On a Spector ultrapower of the Solovay model

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Abstract

We prove that a Spector–like ultrapower extension \( \mathcal{M} \) of a countable Solovay model \( \mathcal{M} \) (where all sets of reals are Lebesgue measurable) is equal to the set of all sets constructible from reals in a generic extension \( \mathcal{M}[\alpha] \) where \( \alpha \) is a random real over \( \mathcal{M} \). The proof involves an almost everywhere uniformization theorem in the Solovay model.

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Introduction

Let \( U \) be an ultrafilter in a transitive model \( M \) of ZF. Assume that an ultrapower of \( M \) via \( U \) is to be defined. The first problem we meet is that \( U \) may not be an ultrafilter in the universe because not all subsets of the index set belong to \( M \).

We can, of course, extend \( U \) to a true ultrafilter, say \( U' \), but this may cause additional trouble. Indeed, if \( U \) is a special ultrafilter in \( M \) certain properties of which were expected to be exploit, then most probably these properties do not transfer to \( U' \); assume for instance that \( U \) is countably complete in \( M \) and \( M \) itself is countable. Therefore, it is better to keep \( U \) rather than any of its extensions in the universe, as the ultrafilter.

If \( M \) models ZFC, the problem can be solved by taking the inner ultrapower. In other words, we consider only those functions \( f : I \to M \) (where \( I \in M \) is the carrier of \( U \)) which belong to \( M \) rather than all functions \( f \in M^I \), to define the ultrapower. This version, however, depends on the axiom of choice in \( M \); otherwise the proofs of the basic facts about ultrapowers (e.g. Loś theorem) will not work.

The “choiceless” case can be handled by a sophisticated construction of Spector [1991], which is based on ideas from both forcing and the ultrapower technique. As presented in Kanovei and van Lambalgen [1994], this construction proceeds as follows. One has to add to the family of functions \( F_0 = M^I \cap M \) a number of new functions \( f \in M^I \setminus M \), which are intended to be choice functions whenever we need such in the ultrapower construction.

In this paper, we consider a very interesting choiceless case: \( M \) is a Solovay model of ZF plus the principle of dependent choice, in which all sets of reals are Lebesque measurable, and the ultrafilter \( L \) on the set \( I \) of Vitali degrees of reals in \( M \), generated by sets of positive measure.
1 On a.e. uniformization in the Solovay model

In this section, we recall the uniformization properties in a Solovay model. Thus let $\mathcal{M}$ be a countable transitive Solovay model for Dependent Choices plus “all sets are Lebesgue measurable”, as it is defined in Solovay [1970], – the ground model. The following known properties of such a model will be of particular interest below.

**Property 1** [True in $\mathcal{M}$]

$V = \mathbb{L}(\text{reals})$; in particular every set is real–ordinal–definable.

To state the second property, we need to introduce some notation.

Let $\mathcal{N} = \omega^\omega$ denote the Baire space, the elements of which will be referred to as real numbers or reals.

Let $P$ be a set of pairs such that $\text{dom } P \subseteq \mathcal{N}$ (for instance, $P \subseteq \mathcal{N}^2$). We say that a function $f$ defined on $\mathcal{N}$ uniformizes $P$ a.e. (almost everywhere) iff the set

$$\{ \alpha \in \text{dom } P : \langle \alpha, f(\alpha) \rangle \notin P \}$$

has null measure. For example if the projection $\text{dom } P$ is a set of null measure in $\mathcal{N}$ then any $f$ uniformizes a.e. $P$, but this case is not interesting. The interesting case is the case when $\text{dom } P$ is a set of full measure, and then $f$ a.e. uniformizes $P$ iff for almost all $\alpha$, $\langle \alpha, f(\alpha) \rangle \in P_\alpha$.

**Property 2** [True in $\mathcal{M}$]

Any set $P \in \mathcal{M}$, $P \subseteq \mathcal{N}^2$, can be uniformized a.e. by a Borel function. (This implies the Lebesgue measurability of all sets of reals, which is known to be true in $\mathcal{M}$ independently.)

This property can be expanded (with the loss of the condition that $f$ is Borel) on the sets $P$ which do not necessarily satisfy $\text{dom } P \subseteq \mathcal{N}$.

**Theorem 3** In $\mathcal{M}$, any set $P$ with $\text{dom } P \subseteq \mathcal{M}$ admits an a.e. uniformisation.

**Proof** Let $P$ be an arbitrary set of pairs such that $\text{dom } P \subseteq \mathcal{N}$ in $\mathcal{M}$. Property [1] implies the existence of a function $D : (\text{Ord } \cap \mathcal{M}) \times (\mathcal{N} \cap \mathcal{M})$ onto $\mathcal{M}$ which is $\in$-definable in $\mathcal{M}$.

