Saddle Point Optimality Conditions in Fuzzy Optimization Problems
Hassan Mishmast Nehi and Ali Daryab

Abstract

The Karush–Kuhn–Tucker (KKT) optimality conditions and saddle point optimality conditions in fuzzy programming problems have been studied in literature by various authors under different conditions. In this paper, by considering a partial order relation on the set of fuzzy numbers, and convexity with differentiability of fuzzy mappings, we have obtained the Fritz John (FJ) constraint qualification and KKT necessary conditions for a fuzzy optimization problem with fuzzy coefficients, for first time. Owing to the help of the KKT optimality conditions, we then discuss, the saddle point optimality conditions, associated with a fuzzy optimization problem under convexity and differentiability of fuzzy mappings.

Keywords: Comparable fuzzy mapping, Convex fuzzy mapping, Differentiable fuzzy mapping, Fuzzy Lagrange mapping, Fuzzy numbers, Saddle point.

1. Introduction

Fuzzy mathematical programming was developed for formulating real world problems where the problems are usually vague, imprecise, and not well defined. The imprecise and fuzzy data occurring in the optimization problems, will be categorized as the fuzzy optimization problems.

Bellman and Zadeh [2] first proposed the basic concept of fuzzy decision making. Zimmermann [17] formulated fuzzy linear programming problems by the use of both the minimum operator and the product operator. Since that, there come out a lot of papers to investigate the fuzzy optimization problems edited by Slowinski [10] and Delgado et al. [3] summarized the main ideas on this topic. Lai and Hwang [7, 8] also gave the insightful survey.

The concept of saddle point for a fuzzy mapping and its optimality conditions have been discussed by researchers such as in [6, 11, 12, 16]. The saddle point optimality conditions for fuzzy optimization problems were obtained in [11] by using a special kind of partial ordering on the set of fuzzy numbers and introducing the fuzzy scalar product and a solution concept that is essentially similar to the notion of Pareto solution in multiobjective optimization problems. In [16], the concept of saddle point for a fuzzy mapping were discussed and the obtained results were used to the Lagrangian dual of fuzzy programming by using the fuzzy scalar and subdifferential of convex fuzzy mappings. In [12], the saddle point optimality conditions were discussed by considering a general partial ordering on the set of fuzzy numbers and introducing two solution concepts for fuzzy optimization problems. In [6], firstly by considering a total ordering on the set of fuzzy numbers the fuzzy Lagrangian function of a fuzzy optimization problem were proposed, and then, the saddle point of fuzzy Lagrangian function, and its optimality condition were discussed.

In this paper, the saddle point optimality conditions in fuzzy optimization problem with fuzzy coefficients, by considering a partial order relation on the set of fuzzy numbers and base on the concept of convexity with differentiability of fuzzy mappings are investigated.

In section 2 we provide some basic properties of fuzzy numbers. In section 3 we introduce some concepts of the fuzzy differential calculus which will be needed in the sequel. In the paper we accept the concept of differentiability and convexity of fuzzy mapping due to Panigrahi et al. [9]. In section 4, we first develop the Fritz John constraint qualification for a fuzzy minimization problem, and then we drive the KKT necessary optimality conditions without any convexity assumption. Under suitable convexity assumption, the KKT necessary optimality conditions are also sufficient for optimality, which the results are presented in this section. In section 5, the saddle point problem associated with a fuzzy optimization problem, and its optimality condition is discussed. In this section we show that a saddle point of the fuzzy Lagrangian mapping associated with the fuzzy optimization problem (if it exists) yields an optimal solution to the fuzzy optimization problem and that under certain conditions an optimal solution to the fuzzy optimization problem provides a saddle point of the associated fuzzy Lagrangian mapping. The conditions for existence of a saddle point and its relation with the KKT conditions are then driven.
2. Fuzzy Numbers

In this section we give some definitions and properties from fuzzy numbers.

Definition 2.1 [9]: Let \( R \) denote the set of all real numbers. A fuzzy number is a mapping \( \tilde{a}: R \rightarrow [0,1] \) with the following properties:

1. \( \tilde{a} \) is normal, that is, there exists \( x_0 \in R \) such that \( \tilde{a}(x_0) = 1 \).
2. \( \tilde{a} \) is upper semi-continuous, that is, the set \( \{ x \in R : \tilde{a}(x) \geq \alpha \} \) for all \( \alpha \in [0,1] \) is a closed subset in \( R \).
3. \( \tilde{a} \) is convex, that is, for all \( x, y \in R, \lambda \in [0,1], \alpha \), \( \tilde{a}(\lambda x + (1-\lambda)y) \geq \min\{\tilde{a}(x),\tilde{a}(y)\} \).
4. The support of \( \tilde{a} \), \( \text{Supp} \tilde{a} = \{ x \in R : \tilde{a}(x) > 0 \} \) and its closure \( \text{cl}(\text{Supp} \tilde{a}) \) is compact.

Let \( F(R) \) be the set of all fuzzy numbers on \( R \). The \( \alpha \)-level set of a fuzzy number \( \tilde{a} \in F(R), 0 \leq \alpha \leq 1 \), denoted by \( \tilde{a}[\alpha] \), is defined as

\[
\tilde{a}[\alpha] = \begin{cases} 
\{ x \in R : \tilde{a}(x) \geq \alpha \} & \text{if } 0 < \alpha < 1, \\
\text{cl}(\text{Supp} \tilde{a}) & \text{if } \alpha = 0. 
\end{cases}
\]

It is clear that the \( \alpha \)-level set of a fuzzy number is a closed and bounded interval \([a_\alpha, a'\alpha]\), where \( a_\alpha \) denotes the left-hand endpoint of \( \tilde{a}[\alpha] \) and \( a'\alpha \) denotes the right-hand endpoint of \( \tilde{a}[\alpha] \).

Also each \( y \in R \) can be regarded as a fuzzy number \( \tilde{y} \) defined by

\[
\tilde{y}(t) = \begin{cases} 
1 & \text{if } t = y, \\
0 & \text{if } t \neq y. 
\end{cases}
\]

From this characteristic of fuzzy numbers, we see that a fuzzy number is determined by the endpoints of the intervals \( \tilde{a}[\alpha] \). Thus a fuzzy number \( \tilde{a} \) can be identified by a parameterized triple

\[
(a_\alpha, a'\alpha, \alpha) : 0 \leq \alpha \leq 1.
\]

This leads to the following characteristic of a fuzzy number in terms of the two "endpoint" functions \( a_\alpha \) and \( a'\alpha \).

Lemma 2.2 [5]: Assume that \( I = [0,1] \), and \( a_\alpha : I \rightarrow R \) and \( a'\alpha : I \rightarrow R \) satisfy the conditions:

1. \( a_\alpha : I \rightarrow R \) is a bounded increasing function,
2. \( a'\alpha : I \rightarrow R \) is a bounded decreasing function,
3. \( a_\alpha(0) \leq a'\alpha(0) \),
4. \( \lim_{k \rightarrow 0^+} a_\alpha(k) = a_\alpha(0) \), and \( \lim_{k \rightarrow 0^+} a'\alpha(k) = a'\alpha(0) \),
5. \( \tilde{a}(0) = a_\alpha(0) \) and \( \tilde{a}'(0) = a'\alpha(0) \).

Then \( \tilde{a} : R \rightarrow I \) defined by

\[
\tilde{a}(x) = \sup\{ a_\alpha : a_\alpha(\alpha) \leq x \leq a'\alpha(\alpha) \}
\]

is a fuzzy number with parameterization given by \( \{ (a_\alpha, a'\alpha, \alpha) : 0 \leq \alpha \leq 1 \} \). Moreover, if \( \tilde{a} : R \rightarrow I \) is a fuzzy number with parameterization given by \( \{ (a_\alpha, a'\alpha, \alpha) : 0 \leq \alpha \leq 1 \} \), then functions \( a_\alpha \) and \( a'\alpha \) satisfy conditions (1)-(5).

Using the extension principle presented by Zadeh [13-15], the fuzzy addition for two any \( \tilde{a}, \tilde{b} \in F(R) \) is defined as follows:

\[
(\tilde{a} + \tilde{b})(x) = \sup \{ \tilde{a}(y), \tilde{a}(x-y) \}, \ x \in R.
\]

We also define for every \( \tilde{a} \in F(R) \) the scalar multiplication as follows:

\[
(k\tilde{a})(x) = \begin{cases} 
\tilde{a}(x/k) & k > 0, \\
0 & k = 0.
\end{cases}
\]

where \( \tilde{0} \in F(R) \) and \( x \in R \). To deal with subtraction, Goetschel and Voxman [5] define the opposite of a fuzzy number \( \tilde{a} \) by

\[
(\tilde{a}^\alpha)(x) = \{ (a_\alpha, a'\alpha, \alpha) : 0 \leq \alpha \leq 1 \}.
\]

That is

\[
-\tilde{a}(x) = (\tilde{a}(-x)).
\]

We accept the subtraction of fuzzy numbers as defined by Dubois and Prade [4]. For this, define the opposite of a fuzzy number \( \tilde{a} \) to be the fuzzy number \( -\tilde{a} \) satisfying

\[
(-\tilde{a})(x) = \tilde{a}(-x).
\]

In other words, if \( \tilde{a} \) is represented by the parametric form \( \{ (a_\alpha, a'\alpha, \alpha) : 0 \leq \alpha \leq 1 \} \), then \( -\tilde{a} \) is represented by the corresponding parametric form \( \{ (-a'\alpha, a_\alpha, \alpha) : 0 \leq \alpha \leq 1 \} \).

