex function $f(x)=x^s$. Equality in (2.1) holds if and only if $s=1$. □

Corollary 2. *In Corollary 1, if we take $|f'|\leq M$, we get*

$$\left|\frac{vf(v)-uf(u)}{2(v-u)}+\frac{1}{2}f\left(\frac{u+v}{2}\right)-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right|\leq M\frac{(u+v)}{2(s+1)}$$

Theorem 5. *Let $f:J\to\mathbb{R}$ be a differentiable function on J°. If $|f'|^q$ is convex on $[u,v]$ and $q>1$ with $1/p+1/q=1$, then*

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}L_p(u,v)\left(\frac{|f'(x)|^q+|f'(v)|^q}{2}\right)^{\frac{1}{q}}+\frac{(x-u)^2}{(v-u)^2}L_p(u,v)\left(\frac{|f'(x)|^q+|f'(u)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

for each $x\in[u,v]$.

Proof. Using Lemma 1 and using the well-known Hölder's inequality and since $|f'|^q$ is convex on $[u,v]$, we establish

(2.2)

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\int_0^1(tu+(1-t)v)\left|f'(tx+(1-t)v)\right|dt \\
& +\frac{(x-u)^2}{(v-u)^2}\int_0^1(tv+(1-t)u)\left|f'(tx+(1-t)u)\right|dt \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\left(\int_0^1(tu+(1-t)v)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)v)\right|^q dt\right)^{\frac{1}{q}} \\
& +\frac{(x-u)^2}{(v-u)^2}\left(\int_0^1(tv+(1-t)u)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)u)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is convex on $[u,v]$, by Hadamard's inequality, we have

$$\int_0^1\left|f'(tx+(1-t)v)\right|^q dt \leq \frac{|f'(x)|^q+|f'(v)|^q}{2} \qquad (2.3)$$

$$\int_0^1\left|f'(tx+(1-t)u)\right|^q dt \leq \frac{|f'(x)|^q+|f'(u)|^q}{2} \qquad (2.4)$$

It can be easily seen that

$$\int_0^1 (tu+(1-t)v)^p\,dt = \int_0^1 (tv+(1-t)u)^p\,dt = \frac{v^{p+1}-u^{p+1}}{(v-u)(p+1)} = L_p^p(u,v) \tag{2.5}$$

If expressions (2.3)-(2.5) are written in (2.2), we obtain

$$\begin{aligned}&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2} - \frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\ \leq\ & \frac{(v-x)^2}{(v-u)^2}L_p(u,v)\left(\frac{|f'(x)|^q+|f'(v)|^q}{2}\right)^{\frac{1}{q}} + \frac{(x-u)^2}{(v-u)^2}L_p(u,v)\left(\frac{|f'(x)|^q+|f'(u)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

The proof is completed. □

Theorem 6. *Let $f : J \to \mathbb{R}$ be a differentiable function on J°. If $|f'|^q$ is s-convex on $[u,v]$ for some fixed $s \in (0,1]$ and $q > 1$ with $1/p + 1/q = 1$, then*

$$\begin{aligned}&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2} - \frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\ \leq\ & \frac{(v-x)^2}{(v-u)^2}L_p(u,v)\left(\frac{|f'(x)|^q+|f'(v)|^q}{s+1}\right)^{\frac{1}{q}} + \frac{(x-u)^2}{(v-u)^2}L_p(u,v)\left(\frac{|f'(x)|^q+|f'(u)|^q}{s+1}\right)^{\frac{1}{q}}.\end{aligned}$$

for each $x \in [u,v]$.