We argue in $\mathcal{M}$. Let, for $\alpha \in \mathcal{N}$, $\xi(\alpha)$ denote the least ordinal $\xi$ such that

$$\exists \gamma \in \mathcal{N} [ \langle \alpha, D(\xi, \gamma) \rangle \in P ] .$$

(It follows from the choice of $D$ that $\xi(\alpha)$ is well defined for all $\alpha \in \mathcal{N}$.) It remains to apply Property 2 to the set $P' = \{ \langle \alpha, \gamma \rangle \in \mathcal{N}^2 : \langle \alpha, D(\xi(\alpha), \gamma) \rangle \in P \}$.
2 The functions to get the Spector ultrapower

We use a certain ultrafilter over the set of Vitali degrees of reals in $\mathcal{M}$, the initial Solovay model, to define the ultrapower.

Let, for $\alpha, \alpha' \in \mathbb{N}$, $\alpha \ vit \ \alpha'$ if and only if $\exists m \ \forall k \geq m \ (\alpha(k) = \alpha'(k))$, (the Vitali equivalence).

- For $\alpha \in \mathbb{N}$, we set $\underline{\alpha} = \{\alpha' : \alpha' \ vit \ \alpha\}$, the Vitali degree of $\alpha$.
- $\underline{\mathbb{N}} = \{\alpha : \alpha \in \mathbb{N}\}$; $i, j$ denote elements of $\underline{\mathbb{N}}$.

As a rule, we shall use underlined characters $\underline{f}, \underline{F}, \ldots$ to denote functions defined on $\underline{\mathbb{N}}$, while functions defined on $\mathbb{N}$ itself will be denoted in the usual manner.

Define, in $\mathcal{M}$, an ultrafilter $\mathcal{L}$ over $\underline{\mathbb{N}}$ by: $X \subseteq \underline{\mathbb{N}}$ belongs to $\mathcal{L}$ iff the set $X = \{\alpha \in \mathbb{N} : \alpha \in X\}$ has full Lebesgue measure. It is known (see e.g. van Lambalgen [1992], Theorem 2.3) that the measurability hypothesis implies that $\mathcal{L}$ is $\kappa$-complete in $\mathcal{M}$ for all cardinals $\kappa$ in $\mathcal{M}$.

One cannot hope to define a good $\mathcal{L}$-ultrapower of $\mathcal{M}$ using only functions from $F_0 = \{f \in \mathcal{M} : \text{dom } f = \mathbb{N}\}$ as the base for the ultrapower. Indeed consider the identity function $i \in \mathcal{M}$ defined by $i(i) = i$ for all $i \in \mathbb{N}$. Then $i(i)$ is nonempty for all $i \in \underline{\mathbb{N}}$ in $\mathcal{M}$, therefore to keep the usual properties of ultrapowers we need a function $\underline{f} \in F_0$ such that $\underline{f}(i) \in i$ for almost all $i \in \underline{\mathbb{N}}$, but Vitali showed that such a choice function yields a nonmeasurable set.

Thus at least we have to add to $F_0$ a new function $\underline{f}$, not an element of $\mathcal{M}$, which satisfies $\underline{f}(i) \in i$ for almost all $i \in \underline{\mathbb{N}}$. Actually it seems likely that we have to add a lot of new functions, to handle similar situations, including those functions the existence of which is somehow implied by the already added functions. A general way how to do this, extracted from the exposition in Spector [1991], was presented in Kanovei and van Lambalgen [1994]. However in the case of the Solovay model the a.e. uniformization theorem (Theorem 3) allows to add essentially a single new function, corresponding to the $i$-case considered above.

The generic choice function for the identity

Here we introduce a function $\tau$ defined on $\mathbb{N} \cap \mathcal{M}$ and satisfying $\tau(i) \in i$ for all $i \in \mathbb{N} \cap \mathcal{M}$. $\tau$ will be generic over $\mathcal{M}$ for a suitable notion of forcing.

The notion of forcing is introduced as follows. In $\mathcal{M}$, let $\mathbb{P}$ be the set of all functions $p$ defined on $\underline{\mathbb{N}}$ and satisfying $p(i) \subseteq i$ and $p(i) \neq \emptyset$ for all $i$. (For example $i \in \mathbb{P}$.) We order $\mathbb{P}$ so that $p$ is stronger than $q$ iff $p(i) \subseteq q(i)$ for all $i$.