Definition 2.3 [9]: For \( \tilde{a}, \tilde{b} \in F(R) \), we say that \( \tilde{a} \preceq \tilde{b} \) if for each \( \alpha \in [0,1] \), \( a_\alpha \leq b_\alpha \) and \( a'\alpha \leq b'\alpha \). If \( \tilde{a} \preceq \tilde{b} \), then \( \tilde{a} = \tilde{b} \). We say that \( \tilde{a} \prec \tilde{b} \) if \( \tilde{a} \preceq \tilde{b} \) and \( \exists \alpha \in [0,1] \) such that \( a_\alpha \leq b_\alpha \) or \( a'\alpha \leq b'\alpha \). For \( \tilde{a}, \tilde{b} \in F(R) \), if either \( \tilde{a} \preceq \tilde{b} \) or \( \tilde{b} \preceq \tilde{a} \), then we say that \( \tilde{a} \) and \( \tilde{b} \) are comparable, otherwise non-comparable.

Note that \( \preceq \) is a partial order relation on \( F(R) \). Sometimes we may write \( \tilde{b} \succeq \tilde{a} \) instead of \( \tilde{a} \preceq \tilde{b} \).

Definition 2.4: For \( \tilde{a} \in F(R) \), we say that \( \tilde{a} \preceq \tilde{0} \) if for each \( \alpha \in [0,1] \), \( a_\alpha \geq 0 \) and \( a'\alpha \geq 0 \). Similarly, \( \tilde{a} \preceq \tilde{0} \) if \( -\tilde{a} \preceq \tilde{0} \).

Definition 2.5 [9]: A fuzzy number \( \tilde{a} = (a_\alpha, a'\alpha, \alpha, \alpha) \) is said to be a triangular fuzzy number if \( a_1 = a' \) and for each \( \alpha \in [0,1] \) both \( a_\alpha, a'\alpha \) are linear. We denote a triangular fuzzy
number by \( \langle a, 0 \rangle, a(1), a^*(0) \).

For example for the fuzzy number \( \bar{a} = (1, 3, 5) \), we have \( \bar{a}[\alpha] = [1 + 2\alpha, 5 - 2\alpha] \) for \( \alpha \in [0, 1] \).

### 3. Definitions of Fuzzy Differential Calculus

We quote some elementary definitions of fuzzy differential calculus in this section.

**Definition 3.1:** \( \bar{a} \) is said to be an \( n \)-dimensional fuzzy vector if the components of \( \bar{a} \) are composed by \( n \) fuzzy numbers, denote by \( \bar{a} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)^\prime \). The set of all \( n \)-dimensional fuzzy vectors is denoted by \( F^n(R) \).

The \( \alpha \)-cut set of a fuzzy vector \( \bar{a} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)^\prime \) is defined as
\[
\bar{a}[\alpha] = (\bar{a}_1[\alpha], \bar{a}_2[\alpha], \ldots, \bar{a}_n[\alpha])^\prime,
\]
and
\[
a_\alpha(a) = (a_1(\alpha), a_2(\alpha), \ldots, a_n(\alpha))^\prime,
\]
\[
a^\alpha(a) = (a_1^\alpha(a), a_2^\alpha(a), \ldots, a_n^\alpha(a))^\prime.
\]

**Definition 3.2:** Let \( \bar{a} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)^\prime \in F^n(R) \) and \( x = (x_1, x_2, \ldots, x_n) \in R^n \) be an \( n \)-dimensional fuzzy vector and an \( n \)-dimensional real vector, respectively. We define the product of a fuzzy vector with a real vector as
\[
\bar{a}^\prime x = \sum_{i=1}^{n} \bar{a}_i x_i,
\]
which is a fuzzy number.

**Definition 3.3:** For a fuzzy vector \( \bar{a} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)^\prime \in F^n(R) \), we say that \( \bar{a}_i \geq 0 \), if for \( i = 1, 2, \ldots, n \), \( \bar{a}_i \geq 0 \). Similarly, \( \bar{a} \geq 0 \), if for \( i = 1, 2, \ldots, n \), \( \bar{a}_i \geq 0 \).

**Definition 3.4** [9]: Let \( \tilde{f} : \Omega \rightarrow F(R) \) be a fuzzy mapping, where \( \Omega \subseteq R^n \) and \( F(R) \) is the set of fuzzy numbers. The \( \alpha \)-cut of \( \tilde{f} \) at \( x \in \Omega \), denoted by \( \tilde{f}(x)[\alpha] = \{ f(x, \alpha), f^\ast(x, \alpha) \} \) which is a closed and bounded interval, where \( f(x, \alpha) = \min \{ \tilde{f}(x)[\alpha] \} \) and \( f^\ast(x, \alpha) = \max \{ \tilde{f}(x)[\alpha] \} \). Thus, \( \tilde{f} \) can be understood by the two functions \( f(x, \alpha) \) and \( f^\ast(x, \alpha) \) which are functions from \( \Omega \times [0, 1] \) to the set of real numbers \( R \), \( f(x, \alpha) \) is a bounded increasing function of \( \alpha \) and \( f^\ast(x, \alpha) \) is a bounded decreasing function of \( \alpha \). Moreover, \( f(x, \alpha) \leq f^\ast(x, \alpha) \) for each \( \alpha \in [0, 1] \).

**Definition 3.5** [9]: Let \( \tilde{f} : \Omega \subseteq R^n \rightarrow F(R) \) be a fuzzy mapping. Then, \( \tilde{f} \) is said to be continuous at \( x \in \Omega \), if for each \( \alpha \in [0, 1] \), both \( f(x, \alpha) \) and \( f^\ast(x, \alpha) \) are continuous functions of \( x \).

**Definition 3.6:** Let \( \tilde{f} \), be a fuzzy mapping from the set of all real numbers \( R \) to the set of all fuzzy numbers, let \( \tilde{f}(t)[\alpha] = \{ f(t, \alpha), f^\ast(t, \alpha) \} \). Assume that the partial derivatives of \( f(\cdot, \alpha) \) and \( f^\ast(\cdot, \alpha) \) with respect to \( t \) exist and \( f(\cdot, \alpha) \) and \( f^\ast(\cdot, \alpha) \) for each \( \alpha \in [0, 1] \) are differentiable at \( t \in R \). \( f(t, \alpha) \) and \( f^\ast(t, \alpha) \) are differentiable functions of \( t \) for each \( \alpha \in [0, 1] \), let \( \Gamma(t) = [f(t, \alpha), f^\ast(t, \alpha)] \). If \( \Gamma(t) \) defines the \( \alpha \)-cut of a fuzzy number for each \( t \in R \), then \( \tilde{f}(t) \) is said to be differentiable and is written as \( \tilde{f}(t)[\alpha] = \Gamma(t, \alpha) = [f(t, \alpha), f^\ast(t, \alpha)] \), for all \( t \in R \), \( \alpha \in [0, 1] \).

Throughout this paper we have accepted the fuzzy differentiability concept for a fuzzy mapping due to Panigrahi et al. [9].

**Definition 3.7** [9]: Let \( \tilde{f} : \Omega \rightarrow F(R) \) be a fuzzy mapping, where \( \Omega \) is an open subset of \( R^n \). Let \( x = (x_1, x_2, \ldots, x_n) \in \Omega \). Let \( D_{x_i} \) for \( i = 1, 2, \ldots, n \) stand for the "partial differentiation" with respect to the \( i \)th variable \( x_i \). Assume that for all \( \alpha \in [0, 1] \), \( f(x, \alpha) \), \( f^\ast(x, \alpha) \) have continuous partial derivatives so that \( D_{x_i} f(x, \alpha) \), \( D_{x_i} f^\ast(x, \alpha) \) are continuous. Define for \( i = 1, 2, \ldots, n \), and \( \alpha \in [0, 1] \)
\[
D_{x_i} \tilde{f}(x)[\alpha] = [D_{x_i} f(x, \alpha), D_{x_i} f^\ast(x, \alpha)]
\]
If each for \( i = 1, 2, \ldots, n \), (1) defines the \( \alpha \)-cut of a fuzzy number, then we will say that the gradient of the fuzzy mapping \( \tilde{f} \) at \( x \) exists and we write
\[
\nabla \tilde{f}(x) = (\tilde{D}_{x_1} \tilde{f}(x), \tilde{D}_{x_2} \tilde{f}(x), \ldots, \tilde{D}_{x_n} \tilde{f}(x))^\prime.
\]
Thus, from Lemma 2.2, the sufficient conditions that the gradient of \( \tilde{f} \) at \( x \) exists are for each \( i = 1, 2, \ldots, n \), \( \alpha \in [0, 1] \),
1. The partial derivatives of \( f_i(x, \alpha) \) and \( f^\ast_i(x, \alpha) \) with respect to \( x_i \) exist,
2. \( D_{x_i} f_i(x, \alpha) \) is a continuous increasing function of \( \alpha \),
3. \( D_{x_i} f^\ast_i(x, \alpha) \) is a continuous decreasing function of \( \alpha \),
4. \( D_{x_i} f_i(x, \alpha) \leq D_{x_i} f^\ast_i(x, \alpha) \).