Proof. The proof of Theorem 6 is similar to Theorem 5. □

Corollary 3. *In Theorem 6,*

i) if we take $x = u$ or $x = v$, we get

$$\left|\frac{vf(v)-uf(u)}{v-u} - \frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \leq L_p(u,v)\left(\frac{|f'(u)|^q+|f'(v)|^q}{s+1}\right)^{\frac{1}{q}}. \tag{2.6}$$

ii) if we take $x = \frac{u+v}{2}$ and since $\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \leq 1,\ s \in (0,1]$, we get

$$\left|\frac{vf(v)-uf(u)}{2(v-u)} + \frac{1}{2}f\left(\frac{u+v}{2}\right) - \frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right|$$

$$\leq \frac{L_p(u,v)}{4}\left[\left(\frac{\left|f'\left(\frac{u+v}{2}\right)\right|^q+|f'(v)|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{\left|f'\left(\frac{u+v}{2}\right)\right|^q+|f'(u)|^q}{s+1}\right)^{\frac{1}{q}}\right] \tag{2.7}$$

$$\leq \frac{L_p(u,v)}{4}\left[\left(\left|f'\left(\frac{u+v}{2}\right)\right|^q+|f'(v)|^q\right)^{\frac{1}{q}} + \left(\left|f'\left(\frac{u+v}{2}\right)\right|^q+|f'(u)|^q\right)^{\frac{1}{q}}\right] \tag{2.8}$$

Proposition 2. *Let $u, v \in J^\circ,\ 0 < u < v$ and $s \in (0,1]$, then*

$$\begin{aligned}&|(s-1)L_s^s(u,v) + A^s(u,v)| \\ \leq\ & \frac{L_p(u,v)}{2}\left(\frac{s}{s+1}\right)^{\frac{1}{q}}\left[\left(\left(\frac{u+v}{2}\right)^{(s-1)q} + v^{(s-1)q}\right)^{\frac{1}{q}} + \left(\left(\frac{u+v}{2}\right)^{(s-1)q} + u^{(s-1)q}\right)^{\frac{1}{q}}\right]\end{aligned}$$

Proof. The proof follows from (2.7) applied to the s-convex function $f(x) = x^s$. Equality in (2.1) holds if and only if $s = 1$ and $u = v$. □

Theorem 7. *Let $f : J \to \mathbb{R}$ be a differentiable function on J°. If $|f'|^q$ is s-convex on $[u, v]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then*

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}A^{1-\frac{1}{q}}(u,v)\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(((s+1)u+v)|f'(x)|^q+((s+1)v+u)|f'(v)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{(x-u)^2}{(v-u)^2}A^{1-\frac{1}{q}}(u,v)\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(((s+1)v+u)|f'(x)|^q+((s+1)u+v)|f'(u)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

for each $x \in [u, v]$.

Proof. Using Lemma 1 and the well-known power mean inequality and since $|f'|^q$ is s-convex on $[u, v]$, we establish

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\int_0^1 (tu+(1-t)v)\,|f'(tx+(1-t)v)|\,dt+\frac{(x-u)^2}{(v-u)^2}\int_0^1 (tv+(1-t)u)\,|f'(tx+(1-t)u)|\,dt \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\left(\int_0^1 (tu+(1-t)v)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (tu+(1-t)v)\,|f'(tx+(1-t)v)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{(x-u)^2}{(v-u)^2}\left(\int_0^1 (tv+(1-t)u)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (tv+(1-t)u)\,|f'(tx+(1-t)u)|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\left(\int_0^1 (tu+(1-t)v)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (tu+(1-t)v)\left(t^s|f'(x)|^q+(1-t)^s|f'(v)|^q\right)dt\right)^{\frac{1}{q}} \\
&+\frac{(x-u)^2}{(v-u)^2}\left(\int_0^1 (tv+(1-t)u)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (tv+(1-t)u)\left(t^s|f'(x)|^q+(1-t)^s|f'(u)|^q\right)dt\right)^{\frac{1}{q}} \\
=\ & \frac{(v-x)^2}{(v-u)^2}\left(\frac{u+v}{2}\right)^{1-\frac{1}{q}}\left(\frac{(s+1)u+v}{(s+1)(s+2)}|f'(x)|^q+\frac{(s+1)v+u}{(s+1)(s+2)}|f'(v)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{(x-u)^2}{(v-u)^2}\left(\frac{u+v}{2}\right)^{1-\frac{1}{q}}\left(\frac{(s+1)v+u}{(s+1)(s+2)}|f'(x)|^q+\frac{(s+1)u+v}{(s+1)(s+2)}|f'(u)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