If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathcal{M}$, $G$ defines a function $\tau$ by

$$\tau(i) = \text{the single element of } \bigcap_{p \in G} p(i)$$

1Or, equivalently, the collection of all sets $X \subseteq \mathbb{N}$ which have a nonempty intersection with every Vitali degree. Perhaps this forcing is of separate interest.
for all \( i \in \mathbb{N} \cap M \). Functions \( r \) defined this way will be called \( \mathbb{P} \)-generic over \( M \). Let us fix such a function \( r \) for the remainder of this paper.

The set of functions used to define the ultrapower

We let \( F \) be the set of all superpositions \( f \circ r \) where \( r \) is the generic function fixed above while \( f \in M \) is an arbitrary function defined on \( \mathbb{N} \cap M \). Notice that in particular any function \( f \in M \) defined on \( \mathbb{N} \cap M \) is in \( F \): take \( f(\alpha) = f(r(\alpha)) \).

To see that \( F \) can be used successfully as the base of an ultrapower of \( M \), we have to check three fundamental conditions formulated in Kanovei and van Lambalgen [1994].

**Proposition 4** [Measurability] Assume that \( E \in M \) and \( f_1, \ldots, f_n \in F \). Then the set \( \{ i \in \mathbb{N} \cap M : E(f_1(i), \ldots, f_n(i)) \} \) belongs to \( M \).

**Proof** By the definition of \( F \), it suffices to prove that \( \{ i : r(i) \in E \} \in M \) for any set \( E \in M \), \( E \subseteq \mathbb{N} \). By the genericity of \( r \), it remains then to prove the following in \( M \): for any \( p \in \mathbb{P} \) and any set \( E \subseteq \mathbb{N} \), there exists a stronger condition \( q \) such that, for any \( i \), either \( q(i) \subseteq E \) or \( q(i) \cap E = \emptyset \). But this is obvious. \( \square \)

**Corollary 5** Assume that \( V \in M \), \( V \subseteq \mathbb{N} \) is a set of null measure in \( M \). Then, for \( \mathcal{L} \)-almost all \( i \), we have \( r(i) \notin V \).

**Proof** By the proposition, the set \( I = \{ i : r(i) \in V \} \) belongs to \( M \). Suppose that, on the contrary, \( I \in \mathcal{L} \). Then \( A = \{ \alpha : \alpha \in I \} \) is a set of full measure. On the other hand, since \( r(i) \in i \), we have \( A \subseteq \bigcup_{\beta \in V} \beta \), where the right–hand side is a set of null measure because \( V \) is such a set, contradiction. \( \square \)

**Proposition 6** [Choice] Let \( f_1, \ldots, f_n \in F \) and \( W \in M \). There exists a function \( f \in F \) such that, for \( \mathcal{L} \)-almost all \( i \in \mathbb{N} \cap M \), it is true in \( M \) that

\[
\exists x \ W(f_1(i), \ldots, f_n(i), x) \rightarrow W(f_1(i), \ldots, f_n(i), f(i)) .
\]

**Proof** This can be reduced to the following: given \( W \in M \), there exists a function \( f \in F \) such that, for \( \mathcal{L} \)-almost all \( i \in \mathbb{N} \cap M \),

\[
\exists x \ W(r(i), x) \rightarrow W(r(i), f(i)) \tag{*}
\]

in \( M \).

\( \text{To make things clear, } f \circ r(i) = f(r(i)) \text{ for all } i . \)

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We argue in $\mathcal{M}$. Choose $p \in \mathbb{P}$. and let $p'(i) = \{\beta \in p(i) : \exists x W(\beta, x)\}$, and $X = \{i : p'(i) \neq \emptyset\}$. If $X \not\in \mathcal{L}$ then an arbitrary $f$ defined on $\mathbb{N}$ will satisfy (*), therefore it is assumed that $X \in \mathcal{L}$. Let

$$q(i) = \begin{cases} p'(i) & \text{iff } i \in X \\ p(i) & \text{otherwise} \end{cases}$$

for all $i \in \mathbb{N}$; then $q \in \mathbb{P}$ is stronger than $p$. Therefore, since $r$ is generic, one may assume that $r(i) \in q(i)$ for all $i$.