Note that \( \nabla \tilde{f}(x) \) is an \( n \)-dimensional fuzzy vector. A fuzzy mapping \( \tilde{f} \) is said to be differentiable at \( x \) if \( \nabla \tilde{f}(x) \) exists and both \( f_i(x, \alpha) \), \( f^\ast_i(x, \alpha) \) for each \( \alpha \in [0, 1] \) are differentiable at \( x \).

**Definition 3.8:** \( A \) is said to be a fuzzy matrix if the entries of \( A \) are composed by fuzzy numbers, denote by
The $\alpha$-cut set of the fuzzy matrix $\tilde{A}$ is defined as

$$\tilde{A}[\alpha] = \left[ \tilde{a}_{i1}[\alpha] \cdots \tilde{a}_{in}[\alpha] \right]$$

and

$$A_\alpha(\alpha) = \left[ a_{i1}(\alpha) \cdots a_{in}(\alpha) \right]$$

$$A^\alpha(\alpha) = \left[ a_{i1}^\alpha(\alpha) \cdots a_{in}^\alpha(\alpha) \right]$$

**Definition 3.9 [9]:** Let $\tilde{f} : \Omega \to F(R)$ be a fuzzy mapping, where $\Omega$ is an open subset of $R^n$. Let $x = (x_1, x_2, \ldots, x_n) \in \Omega$ and $D_{x_{ij}}$, for $i, j = 1, 2, \ldots, n$ stand for the "second-order partial differentiation" with respect to the $i$th variable $x_i$ and $j$th variable $x_j$. Assume that $\nabla f(x)$ exists and for all $\alpha \in [0,1]$, $f(x, \alpha)$, $f^*(x, \alpha)$ have continuous second-order partial derivatives so that $D^2 x_{ij} f(x, \alpha)$, $D^2 x_{ij} f^*(x, \alpha)$ are continuous (here $D^2 x_{ij} f(x, \alpha) = D_{x_{ij}} f(x, \alpha)$, and $D^2 x_{ij} f^*(x, \alpha) = D_{x_{ij}} f^*(x, \alpha)$). Define for $i, j = 1, 2, \ldots, n$, and $\alpha \in [0,1]$

$$D^2 x_{ij} \tilde{f}(x)[\alpha] = [D^2 x_{ij} f(x, \alpha), D^2 x_{ij} f^*(x, \alpha)]$$

If for each $i, j = 1, 2, \ldots, n$, (2) defines the $\alpha$-cut of a fuzzy number, then we define the Hessian of the fuzzy mapping (in the matrix notation) as follows:

$$\nabla^2 \tilde{f}(x) = (D^2 x_{ij} \tilde{f}(x))[\alpha], \quad i, j = 1, 2, \ldots, n.$$  

We will say that $\tilde{f}$ is twice differentiable at $x$, if the Hessian of the fuzzy mapping exists and both $f(x, \alpha)$, $f^*(x, \alpha)$ are twice differentiable at $x$. The sufficient conditions that the Hessian of the fuzzy mapping $\tilde{f}$ at $x$ exists are:

1. $\nabla f(x)$ exists,
2. the second-order partial derivatives of $f(x, \alpha)$ and $f^*(x, \alpha)$ with respect to $x_i$, $x_j$ exist,
3. $D^2 x_{ij} f(x, \alpha)$ is a continuous increasing function of $\alpha$,
4. $D^2 x_{ij} f^*(x, \alpha)$ is a continuous decreasing function of $\alpha$.

5. $D^2 x_{ij} f(x, \alpha) \leq D^2 x_{ij} f^*(x, \alpha)$.

### 4. The Optimality Conditions of Fuzzy Minimization Problems

Let $\Omega$ be an open set in $R^n$, and $\tilde{f} : \Omega \to F(R)$ be a fuzzy mapping. Also let $\tilde{g} : \Omega \to F^m(R)$ be an $m$-dimensional fuzzy mapping.

Consider the following fuzzy programming problem, which we call the fuzzy minimization problem with fuzzy coefficients:

**Minimize** $\tilde{f}(x)$

**subject to** $\tilde{g}(x) \leq \tilde{0}$

where $x \in \Omega$, $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_m)$, and $\tilde{g}_i(x) \leq \tilde{0}$ for each $i = 1, 2, \ldots, m$.

**Definition 4.1 [9]:** Let $\tilde{f} : \Omega \subseteq R^n \to F(R)$ be a function with fuzzy coefficients. Then, $\tilde{f}$ is said to be a comparable fuzzy mapping if for each pair $x_1, x_2 \in \Omega$, $\tilde{f}(x_1)$ and $\tilde{f}(x_2)$ are comparable. Otherwise, $\tilde{f}$ is said to be a non-comparable fuzzy mapping. Let $E$ denote the set of all comparable fuzzy mappings.

**Example 4.2:** Consider the fuzzy mapping $\tilde{f} : R^2 \to F(R)$, defined by $\tilde{f}(x, y) = (1, 3, 5)(x_1 - x_2)$. The $\alpha$-cut is given by $\tilde{f}(x_1, x_2)[\alpha] = [1 + 2\alpha, 5 - 2\alpha]x_1 - x_2$ for each $\alpha \in [0,1]$. Then, for $\alpha \in [0,1]$ we have, $f(x_1, x_2) = (1 + 2\alpha)(x_1 - x_2)$ and $f^*(x_1, x_2) = (5 - 2\alpha)(x_1 - x_2)$. Then, for each pair $x = (x_1, x_2) \in R^2$, $y = (y_1, y_2) \in R^2$, where $x \neq y$, and for $\alpha \in [0,1]$, we can write $f(x_1, x_2) = (1 + 2\alpha)(x_1 - x_2)$ and $f^*(x_1, x_2) = (5 - 2\alpha)(y_1 - y_2)$.

**Example 4.3:** Consider the fuzzy mapping $\tilde{g} : R^2 \to F(R)$, where $\tilde{g}_i(x_1, x_2) = (1, 3, 5)x_1^2 - (1, 3, 5)x_2$. Then the $\alpha$-cut for each $\alpha \in [0,1]$ is given by $\tilde{g}_i(x_1, x_2)[\alpha] = [1 + 2\alpha, 5 - 2\alpha]x_1^2 - [5 + 2\alpha, -1 - 2\alpha]x_2$.

Hence, $g_i(x_1, x_2) = (1 + 2\alpha)x_1^2 + (2\alpha - 5)x_2$, and also $g^*_i(x_1, x_2) = (5 - 2\alpha)x_1^2 - (1 - 2\alpha)x_2$. Now let $x = (1, 2), y = (0, 1)$. Then, $g_i(1, 2, \alpha) = 6\alpha - 9$ and $g^*_i(1, 2, \alpha) = 3 - 6\alpha$. Thus, $g_i(1, 2) = (-9, -3, 3)$, $g^*_i(0, 1) = (-5, -3, -1)$. Clearly,
\( \bar{g}(1,2) \) and \( \bar{g}(0,1) \) are not comparable. Hence, \( \bar{g} \) is a non-comparable fuzzy mapping.

We are going to obtain Fritz John (FJ) constraint qualification and the Karush–Kuhn–Tucker (KKT) optimality conditions for the problem (3).

**Definition 4.4:** Let \( S = \{ x \in \Omega : \bar{g}(x) \leq 0 \} \) be the feasible region for the problem (3), and let \( x_0 \in S \). The cone of feasible direction of \( S \) at \( x_0 \), denoted by \( \mathcal{D} \), is defined as
\[
\mathcal{D} = \{ d : d \neq 0, \quad x_0 + \delta d \in S, \quad \forall \delta \in (0, \delta), \quad \delta > 0 \},
\]
each non-zero vector \( d \in \mathcal{D} \) is called a feasible direction.

Now, we develop a necessary optimality condition for the problem (3).

**Theorem 4.5:** Consider the problem (3). Let \( x_0 \) be a feasible solution. Suppose that \( \bar{F} : \Omega \subseteq \mathbb{R}^n \rightarrow F(R) \) is differentiable at \( x_0 \), and \( \bar{F} \in \mathcal{E} \). If \( x_0 \) is a local optimal solution, then \( F_0 \cap \mathcal{D} = \emptyset \), where \( F_0 = \{ d : \nabla \bar{F}(x_0)^{\top} d < 0 \} \), and \( \mathcal{D} \) is the cone of feasible directions of \( S \) at \( x_0 \).

**Proof:** By contradiction, suppose that there exists a vector \( d \in F_0 \cap \mathcal{D} \). Then, \( d \in F_0 \), \( d \in \mathcal{D} \).