The proof is completed. □

Theorem 8. *Let $f : J \to \mathbb{R}$ be a differentiable function on J°. If $|f'|^q$ is convex on $[u,v]$ and $q \geq 1$, then*

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}A^{1-\frac{1}{q}}(u,v)\left(\frac{1}{6}\right)^{\frac{1}{q}}\left((2u+v)|f'(x)|^q+(2v+u)|f'(v)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{(x-u)^2}{(v-u)^2}A^{1-\frac{1}{q}}(u,v)\left(\frac{1}{6}\right)^{\frac{1}{q}}\left((2v+u)|f'(x)|^q+(2u+v)|f'(u)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

for each $x \in [u,v]$.

Proof. In Theorem 7, if we take $s = 1$, then the assertion is proved. □

Corollary 4. *In Theorem 7,*

i) if we take $x = u$ or $x = v$, we get

$$\begin{aligned}
&\left|\frac{(v-u)(vf(v)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & A^{1-\frac{1}{q}}(u,v)\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(((s+1)u+v)|f'(u)|^q+((s+1)v+u)|f'(v)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

ii) if we choose $x = u$ or $x = v$ and $s = q = 1$, we get

$$\left|\frac{(v-u)(vf(v)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \leq \frac{1}{6}\left((2u+v)|f'(u)|+(2v+u)|f'(v)|\,dt\right)$$

iii) if we take $x = \frac{u+v}{2}$ and since $\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}} \leq 1$, $s \in (0,1]$, we get

$$\begin{aligned}
&\left|\frac{vf(v)-uf(u)}{2(v-u)}+\frac{1}{2}f\left(\frac{u+v}{2}\right)-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
&\qquad (2.9) \\
\leq\ & \frac{1}{4}A^{1-\frac{1}{q}}(u,v)\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(((s+1)u+v)\left|f'\left(\frac{u+v}{2}\right)\right|^q+((s+1)v+u)|f'(v)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{1}{4}A^{1-\frac{1}{q}}(u,v)\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(((s+1)v+u)\left|f'\left(\frac{u+v}{2}\right)\right|^q+((s+1)u+v)|f'(u)|^q\,dt\right)^{\frac{1}{q}} \\
&\qquad (2.10) \\
\leq\ & \frac{1}{4}A^{1-\frac{1}{q}}(u,v)\left(((s+1)u+v)\left|f'\left(\frac{u+v}{2}\right)\right|^q+((s+1)v+u)|f'(v)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{1}{4}A^{1-\frac{1}{q}}(u,v)\left(((s+1)v+u)\left|f'\left(\frac{u+v}{2}\right)\right|^q+((s+1)u+v)|f'(u)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

Proposition 3. *Let* $u,v\in J^{\circ},\ 0<u<v$ *and* $s\in(0,1]$, *then*

$$\begin{aligned}
&\left|(s-1)\,L_s^s(u,v)+A^s(u,v)\right| \\
\leq\ & \left(\frac{s^q A^{q-1}(u,v)}{(s+1)(s+2)2^q}\right)^{\frac{1}{q}}\left(((s+1)\,u+v)\,A^{(s-1)q}(u,v)+((s+1)\,v+u)\,v^{(s-1)q}dt\right)^{\frac{1}{q}} \\
&+\left(\frac{s^q A^{q-1}(u,v)}{(s+1)(s+2)2^q}\right)^{\frac{1}{q}}\left(((s+1)\,v+u)\,A^{(s-1)q}(u,v)+((s+1)\,u+v)\,u^{(s-1)q}dt\right)^{\frac{1}{q}}
\end{aligned}$$

Proof. The proof follows from (2.9) applied to the s-convex function $f(x)=x^s$. □

Theorem 9. *Let* $f:J\to\mathbb{R}$ *be a differentiable function on* J°. *If* $|f'|^q$ *is* s*-concave on* $[u,v]$ *for some fixed* $s\in(0,1]$ *and* $q>1$ *with* $1/p+1/q=1$, *then*

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{2^{\frac{s-1}{q}}L_p(u,v)}{(v-u)^2}\left[(v-x)^2\left|f'\left(\frac{x+v}{2}\right)\right|+(x-u)^2\left|f'\left(\frac{x+u}{2}\right)\right|\right]
\end{aligned}$$

for each $x\in[u,v]$.