Furthermore, DC in the Solovay model $\mathcal{M}$ implies that for every $i \in X$ the following is true: there exists a function $\phi$ defined on $q(i)$ and such that $W(\beta, \phi(\beta))$ for every $\beta \in q(i)$. Theorem 3 provides a function $\Phi$ such that for almost all $\alpha$ the following is true: the value $\Phi(\alpha, \beta)$ is defined and satisfies $W(\beta, \Phi(\alpha, \beta))$ for all $\beta \in q(\alpha)$. Then, by Corollary 3, we have

$$\text{for all } \beta \in q(r(i)), \ W(\beta, \Phi(r(i), \beta))$$

for almost all $i$. However, $r(i) = i$ for all $i$. Applying the assumption that $r(i) \in q(i)$ for all $i$, we obtain $W(\overline{r(i)}, \Phi(r(i), r(i)))$ for almost all $i$. Finally the function $f(i) = \Phi(r(i), r(i))$ is in $\mathcal{F}$ by definition. $\square$

**Proposition 7** [Regularity] For any $f \in \mathcal{F}$ there exists an ordinal $\xi \in \mathcal{M}$ such that for $\mathcal{L}$-almost all $i$, if $f(i)$ is an ordinal then $f(i) = \xi$.

**Proof** To prove this statement, assume that $f = f \circ r$ where $f \in \mathcal{M}$ is a function defined on $\mathbb{N}$ in $\mathcal{M}$.

We argue in $\mathcal{M}$. Consider an arbitrary $p \in \mathbb{P}$. We define a stronger condition $p'$ as follows. Let $i \in \mathbb{N}$. If there does not exist $\beta \in p(i)$ such that $f(\beta)$ is an ordinal, we put $p'(i) = p(i)$ and $\xi(i) = 0$. Otherwise, let $\xi(i)$ be the least ordinal $\xi$ such that $f(\beta) = \xi$ for some $\beta \in p(i)$. We set $p'(i) = \{\beta \in p(i) : f(\beta) = \xi(i)\}$.

Notice that $\xi(i)$ is an ordinal for all $i \in \mathbb{N}$. Therefore, since the ultrafilter $\mathcal{L}$ is $\kappa$-complete in $\mathcal{M}$ for all $\kappa$, there exists a single ordinal $\xi \in \mathcal{M}$ such that $\xi(i) = \xi$ for almost all $i$.

By genericity, we may assume that actually $r(i) \in p'(i)$ for all $i \in \mathbb{N} \cap \mathcal{M}$. Then $\xi$ is as required. $\square$

**The ultrapower**

Let $\mathcal{N} = \text{Ult}_{\mathcal{L}} \mathcal{F}$ be the ultrapower. Thus we define:

- $f \approx g$ iff $\{i : f(i) = g(i)\} \in \mathcal{L}$ for $f, g \in \mathcal{F}$;
- $[f] = \{g : g \approx f\}$ (the $\mathcal{L}$-degree of $f$).
• \([f] \in^* [g]\) iff \(\{i : f(i) \in g(i)\} \in \mathcal{L}\);

• \(\mathfrak{N} = \{[f] : f \in \mathcal{F}\}\), equipped with the above defined membership \(\in^*\).

Theorem 8  \(\mathfrak{N}\) is an elementary extension of \(\mathcal{M}\) via the embedding which associates \(x^* = [\mathfrak{N} \times \{x\}]\) with any \(x \in \mathcal{M}\). Moreover \(\mathfrak{N}\) is wellfounded and the ordinals in \(\mathcal{M}\) are isomorphic to the \(\mathcal{M}\)-ordinals via the mentioned embedding.

Proof  See Kanovei and van Lambalgen [1994].

Comment. Propositions 4 and 6 are used to prove the Loś theorem and the property of elementary embedding. Proposition 7 is used to prove the wellfoundedness part of the theorem.

3  The nature of the ultrapower

Theorem 8 allows to collapse \(\mathfrak{N}\) down to a transitive model \(\widehat{\mathfrak{N}}\); actually \(\widehat{\mathfrak{N}} = \{\widehat{X} : X \in \mathfrak{N}\}\) where

\[\widehat{X} = \{\widehat{Y} : Y \in \mathfrak{N} \text{ and } Y \in^* X\}\] .

The content of this section will be to investigate the relations between \(\mathcal{M}\), the initial model, and \(\widehat{\mathfrak{N}}\), the (transitive form of its) Spector ultrapower. In particular it is interesting how the superposition of the “asterisk” and “hat” transforms embeds \(\mathcal{M}\) into \(\widehat{\mathfrak{N}}\).

Lemma 9  \(x \mapsto x^*\) is an elementary embedding \(\mathcal{M}\) into \(\widehat{\mathfrak{N}}\), equal to identity on ordinals and sets of ordinals (in particular on reals).

Proof  Follows from what is said above.