Since \( d \in F_0 \), then by definition of \( F_0 \) we have
\[
\nabla \bar{F}(x_0)^{\top} d < 0,
\]
which implies that for each \( \alpha \in [0, 1] \)
\[
\nabla f_i(x_0, \alpha)^{\top} d \leq 0, \quad \nabla f_i^{\ast}(x_0, \alpha)^{\top} d \leq 0,
\]
and there exists an \( \alpha_0 \in [0, 1] \) such that \( \nabla f_i(x_0, \alpha_0)^{\top} d < 0 \) or \( \nabla f_i^{\ast}(x_0, \alpha_0)^{\top} d < 0 \). Thus by Theorem 3.12 (Panigrahi et al [9]), there exists \( \delta > 0 \) such that
\[
\bar{F}(x_0 + \delta d) < \bar{F}(x_0) \quad \forall \alpha \in (0, \delta_1). \quad (4)
\]
Furthermore, since \( d \in \mathcal{D} \), by Definition 4.4, there exists \( \delta > 0 \) such that
\[
x_0 + \delta d \in \mathcal{D} \quad \forall \alpha \in (0, \delta_1). \quad (5)
\]
Now, let \( x = x_0 + \delta d \) for each \( \alpha \in (0, \delta_1) \), where \( \delta = \min(\delta_1, \delta_2) \), then by (5) we have \( \bar{F}(x) < \bar{F}(x_0) \). This is not compatible with the assumption that \( x_0 \) is a local optimal solution to the problem (3). The proof is complete.

It is necessary to mention that in necessary condition for local optimality at \( x_0 \) is that, \( F_0 \cap \mathcal{D} = \emptyset \), \( \mathcal{D} \) is the cone of feasible directions, which is not necessarily defined in terms of the gradients of the mappings involved. This precludes us from converting the geometric optimality condition \( F_0 \cap \mathcal{D} = \emptyset \) into a more usable algebraic statement involving equations. As Lemma 4.6 below indicates, we can define an open cone \( G_0 \) in terms of the gradients of the binding constraints at \( x_0 \), such that \( G_0 \subseteq D \). Since \( F_0 \cap \mathcal{D} = \emptyset \) must hold at \( x_0 \), and since \( G_0 \subseteq D \), then \( F_0 \cap G_0 = \emptyset \) is also a necessary optimality condition at \( x_0 \).

**Lemma 4.6:** Let \( x_0 \in S \), be a feasible point and \( I = \{ i : \bar{g}_i(x_0) = 0 \} \) be the index set for the binding or active constraints, and assume that \( \bar{g}_i \) for \( i \in I \) are differentiable at \( x_0 \) and that the \( \bar{g}_i \)'s for \( i \notin I \) are continuous at \( x_0 \). Define the set \( G_0 = \{ d : \nabla \bar{g}_i(x_0)^{\top} d < 0 \} \), for each \( i \in I \). Then we have \( G_0 \subseteq D \).

**Proof:** Let \( d \in G_0 \), then \( d \neq 0 \). Since \( x_0 \in \Omega \), \( \Omega \) is an open set in \( \mathbb{R}^n \), there exists a \( \delta_0 > 0 \) such that \( x_0 + \delta d \in \Omega \) for each \( \lambda \in (0, \delta) \).

Also, since \( \bar{g}_i \) is continuous at \( x_0 \) for \( i \notin I \), then we have for each \( \alpha \in [0, 1] \)
\[
g_i(x_0, \alpha) \leq 0, \quad g_i^{\ast}(x_0, \alpha) \leq 0. \quad (6)
\]
and there exists an \( \alpha_0 \in [0, 1] \) such that \( g_i(x_0, \alpha_0) < 0 \) or \( g_i^{\ast}(x_0, \alpha_0) < 0 \). Without loss of generality assume that
\[
g_i(x_0, \alpha_0) < 0 \quad \forall i \notin I. \quad (6)
\]
Furthermore, since \( \bar{g}_i \) is continuous at \( x_0 \) for \( i \notin I \), by Definition 3.5, both functions \( g_i(x, \alpha) \), \( g_i^{\ast}(x, \alpha) \) are continuous for each \( \alpha \in [0, 1] \), \( i \notin I \). Thus by (7), there exist \( \delta_2, \delta_1 > 0 \) such that for each \( \alpha \in [0, 1] \)
\[
g_i(x_0 + \lambda d, \alpha) \leq 0 \quad \forall \lambda \in (0, \delta_2) \quad \text{and for} \quad i \notin I, \quad (9)
\]
and also by continuity of function \( g_i(x, \alpha) \) for \( i \notin I \), and by (8), there exist a \( \delta_4 > 0 \) such that
\[
g_i^{\ast}(x_0 + \lambda d, \alpha_0) < 0 \quad \forall \alpha \in (0, \delta_4) \quad \text{and for} \quad i \notin I. \quad (10)
\]
From (9)-(11), we get
\[
g_i(x_0 + \lambda d, \alpha_0) \leq \bar{g}_i(x_0) = 0 \quad \forall \alpha \in (0, \delta_4) \quad \text{and for} \quad i \notin I. \quad (12)
\]
where \( \delta = \min(\delta_2, \delta_1, \delta_4) \).

Furthermore, since \( d \in G_0 \), then \( \nabla \bar{g}_i(x_0)^{\top} d \leq 0 \) for each \( i \in I \); and by Theorem 3.12 (Panigrahi et al [9]), there exists \( \delta > 0 \) such that
\[
\bar{g}_i(x_0 + \delta d, \alpha_0) < \bar{g}_i(x_0) = 0 \quad \forall \alpha \in (0, \delta) \quad \text{and for} \quad i \notin I. \quad (13)
\]
Now, from (6), (12) and (13), we get \( x_0 + \lambda d \in \Omega \) for each \( \lambda \in (0, \delta) \), where \( \delta = \min(\delta_2, \delta_1, \delta_4) \), and \( \mathcal{D} \) since \( d \neq 0 \). We have shown that \( d \in G_0 \) implies that \( d \in \mathcal{D} \), hence, \( G_0 \subseteq D \). The proof is complete.

**Theorem 4.7:** Let \( x_0 \) be a feasible point of (3), and denote \( I = \{ i : \bar{g}_i(x_0) = 0 \} \). Furthermore, suppose that \( \bar{F} \) and \( \bar{g}_i \) for \( i \in I \) are differentiable at \( x_0 \) and that \( \bar{g}_i \) for \( i \notin I \) are continuous at \( x_0 \). If \( x_0 \) is a local optimal solution, then \( F_0 \cap G_0 = \emptyset \), where \( F_0 = \{ d : \nabla \bar{F}(x_0)^{\top} d < 0 \} \) and \( G_0 = \{ d : \nabla \bar{g}_i(x_0)^{\top} d < 0 \} \), for each \( i \in I \).
Proof: The result follows from Theorem 4.5 and Lemma 4.6, immediately.

Now, since both $F_0$ and $G_0$ are defined in terms of the gradient vectors, we will use the condition $F_0 \cap G_0 = \emptyset$ in this section to develop the constraint qualification credited to Fritz John (FJ). With mild additional assumption, the conditions reduce to the well-known Karush-Kuhn-Tucker (KKT) optimality conditions.

**Theorem 4.8:** (The Fritz John constraint qualification). Consider the problem (3), where $\Omega$ is a non-empty open set in $R^n$, and let $\tilde{f}: \Omega \subseteq R^n \to F(R)$ is a fuzzy mapping, and $\tilde{g}: \Omega \subseteq R^n \to F^m(R)$ is an $m$-dimensional fuzzy mapping. Let $x_0$ be a feasible solution, and denote $I = \{ i : \tilde{g}_i(x_0) = \tilde{0} \}$. Furthermore, suppose that $\tilde{f}$ and $\tilde{g}_i$ for $i \in I$ are differentiable at $x_0$ and that $\tilde{g}_i$ for $i \notin I$ are continuous at $x_0$, and we also have for each $i \in I$, $j = 1, 2, \ldots, n$ and $\alpha \in [0, 1]$

$$D_i f_j(x_0)(\alpha) = [h_i(\alpha)f_j(x_0), h^*(\alpha)f_j(x_0)]$$

(14)

$$D_i g_i(x_0)\alpha = [h_i(\alpha)g_i(x_0), h^*(\alpha)g_i(x_0)]$$

(15)

which in (14), (15), both $h_i(\alpha)$, $h^*(\alpha)$ are functions in terms of $\alpha$, and both are positive (or negative) for each $\alpha \in [0, 1]$, at the same time. If $x_0$ is a local optimal solution, then there exist scalars $u_0$ and $u_i$ for $i \in I$ such that

$$u_0 \tilde{f}(x_0) + \sum_{i \in I} u_i \tilde{g}_i(x_0) = \tilde{0}$$

(16)

$$u_0, u_i \geq 0, \quad (u_0, u_i) \neq (0, 0)$$

where $u_i$ is the vector whose components are $u_i$ for $i \in I$. Furthermore, if $\tilde{g}_i$ for $i \notin I$ are also differentiable at $x_0$, then the foregoing conditions can be written in the following equation form:

$$u_0 \tilde{f}(x_0) + \sum_{i \in I} u_i \tilde{g}_i(x_0) = \tilde{0}$$

$$u_0 \tilde{g}_i(x_0) = 0, \quad \text{and} \quad u_0, u_i \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, m$$

(17)

where $u$ is the vector whose components are $u_i$ for $i = 1, 2, \ldots, m$.