Proof. Using Lemma 1 and using the Hölder inequality and since $|f'|^q$ is s-concave on $[u,v]$ and using the inequality (1.4), we establish

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\int_0^1 (tu+(1-t)\,v)\,|f'(tx+(1-t)\,v)|\,dt+\frac{(x-u)^2}{(v-u)^2}\int_0^1 (tv+(1-t)\,u)\,|f'(tx+(1-t)\,u)|\,dt \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}\left(\int_0^1 (tu+(1-t)\,v)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)\,v)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{(x-u)^2}{(v-u)^2}\left(\int_0^1 (tv+(1-t)\,u)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)\,u)|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(v-x)^2}{(v-u)^2}L_p(u,v)\left(2^{s-1}\left|f'\left(\frac{x+v}{2}\right)\right|^q\right)^{\frac{1}{q}}+\frac{(x-u)^2}{(v-u)^2}L_p(u,v)\left(2^{s-1}\left|f'\left(\frac{x+u}{2}\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

The proof is completed. □

Theorem 10. *Let* $f:J\to\mathbb{R}$ *be a differentiable function on* J°. *If* $|f'|^q$ *is concave on* $[u,v]$ *and* $q>1$ *with* $1/p+1/q=1$, *then*

$$\begin{aligned}
&\left|\frac{(v-x)(vf(v)-uf(x))+(x-u)(vf(x)-uf(u))}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \\
\leq\ & \frac{L_p(u,v)}{(v-u)^2}\left[(v-x)^2\left|f'\left(\frac{x+v}{2}\right)\right|+(x-u)^2\left|f'\left(\frac{x+u}{2}\right)\right|\right]
\end{aligned}$$

for each $x\in[u,v]$.

Proof. Using Lemma 1 and using the Hölder inequality and since $|f'|^q$ is concave on $[u,v]$ and using Hadamard's inequality for concave functions, we complete the proof. Or, in Theorem 9, if we take $s=1$, then the assertion is also proved. □

Corollary 5. *In Theorem 9,*

i) if we take $x=u$ or $x=v$, we get

$$\left|\frac{(v-u)\left(vf(v)-uf(u)\right)}{(v-u)^2}-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \leq 2^{\frac{s-1}{q}} L_p(u,v)\left|f'\left(\frac{u+v}{2}\right)\right|$$

ii) if we take $x=\frac{u+v}{2}$, we get

$$\left|\frac{vf(v)-uf(u)}{2(v-u)}+\frac{1}{2}f\left(\frac{u+v}{2}\right)-\frac{1}{v-u}\int_u^v f(\mu)\,d\mu\right| \leq \frac{2^{\frac{s-1}{q}}L_p(u,v)}{4}\left[\left|f'\left(\frac{u+3v}{4}\right)\right|+\left|f'\left(\frac{v+3u}{4}\right)\right|\right] \tag{2.11}$$

Proposition 4. *Let $u,v\in J^{\circ}$, $0<u<v$ and $s=1$, then*

$$\left|\frac{v\sin v-u\sin u+2\cos v-2\cos u}{v-u}+\sin\left(A(u,v)\right)\right| \leq \frac{L_p(u,v)}{2}\left[\left|\cos\left(\frac{u+3v}{4}\right)\right|+\left|\cos\left(\frac{v+3u}{4}\right)\right|\right]$$

Proof. The proof follows from (2.11) applied to the concave function $f:[0,\pi]\to[0,1]$, $f(x)=\sin x$. □

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