Thus \(\widehat{\mathfrak{N}}\) contains all reals in \(\mathcal{M}\). We now show that \(\widehat{\mathfrak{N}}\) also contains some new reals. We recall that \(r \in \mathcal{I}\) is a function satisfying \(r(i) \in i\) for all \(i \in \mathcal{N} \cap \mathcal{M}\).

Let \(a = [\hat{r}]\). Notice that by Loś \([r]\) is a real in \(\mathfrak{N}\), therefore \(a\) is a real in \(\widehat{\mathfrak{N}}\).

Lemma 10  \(a\) is random over \(\mathcal{M}\).

Proof  Let \(B \subseteq \mathcal{N}\) be a Borel set of null measure coded in \(\mathcal{M}\); we prove that \(a \notin B\). Being of measure 0 is an absolute notion for Borel sets, therefore \(B \cap \mathcal{M}\) is a null set in \(\mathcal{M}\) as well. Corollary 3 implies that for \(\mathcal{L}\)-almost all \(i\), we have \(r(i) \notin B\). By Loś, \(\neg (([r] \in^* B^*)\) in \(\mathfrak{N}\). Then \(a \notin \hat{B}^*\) in \(\widehat{\mathfrak{N}}\). However, by the absoluteness of the Borel coding, \(\hat{B}^* = B \cap \mathfrak{N}\), as required.

Thus \(\widehat{\mathfrak{N}}\) contains a new real number \(a\). It so happens that this \(a\) generates all reals in \(\widehat{\mathfrak{N}}\).
Lemma 11  The reals of $\hat{\mathcal{N}}$ are exactly the reals of $\mathcal{M}[a]$.

Proof  It follows from the known properties of random extensions that every real in $\mathcal{M}[a]$ can be obtained as $F(a)$ where $F$ is a Borel function coded in $\mathcal{M}$. Since $a$ and all reals in $\mathcal{M}$ belong to $\hat{\mathcal{N}}$, we have the inclusion $\supseteq$ in the lemma.

To prove the opposite inclusion let $\beta \in \hat{\mathcal{N}} \cap \mathcal{N}$. Then by definition $\beta = [\hat{F}]$, where $F \in \mathcal{F}$. In turn $F = f \circ \tau$, where $f \in \mathcal{M}$ is a function defined on $\mathcal{N} \cap \mathcal{M}$. We may assume that in $\mathcal{M}$ $f$ maps reals into reals. Then, first, by Property 2, $f$ is a.e. equal in $\mathcal{M}$ to a Borel function $g = B_{\gamma}$ where $\gamma \in \mathcal{N} \cap \mathcal{M}$ and $B_{\gamma}$ denotes, in the usual manner, the Borel subset (of $\mathcal{N}^2$ in this case) coded by $\gamma$. Corollary 3 shows that we have $F(i) = B_{\gamma}(\tau(i))$ for $\mathcal{L}$-almost all $i$. In other words, $F(i) = B_{\gamma}(\tau(i))$ for $\mathcal{L}$-almost all $i$. By Loś, this implies $[F] = B_{[\gamma]}([\tau])$ in $\mathcal{N}$, therefore $\beta = B_{\gamma}(a)$ in $\hat{\mathcal{N}}$. By the absoluteness of Borel coding, we have $\beta \in \mathcal{L}_{[\gamma,a]}$, therefore $\beta \in \mathcal{M}[a]$.

We finally can state and prove the principal result.

Theorem 12  $\hat{\mathcal{N}} \subseteq \mathcal{M}[a]$ and $\hat{\mathcal{N}}$ coincides with $\mathcal{L}^\mathcal{M}[a](\text{reals})$, the smallest subclass of $\mathcal{M}[a]$ containing all ordinals and all reals of $\mathcal{M}[a]$ and satisfying all the axioms of ZF.

Proof  Very elementary. Since $\mathcal{V} = \mathcal{L}(\text{reals})$ is true in $\mathcal{M}$, the initial Solovay model, this must be true in $\hat{\mathcal{N}}$ as well. The previous lemma completes the proof.

Corollary 13  The set $\mathcal{N} \cap \mathcal{M}$ of all “old” reals does not belong to $\hat{\mathcal{N}}$.

Proof  The set in question is known to be non–measurable in the random extension $\mathcal{M}[a]$; thus it would be non–measurable in $\hat{\mathcal{N}}$ as well. However $\hat{\mathcal{N}}$ is an elementary extension of $\mathcal{M}$, hence it is true in $\hat{\mathcal{N}}$ that all sets are measurable.

References