Proof: Since $x_0$ is a local optimal solution for the problem (3), then by Theorem 4.7, there exists no vector $d$ such that $\tilde{f}(x_0) \triangleright 0$ and $\tilde{g}_i(x_0) \triangleright 0$ for each $i \in I$. Now, let $A$ be the fuzzy matrix whose rows are $\tilde{f}(x_0)'$ and $\tilde{g}_i(x_0)'$ for $i \in I$. Then, the necessary optimality condition of Theorem 4.7 is equivalent to the statement that the system $Ad < \tilde{0}$ is inconsistent. Now, let $f(x_0) = (f_1(x_0), f_2(x_0), \ldots, f_n(x_0))$ and $g_i(x_0) = (g_{i1}(x_0), g_{i2}(x_0), \ldots, g_{im}(x_0))$ for $i = 1, 2, \ldots, m$.

But, by definition 2.3, and by (14), (15), $\tilde{A}d \leq \tilde{0}$ is equivalent to $A_{x0} \leq \tilde{0}$, where $A = [f(x_0), g_i(x_0)']^T$ for $i = 1, 2, \ldots, m$, therefore the system $A_{x0} \leq \tilde{0}$ is also inconsistent. By Gordan’s theorem ([1], Theorem 2.4.9), there exists a non-zero vector $p \geq 0$ such that $A'p = 0$, which implies that

$$h_i(\alpha)A'p = 0, \quad h^*(\alpha)A'p = 0, \quad \text{for each} \quad \alpha \in [0, 1].$$

Thus, we have $A_{x0}(\alpha)'p = 0$, $A'p = 0$. Hence, $A'p = \tilde{0}$. Denoting the components of $p$ by $u_0$, $u_i$ for $i \in I$, the first part of the result follows. The equivalent form of the necessary condition is readily obtained by letting $u_i = 0$ for $i \notin I$, and the proof is complete.

Suppose that all the coefficients of the problem (3), are unique positive (or negative) fuzzy number $\tilde{u}$, then we will have

$$\text{Minimize} \quad \tilde{f}(x) = \tilde{u}.f(x), \quad \text{subject to} \quad \tilde{g}(x) = \tilde{u}.g(x) \leq \tilde{0},$$

(18)

then for each $\alpha \in [0, 1]$,

$$f_i(x, \alpha) = u_i(\alpha)f_i(x), \quad f^*(x, \alpha) = u^*(\alpha)f_i(x)$$

thus,

$$Vf_i(x, \alpha) = u_i(\alpha)Vf_i(x), \quad Vf^*(x, \alpha) = u^*(\alpha)Vf_i(x).$$

Therefore, if $\tilde{f}$ is differentiable, hence

$$\tilde{Vf}(x) = \tilde{u}.\tilde{Vf}(x).$$

Similarly, if the fuzzy mapping $\tilde{g}(x)$ is differentiable, then

$$\tilde{Vg}(x) = \tilde{u}.\tilde{Vg}(x).$$

Now, if $x_0$ is a local optimal solution for the problem (16), then by (17), (18), we will have

$$A' = \begin{bmatrix} \tilde{Vf}(x_0) \tilde{Vg}(x_0) \end{bmatrix} = \begin{bmatrix} \tilde{u} \tilde{Vf}(x_0) \tilde{Vg}(x_0) \end{bmatrix},$$

therefore, the assumptions of Theorem 4.6 is satisfied, particularly the conditions (14) and (15). Thus, there exist scalars $u_0$, $u_i$ for $i = 1, 2, \ldots, m$ such that $x_0$ satisfies in FJ constraint qualification conditions.

**Definition 4.9:** we say the collection of fuzzy vectors $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_i \in F^n(R)$, are linear independent, if for each $\alpha \in [0, 1]$ the left-hand and right-hand $\alpha$-level vectors $v_{i1}(\alpha), v_{i2}(\alpha), \ldots, v_{ik}(\alpha)$ and $v^*_1(\alpha), v^*_2(\alpha), \ldots, v^*_k(\alpha)$ are linear independent. Otherwise, we call them linear dependent.

**Theorem 4.10:** (KKT Necessary Optimality Conditions) Let $x_0$ be a feasible solution of (3), and denote $I = \{ i : \tilde{g}_i(x_0) = \tilde{0} \}$. Suppose that $\tilde{f}$ and $\tilde{g}_i$ for $i \in I$ are differentiable at $x_0$ and that $\tilde{g}_i$ for $i \notin I$ are continuous at $x_0$, and $\tilde{f}, \tilde{g}$ satisfy (14), (15), respec-
tively. Furthermore, suppose that \( \widehat{\nabla g_i}(x_0) \) for \( i \in I \) are linearly independent. If \( x_0 \) is a local optimal solution for the problem (3), then there exist scalars \( u_i \) for \( i \in I \) such that

\[
\widehat{\nabla f}(x_0) + \sum_{i=1}^{m} u_i \widehat{\nabla g_i}(x_0) = 0
\]

\[
u_i \geq 0 \quad \text{for } i \in I
\]

In addition to the above assumptions, if \( g_i \) for each \( i \notin I \) is also differentiable at \( x_0 \), then foregoing conditions can be written in the following equivalent form:

\[
\nabla f(x_0) + \sum_{i=1}^{m} u_i \nabla g_i(x_0) = 0
\]

\[
u_i \geq 0 \quad \text{for } i \in I
\]

Proof: By Theorem 4.8, there exist scalars \( u_0 \) and \( \hat{u}_i \) for \( i \in I \), not all equal to zero, such that

\[
u_0 \nabla f(x_0) + \sum_{i=1}^{m} \hat{u}_i \nabla g_i(x_0) = 0,
\]

then, we must have \( u_0 > 0 \). Otherwise, if \( u_0 = 0 \), then by (19) we have

\[
\sum_{i=1}^{m} \hat{u}_i \nabla g_i(x_0, \alpha) = 0, \quad \text{and} \quad \sum_{i=1}^{m} \hat{u}_i \nabla g_i(x_0, \alpha) = 0,
\]

for each \( \alpha \in [0,1] \), where \( \hat{u}_i \) for \( i \in I \), not all equal to zero. This would contradict the assumption of linearity independence of \( \nabla g_i(x_0) \) for \( i \in I \). The first part of the theorem then follows by letting \( u_i = \hat{u}_i / u_0 \) for each \( i \in I \). The equivalent form of the necessary conditions follows by letting \( u_i = 0 \) for \( i \notin I \). This completes the proof.

The scalars \( u_i \) for \( i = 1,2,\ldots,m \), in the Theorem 4.10, are usually called Lagrangian, or Lagrange multipliers.

Remark: Linear independent assumption of \( \nabla g_i(x_0) \) for \( i \in I \) in Theorem 4.10 is called linear independence constraint qualification.

Definition 4.11 [9]: Let \( \bar{f} : \Omega \subseteq \mathbb{R}^n \to F(R) \) be a fuzzy mapping, where \( \Omega \) is a convex subset of \( \mathbb{R}^n \). \( \bar{f} \) is said to be convex on \( \Omega \), if for each \( \alpha \in [0,1] \) both \( f_\alpha(x, \alpha) \), \( f_\alpha^*(x, \alpha) \) are convex on \( \Omega \), that is, for \( 0 \leq \lambda \leq 1, \quad x, y \in \Omega \),

\[
f_\alpha(\lambda x + (1-\lambda)y, \alpha) \leq \lambda f_\alpha(x, \alpha) + (1-\lambda)f_\alpha(y, \alpha)
\]

and

\[
f_\alpha^*(\lambda x + (1-\lambda)y, \alpha) \leq \lambda f^*_\alpha(x, \alpha) + (1-\lambda)f^*_\alpha(y, \alpha)
\]

\( \bar{f} \) is said to be concave if \( -\bar{f} \) is convex.

Example 4.12: Consider the Fuzzy mapping \( \bar{f} \) in Example 4.2. It can be easily checked that both \( f_\alpha(x_1, x_2, \alpha) \) and \( f^*_\alpha(x_1, x_2, \alpha) \) are convex functions for each \( \alpha \in [0,1] \). Thus, \( \bar{f} \) is a convex fuzzy mapping on \( \mathbb{R}^2 \).

Theorem 4.13 [9]: Let \( \bar{f} \) be a fuzzy mapping on an open convex set \( \Omega \subseteq \mathbb{R}^n \). Let \( \bar{f} \) be differentiable at \( x_0 \in \Omega \). If \( \bar{f} \) is convex on \( \Omega \), then for each \( x \in \Omega \) and \( \alpha \in [0,1] \), we have

\[
f_\alpha(x, \alpha) - f_\alpha(x_0, \alpha) \geq \nabla f_\alpha(x_0, \alpha)^T (x - x_0),
\]

\[
f^*_\alpha(x, \alpha) - f^*_\alpha(x_0, \alpha) \geq \nabla f^*_\alpha(x_0, \alpha)^T (x - x_0).
\]

Theorem 4.14: (KKT Sufficient optimality conditions). Consider the problem (3). Let \( x_0 \in \Omega \subseteq \mathbb{R}^n \), let \( \Omega \) be open, and let \( \bar{f} \) and \( \bar{g} \) be differentiable and convex at \( x_0 \). Let \( \bar{f} \in E \) and the Lagrangian mapping \( \bar{L}(x, u) \) be comparable in terms of \( x \). If \( (x_0, u_0) \) satisfies the KKT necessary optimality conditions, then \( x_0 \) is a solution of the problem (3).

Proof: See [9].

Example 4.15: Consider the following fuzzy minimization problem.

Minimize \( \bar{f}(x_1, x_2) = (0, 2, 4)(x_1 - 1)^2 + (0, 2, 4)(x_2 - 2)^2 \)

Subjectto \( g_1(x_1, x_2) = (0, 1, 2)(x_1^2 - x_2) \geq 0 \)

\( g_2(x_1, x_2) = (0, 4, 8)(x_1 + x_2 - 2) \geq 0 \)

where \( \bar{f}, g_1, g_2 \) for each \( \alpha \in [0,1] \) are given by,

\[
f_\alpha(x_1, x_2) = 2\alpha((x_1 - 1)^2 + (x_2 - 2)^2),
\]

\[
f^*_\alpha(x_1, x_2) = (4 - 2\alpha)(x_1^2 - x_2^2) \]

\[
\]

\[
g_1(x_1, x_2) = \alpha(x_1^2 - x_2^2)(2 - \alpha)(x_1^2 - x_2^2),
\]

\[
g_2^*(x_1, x_2) = [\alpha(x_1^2 + x_2^2)(2 - \alpha)](x_1^2 - x_2^2) - 4\alpha(x_1^2 - x_2^2).
\]

Thus, for each \( \alpha \in [0,1] \),

\[
f_\alpha(x_1, x_2, \alpha) = 2\alpha((x_1 - 1)^2 + (x_2 - 2)^2),
\]

\[
f^*_\alpha(x_1, x_2, \alpha) = (4 - 2\alpha)(x_1^2 - x_2^2) \]

\[
\]

\[
g_1^*(x_1, x_2, \alpha) = \alpha(x_1^2 - x_2^2)(2 - \alpha)(x_1^2 - x_2^2),
\]

\[
g_2^*(x_1, x_2, \alpha) = (8 - 4\alpha)(x_1^2 - x_2^2).
\]

Let

\[\bar{L}(x_1, x_2, u_1, u_2) = \bar{f}(x_1, x_2) + u_1 \bar{g}_1(x_1, x_2) + u_2 \bar{g}_2(x_1, x_2)\]

Then,

\[
L_\alpha(x_1, x_2, u_1, u_2, \alpha) = 2\alpha(x_1 - 1)^2 + 2\alpha(x_2 - 2)^2 + u_1 \alpha(x_1^2 + x_2 - 2),
\]

\[
L^*_\alpha(x_1, x_2, u_1, u_2, \alpha) = (4 - 2\alpha)(x_1 - 1)^2 + (4 - 2\alpha)(x_2 - 2)^2 + u_1(2 - \alpha)(x_1^2 - x_2^2) + u_2(8 - 4\alpha)(x_1 + x_2 - 2),
\]

\[
\nabla_x L_\alpha(x_1, x_2, u_1, u_2, \alpha) = 4\alpha(x_1 - 1) + 2u_1\alpha\alpha + 4u_2\alpha,
\]

\[
\nabla_x L^*_\alpha(x_1, x_2, u_1, u_2, \alpha) = 4\alpha(x_2 - 2) - u_1\alpha + 4u_2\alpha.
\]
\[ \nabla_u L(x_1, x_2, u_1, u_2, \alpha) = 2(4 - 2\alpha)(x_1 - 1) + 2u_1(2-\alpha)x_1 + u_2(8 - 4\alpha), \]
\[ \nabla_u L(x_1, x_2, u_1, u_2, \alpha) = 2(4 - 2\alpha)(x_2 - 2) - u_1(2-\alpha) + u_2(8 - 4\alpha), \]

Now for solving the problem (20), by Theorem 4.14, we have to solve
\[ \tilde{L}_u(x_1, x_2, u_1, u_2) = 0 = \tilde{L}_u(x_1, x_2, u_1, u_2), \]
\[ u_1\tilde{g}_1(x_1, x_2) = 0 = u_2\tilde{g}_2(x_1, x_2), \]
\[ \tilde{g}_1(x_1, x_2) \leq 0, \tilde{g}_2(x_1, x_2) \leq 0, u_1, u_2 \geq 0. \]

But, the above system is equivalent to
\[ \tilde{L}_u(x_1, x_2, u_1, u_2, \alpha) = 0 = \tilde{L}_x(x_1, x_2, u_1, u_2, \alpha), \]
\[ u_1\tilde{g}_1(x_1, x_2) = 0 = u_2\tilde{g}_2(x_1, x_2), \]
\[ g_1(x_1, x_2) \leq 0, g_2(x_1, x_2) \leq 0, g_3(x_1, x_2) \leq 0, \]
\[ g_4(x_1, x_2) \leq 0, u_1, u_2 \geq 0. \]

That is to solve
\[ 4\alpha(x_1 - 1) + 2au_1x_1 + 4au_2, 0, \]
\[ 2(4 - 2\alpha)(x_1 - 1) + 2(2-\alpha)u_1x_1 + (8 - 4\alpha)u_2 = 0, \]
\[ 4\alpha(x_2 - 2) - u_1 + 4au_2 = 0, \]
\[ 2(4 - 2\alpha)(x_2 - 2) - (2-\alpha)u_1 + (8 - 4\alpha)u_2 = 0, \]
\[ 4\alpha(x_1^2 - x_2) = 0 = u_1(2-\alpha)(x_1^2 - x_2), \]
\[ 4\alpha(x_1 + x_2 - 2) = 0 = u_2(8 - 4\alpha)(x_1 + x_2 - 2), \]
\[ \alpha(x_1^2 - x_2) \leq 0, \]
\[ (2-\alpha)(x_1^2 - x_2) \leq 0, \]
\[ 4\alpha(x_1 + x_2 - 2) \leq 0, \]
\[ (8 - 4\alpha)(x_1 + x_2 - 2) \leq 0, \]
\[ u_1, u_2 \geq 0. \]

Solving (21)–(31), we get \( x_1 = 1/2, \ x_2 = 3/2, \ u_1 = 0 \) and \( u_2 = 1/2. \) Thus, the minimum value of the problem is found to be \( 1/2(0,2,4). \)

5. Saddle point optimality conditions in fuzzy optimization problems

Consider the problem,
\[ \text{Minimize } \tilde{f}(x) \]
\[ \text{subject to } \tilde{g}(x) \leq 0 \] (32)
where \( \tilde{f}, \tilde{g} \) are differentiable fuzzy mappings of \( x \in \Omega \subseteq R^n. \) It can be shown that under suitable assumptions, the fuzzy optimization problem (32) can be transformed into an equivalent fuzzy saddle point problem.

Let \( \phi(x, u) \) be a comparable fuzzy mapping in terms of \( x \in X \subseteq \Omega \) and \( u \in U \subseteq R^m. \) A point \( (x_0, u_0) \) is said to be a saddle point of \( \phi(x, u) \), if for all \( x \in X \) and \( u \in U \)
\[ \phi(x_0, u_0) \leq \phi(x, u) \leq \phi(x_0, u_0). \]

In other words, a saddle point of the fuzzy mapping \( \phi(x, u) \) is a point \( (x_0, u_0) \) that minimize the fuzzy mapping \( \phi(x, u) \) in \( X \) for fixed \( u_0 \in U, \) and maximize the fuzzy mapping \( \phi(x, u) \) in \( U \) for fixed \( x_0 \in X, \) simultaneously. \( \tilde{v} = \phi(x_0, u_0) \) is then called a fuzzy saddle value, of \( \phi(x, u). \)

The Lagrangian fuzzy mapping associated with the fuzzy optimization (32), is given by,
\[ \tilde{L}(x, u) = \tilde{f}(x) + u^\top \tilde{g}(x) \]
where \( u \in R^m \) is called the vector of Lagrange multipliers and \( \tilde{g}(x) = (\tilde{g}_1(x), \tilde{g}_2(x), ..., \tilde{g}_n(x))^\top. \)

The corresponding fuzzy saddle point problem is to find a pair \( (x_0, u_0) \), such that for all \( x \in \Omega \subseteq R^n, \)
\[ 0 \leq u \leq u_0 \Rightarrow \tilde{f}(x_0) + u^\top \tilde{g}(x_0) \leq \tilde{f}(x) + u^\top \tilde{g}(x), \] (33)

Lemma 5.1: Let \( \tilde{L}(x, u) \) be the fuzzy Lagrangian mapping associated with the fuzzy optimization problem (32). Let \( \tilde{L}(x, u) \in \Omega. \) Suppose that \( (x_0, u_0) \) is a saddle point of \( \tilde{L}(x, u). \) Then, the following results hold true:
1. \( x_0 \) is a feasible solution to the fuzzy minimization problem (32).
2. \( u_0^\top \tilde{g}(x_0) = 0. \)

Proof: 1. Since \( (x_0, u_0) \) is a saddle point of the fuzzy mapping \( \tilde{L}(x, u) \), from (33) we have for all \( x \in \Omega, \alpha \in [0,1] \) and \( u \geq 0 \)
\[ f_i(x_0, \alpha) + u^\top g_i(x_0, \alpha) \leq f_i(x, \alpha) + u^\top g_i(x, \alpha), \]
(34)
and
\[ f^*(x_0, \alpha) + u^\top g^*(x_0, \alpha) \leq f^*(x, \alpha) + u^\top g^*(x, \alpha), \]
(35)
From the left hand inequalities in (34), (35), we have for all \( u \geq 0 \) and \( \alpha \in [0,1], \)
\[ u^\top g_i(x_0, \alpha) \leq u^\top g_i(x_0, \alpha), \]
(36)
and hence, the inequalities (36), (37), are hold true for \( u = u_0 + e_i, \) where \( e_i \) is the ith unit m-vector.

Thus, we have for each \( \alpha \in [0,1] \) and \( i = 1,2, ..., m, \)
\[ g_i(x_0, \alpha) \leq 0, \]
(37)
and hence, the inequalities (36), (37), are hold true for \( u = u_0 + e_i, \) where \( e_i \) is the ith unit m-vector.

Repeating this process for all \( i, \) we get for each \( \alpha \in [0,1], \)
\[ g_i(x_0, \alpha) \leq 0, \] (38)
Let $\mathbf{u}_0 \geq 0$, from (38), we have for each $\alpha \in [0,1]$
\[ u_0 g(x_0, \alpha) \leq 0, \text{ and } u_0^\alpha g'(x_0, \alpha) \leq 0. \] (39)

By (36), (37), with $\mathbf{u} = 0$, we have for each $\alpha \in [0,1]$
\[ u_0 g(x_0, \alpha) \geq 0, \text{ and } u_0^\alpha g'(x_0, \alpha) \geq 0. \] (40)

By (37), (40), we get for each $\alpha \in [0,1]$
\[ u_0^\alpha g(x_0, \alpha) = 0, \text{ and } u_0^\alpha g'(x_0, \alpha) = 0. \] (41)

(41) implies that $u_0^\alpha \tilde{g}(x_0) = \tilde{0}$. This completes the proof.

**Theorem 5.3:** Let $\tilde{L}(x, \mathbf{u})$ be the fuzzy Lagrangian mapping associated with the fuzzy optimization problem (32), where $\tilde{f} \in E$. Let $\tilde{L}(x, \mathbf{u}) \in E$. If $(x_0, \mathbf{u}_0)$ is a saddle point of $\tilde{L}(x, \mathbf{u})$, then $x_0$ is an optimal solution of the problem (32).

**Proof:** By Lemma 5.1, part (1), $x_0$ is a feasible solution of the problem (32). Therefore, it is enough to show that $\tilde{f}(x_0) \preceq \tilde{f}(x)$, for each feasible solution $x$, of the problem (32).

By Lemma 5.1, part (2), the right hand inequality of (33), becomes
\[ \tilde{f}(x_0) \preceq \tilde{f}(x) + u_0^\alpha \tilde{g}(x) \]
which implies that for each feasible point $x$, and for $\alpha = 1$
\[ f_1(x_0, 1) \leq f_1(x, 1) + u_0^\alpha g_1(x, 1) \] (42)

Now, since $u_0^\alpha g(x) \preceq 0$, we have $u_0^\alpha g(x) \leq 0$, which implies that $u_0^\alpha g(x) \leq 0$.

Hence, by (42) we get $f_1(x_0, 1) \leq f_1(x, 1)$. Since $\tilde{f} \in E$, we conclude that $\tilde{f}(x_0) \preceq \tilde{f}(x)$, for each feasible point $x$. The proof is complete.

**Theorem 5.4:** If $f_1, f_2, \ldots, f_m$ are convex fuzzy mappings defined on non-empty set $\Omega \subseteq \mathbb{R}^n$, then $\lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_m f_m$ is a convex fuzzy mapping on $\Omega$ for $\lambda_i \geq 0$ ($i = 1, 2, \ldots, m$).

**Proof:** Since $f_1, f_2, \ldots, f_m$ are convex fuzzy mappings, then by Definition 4.11, the real valued functions, $f_1(x, \alpha), f_2(x, \alpha), \ldots, f_m(x, \alpha)$ and $f_1'(x, \alpha), f_2'(x, \alpha), \ldots, f_m'(x, \alpha)$ are convex for each $\alpha \in [0,1]$. It can be easy to show that the functions $\sum_{i=1}^m \lambda_i f_i(x, \alpha)$ and $\sum_{i=1}^m \lambda_i f_i'(x, \alpha)$ are convex on $\Omega$ for each $\alpha \in [0,1]$ and $\lambda_i \geq 0$ ($i = 1, 2, \ldots, m$). Thus, the fuzzy mapping $\lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_m f_m$ is also a convex mapping, and the proof is complete.

**Theorem 5.5:** Consider the problem (32), where $\Omega$ is a non-empty open convex set in $\mathbb{R}^n$. Let the differentiable fuzzy mappings $\tilde{f}$, $\tilde{g}$ are convex on $\Omega$. Let $x_0$ is a feasible solution of the problem (32), and $\tilde{f}$, $\tilde{g}$ satisfy (14), (15), respectively. Furthermore, suppose that the fuzzy lagrangian mapping $\tilde{L}(x, \mathbf{u})$ associated with the problem (32) is comparable. If $x_0$ is an optimal solution to the problem (32), and the linear independence constraint qualification holds true, then there exists an $u_0 \geq 0$ such that $(x_0, u_0)$ is a saddle point of $\tilde{L}(x, \mathbf{u})$.

**Proof:** Since $x_0$ is an optimal solution to the problem (32) and the linear independence constraint qualification is satisfied, thus the assumptions of the Theorem 4.10 are hold true, and the KKT conditions are applicable which implies that there exists an $u_0 \geq 0$ such that
\[ \tilde{f}(x_0) + u_0^\alpha \tilde{g}(x_0) = \tilde{0} \] (43)
\[ u_0^\alpha \tilde{g}(x_0) = \tilde{0} \] (44)

Now, by Theorem 5.3, the fuzzy mapping $\tilde{L}(x, u_0) = \tilde{f}(x) + u_0^\alpha \tilde{g}(x)$ is convex on $\Omega$, thus by differentiability of the fuzzy mappings $\tilde{f}$, $\tilde{g}$, and by Theorem 4.13, we have for all $x \in \Omega$,
\[ L(x, u_0, 1) \geq L(x, u_0, 1) + \nabla L(x, u_0, 1)^T (x-x_0) \]
But, by (43) we have
\[ \nabla L(x, u_0, 1) = \tilde{f}(x_0, 1) + u_0^\alpha \nabla g(x_0, 1) = 0 \]

Hence, $L(x, u_0, 1) \geq L(x, u_0, 1)$, and since $\tilde{L}(x, \mathbf{u})$ is a comparable fuzzy mapping, thus
\[ L(x, u_0, 1) \leq L(x, u_0, 1) \] (45)

Now, since $L(x, u_0, 1)$ is linear in terms of $\mathbf{u}$, then
\[ L(x, u_0, 1) = L(x, u_0, 1) + \nabla L(x, u_0, 1)^T (\mathbf{u} - u_0) \] (46)
But, we have $\nabla L(x, u_0, 1) = g(x_0, 1)$, and also from (44) we have $u_0^\alpha g(x_0, 1) = 0$, thus
\[ \nabla L(x, u_0, 1)^T (\mathbf{u} - u_0) = \nabla L(x, u_0, 1)^T \mathbf{u} = g(x_0, 1)^T \mathbf{u} \]
Furthermore, since $\mathbf{u} \geq 0$, $g(x_0, 1) \leq 0$, then by (46) we get $L(x, u_0, 1) \leq L(x, u_0, 1)$. Since by assumption $\tilde{L}(x, \mathbf{u}) \in E$, then
\[ \tilde{L}(x, \mathbf{u}) \leq \tilde{L}(x, u_0, 1) \] (47)

From (45), (47), we conclude that $\tilde{L}(x, u_0) \leq \tilde{L}(x, u_0, 1) \leq \tilde{L}(x, u_0)$, $(x_0, u_0)$ is a saddle point of $\tilde{L}(x, \mathbf{u})$. The proof is complete.

We now derive the conditions for existence of a saddle point for a fuzzy mapping $\tilde{\phi}(x, \mathbf{u})$, $x \in \mathbb{R}^n$, $0 \leq \mathbf{u} \in \mathbb{R}^m$.

**Theorem 5.5:** (Necessity) Let $\tilde{\phi}(x, \mathbf{u})$ is a comparable fuzzy mapping, and let $(x_0, u_0)$ is a saddle point of $\tilde{\phi}(x, \mathbf{u})$ for $\mathbf{u} \geq 0$. Suppose that for each $i = 1, 2, \ldots, m$ and $\alpha \in [0,1]$, $\tilde{\phi}(x_0, \mathbf{u})$ satisfies the following condition
\[
\tilde{\phi}(x_0, u_0)[\alpha] = [h(\alpha)\phi(x_0, u_0), h'(\alpha)\phi(x_0, u_0)],
\]
such that both \(h(\alpha), h'(\alpha)\) are functions in terms of \(\alpha\), and both are positive (or negative) for each \(\alpha \in [0,1]\), at the same time. If \(\tilde{\phi}(x, u)\) is a differentiable fuzzy mapping, then \((x_0, u_0)\) satisfies the following conditions:
\[
\begin{align*}
\tilde{\nabla}_0\tilde{\phi}(x_0, u_0) &= 0, \\
\tilde{\nabla}_0\tilde{\phi}(x_0, u_0) &\geq 0, \\
\tilde{\nabla}_0\tilde{\phi}(x_0, u_0)' u_0 &= 0 \\
u_0 &\geq 0
\end{align*}
\]
(49)

**Proof:** Since \((x_0, u_0)\) is a saddle point of \(\tilde{\phi}(x, u)\), thus \(\phi(x, u_0)\) has a local minimum at \(x_0\), and since \(\tilde{\phi}(x, u)\) is differentiable, then by Theorem 3.13 [Panigrahi, 9], we have \(\tilde{\nabla}_0\tilde{\phi}(x_0, u_0) = 0\). Also, since \((x_0, u_0)\) is a saddle point of the fuzzy mapping \(\tilde{\phi}(x, u)\) for \(u \geq 0\), then \(u_0 \geq 0\). But, since \((x_0, u_0)\) is a saddle point of \(\tilde{\phi}(x, u)\), then \((x_0, u_0)\) maximize the fuzzy mapping \(\tilde{\phi}(x, u)\) subject to \(u \geq 0\). In the other words \(u_0\) is the optimal solution to the following constrained fuzzy maximization problem:

\[
\begin{align*}
\text{Maximize} & \quad \tilde{\phi}(x, u) \\
\text{subject to} & \quad u \geq 0
\end{align*}
\]
(50)

Since, \(\tilde{\phi}(x, u)\) satisfies (48), and the constraints of the problem (50) are linear, then \(u_0\) satisfies the KKT necessary conditions, that is, there exists a vector \(v \geq 0\) such that
\[
\begin{align*}
\tilde{\nabla} \tilde{\phi}(x_0, u_0) + v &= 0, \\
v \cdot u_0 &= 0, \quad v, u_0 \geq 0.
\end{align*}
\]
(51)

From (51), it can be easy to conclude that \(\tilde{\nabla} \tilde{\phi}(x_0, u_0) \geq 0\), \(\tilde{\nabla} \tilde{\phi}(x_0, u_0)' u_0 = 0\).

The proof is complete.

**Theorem 5.6:** (Sufficiency) Let \(\tilde{\phi}(x, u)\) is a comparable fuzzy mapping. Let \(\tilde{\phi}(x, u)\) is differentiable at \((x_0, u_0)\). Suppose that the fuzzy mapping \(\tilde{\phi}(x, u_0)\) is convex at \(x_0\). If \(\tilde{\phi}(x, u_0)\) is concave at \(u_0\), and satisfies (48), then the conditions (49) are both necessary and sufficient for \((x_0, u_0)\) to be a saddle point of \(\tilde{\phi}(x, u)\).

**Proof:** Since, the fuzzy mapping \(\tilde{\phi}(x, u)\) is differentiable at \((x_0, u_0)\), then both functions \(\phi(x, u, \alpha)\), \(\phi^*(x, u, \alpha)\) are also differentiable at \((x_0, u_0)\) for each \(\alpha \in [0,1]\). Therefore, since the fuzzy mapping \(\tilde{\phi}(x, u_0)\) is convex at \(x_0\), we have by Theorem 4.13, and for \(\alpha_0 = 1\)
\[
\phi(x, u_0, 1) \geq \phi(x, u_0, 0) + \tilde{\nabla} \phi(x, u_0, 0)' (x - x_0)
\]
(52)

But, since \(\tilde{\nabla}_0\tilde{\phi}(x_0, u_0) = 0\), then we have for each \(\alpha \in [0,1]\)
\[
\nabla_\alpha \phi(x_0, u_0, \alpha) = \nabla \phi(x_0, u_0, 0) \nabla \phi(x_0, u_0, 0)'(x - x_0)
\]
(53)

Thus, by (52), (53), we get
\[
\phi(x, u_0, 1) \geq \phi(x, u_0, 0)
\]
(54)

But, \(\tilde{\nabla}_0\tilde{\phi}(x_0, u_0) \geq 0\) implies that
\[
\nabla_\alpha \phi(x_0, u_0, \alpha) \geq 0
\]
(55)

for each \(\alpha \in [0,1]\). Also, \(\tilde{\nabla}_0\tilde{\phi}(x_0, u_0)' u_0 = 0\) implies that
\[
\nabla_\alpha \phi(x_0, u_0, \alpha)' u_0 = 0
\]
(56)

for each \(\alpha \in [0,1]\). Therefore, since \(u \geq 0\), then by (55), (56), we get
\[
\nabla_\alpha \phi(x_0, u_0, 1)'(u - u_0) \leq 0
\]
(57)

Now, from (54), (57), we get
\[
\phi(x, u_0, 1) \geq \phi(x_0, u_0, 1)
\]
(58)

But, since the fuzzy mapping \(\tilde{\phi}(x, u_0)\) is concave at \(u_0\), then by Definition 2.3, and by Theorem 4.13, we have for \(\alpha_0 = 1\), and for all \(u \geq 0\)
\[
\phi(x, u_0, 1) + \tilde{\nabla}_u \phi(x, u_0, 1)'(u - u_0) \geq \phi(x_0, u_0, 1)
\]
(59)

Thus, by (58), (59), we have for all \(u \geq 0\)
\[
\phi(x_0, u_0, 1) \leq \phi(x_0, u_0, 1) \leq \phi(x_0, u_0, 1)
\]

Hence, since \(\tilde{\phi}(x, u)\) is a comparable fuzzy mapping, then we get
\[
\phi(x_0, u_0) \geq \phi(x_0, u_0) \leq \phi(x_0, u_0)
\]

Thus, \((x_0, u_0)\) is saddle point of the fuzzy mapping \(\tilde{\phi}(x, u)\). The proof is complete.

**Example 5.7:** Consider the same problem in Example 4.15. The fuzzy Lagrangian mapping is then given by,
\[
\tilde{L}(x_1, x_2, u_1, u_2) = f(x_1, x_2) + u_1\tilde{g}(x_1, x_2) + u_2\tilde{g}(x_1, x_2)
\]

We are going to show that \((x_1, x_2, u_1, u_2) = (1/2, 3/2, 0, 1/2)\), is a saddle point of the fuzzy Lagrangian mapping \(\tilde{L}(x, u)\). From Example 4.15, \((x_1, x_2) = (1/2, 3/2)\) is an optimal solution of the problem (20). It is not hard to see that, the other conditions of the Theorem 5.4 are hold true for this problem. Therefore, there exists a vector \(u_0 = (u_1, u_2) \geq 0\) such that \((x_1, x_2, u_1, u_2) = (1/2, 3/2, 0, 1/2)\) is a saddle point of the problem (20). But, from Example 4.15, it can be easily calculated that the point \((x_1, x_2, u_1, u_2) = (1/2, 3/2, 0, 1/2)\) satisfies the conditions (49). Thus by Theorem 5.6, \((1/2, 3/2, 0, 1/2)\) is a saddle point of the fuzzy Lagrangian mapping \(\tilde{L}(x, u)\).

**6. Conclusion**

The Karus-Kuhn-Tucker (KKT) optimality conditions and saddle point optimality conditions in fuzzy programming problems with fuzzy coefficients are sug-
gested in this paper by introducing a partial order relation on the set of fuzzy numbers, and convexity with differentiability of fuzzy mappings. We have obtained the Fritz John (FJ) constraint qualification and KKT necessary conditions for a fuzzy optimization problem with fuzzy coefficients, for first time. Owing to the help of the KKT optimality conditions, we then discuss, the saddle point optimality conditions, associated with a fuzzy optimization problem under convexity and differentiability of fuzzy mappings.

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References


Hassan Mishmast Nehi received his B.Sc. degree in Mathematics at Tarbiat Moallem University, Tehran, Iran, in 1989. He received M.Sc. and Ph.D degrees in Applied Mathematics from university of Kerman, Iran, in 1992 and 2003, respectively. He is currently an Associate professor in Mathematics in faculty of Mathematics at university of Sistan and Baluchestan, Zahedan, Iran. His research interests include operations research, fuzzy optimization and soft computing methods.