Bayesian Estimation of Zenga Curve Using Different Priors In Case of Pareto Distribution

Sangeeta Arora, Kalpana K. Mahajan, Prerna Godura

ABSTRACT — The classical measures of inequality viz. Lorenz curve and related Gini, are based upon comparing the incomes of less fortunate with the mean income of the entire population, which may not be adequate to take into account the increase in disparities between less and more fortunate individuals. In this context, a new curve of inequality known as Zenga curve is introduced which accounts for the relative nature of ‘poor’ and ‘rich’ and is based on the ratio of the lower and upper group means. The Zenga curve represents an alternative to the well-known Lorenz curve, as a measure of economic inequality. The Zenga curve assumes different shapes for different distributions and can serve as a very useful tool to graphically discriminate between them. The Bayesian estimation of this new but important curve of inequality still awaits the attention of researchers though some results in this direction are available in the Classical estimation setup. In this paper, Bayesian estimator for Zenga curve is obtained for Pareto distribution using Jeffreys’ prior, extension of Jeffreys’ prior and conjugate prior under symmetric squared error loss function (SELF). Using simulation techniques, the relative efficiency of the proposed estimators is obtained in terms of expected loss for different configurations of sample sizes and shape parameter for different priors along with real life illustration.

KEYWORDS — Zenga curve, Pareto distribution, Bayes estimation, Squared error loss function.

I. INTRODUCTION

Measures of inequality are widely used to study income inequality and its welfare interpretation. Lorenz curve and Gini index are the two most popular measures of income inequality. Lorenz curve plots the cumulative percentage of total national income against the cumulative percentage of the corresponding population, whereas the associated Gini coefficient or Gini index equals one minus twice the area under the Lorenz curve (Gastwirth 1971, Kakwani 1980). Lorenz ordering is a partial ordering whereas Gini index provides a complete ordering of the income distributions (Lambert 1989, Arora et al. 2011).

However in recent years, a noticeable gap is visible between rich and poor, especially though the poor are getting off as better. To account for these considerations, a new curve of inequality viz. Zenga curve \( Z(p), 0 < p < 1 \), is introduced (Zenga 2007) which involves the ratio of the mean income of the 100 \( p \% \) poorest to that of the 100 \((1-p)\% \) richest and is also related to the Lorenz curve \( L(p) \) in a mathematical form. The corresponding index, the Zenga index, is a newer measure of income inequality which denotes the area under the Zenga curve. Compared with the Classical Gini index, Zenga index gives a more balanced picture of inequality. The Zenga plot based on Zenga curve is introduced in literature for discriminating among possible size distributions for data.

Estimation of these measures of inequality has evoked much interest of researchers for both Classical and Bayesian approach (Beach & Davidson 1983). In the Classical setup, the parametric and non-parametric estimation procedures for measure of inequality have been proposed along with the statistical inference (Moothathu 1985, Arora et al. 2010, Bishop et al. 1989). However, not much work has been done in the Bayesian setup related to Lorenz or other curves of inequality (Bhattacharya et al. 1999, Dyuthi 2002) and particularly in case of Zenga curve no Bayesian estimates are reported in literature.

In the present paper Zenga curve \( Z(p) \) is estimated by following a Bayesian approach and by assuming that the underlying population is from the Pareto distribution. The Pareto distribution has been used in the statistical analysis of socio-economic phenomena since the end of XIX century and also over the last few years a number of authors took it into consideration for making inference about inequality indices from income data (Moothathu 1985, Ganguly et al. 1992).

The choice of an appropriate prior distribution plays an important role in Bayesian estimation. The Bayesian estimation of Gini index in the context of Pareto distribution was considered by Bhattacharya et al. (1999) using two-parameter exponential prior distribution. In the present paper we choose three different priors viz. the Jeffreys’ prior, extension of Jeffreys’ prior and conjugate prior for estimating Zenga curve. The Jeffreys’ and extension of Jeffreys’ prior are non-informative priors while the conjugate priors can be both informative and non-informative.

In Bayesian estimation, different types of symmetrical and asymmetrical loss functions are used viz. linear loss, entropy loss and generalized entropy loss function. The squared error loss function (SELF) also referred to as quadratic loss is the simplest type of loss function which is used in the present paper. In this situation the loss incurred is \( L(\theta) = (\theta - \hat{\theta})^2 \), where \( \hat{\theta} \) is the estimator of \( \theta \).

The outline of this paper is as follows: a brief introduction of the Zenga curve along with the Pareto distribution and Bayesian inference is given in sections 2.1 and 2.2, respectively. Various priors viz. Jeffreys’ prior, extension of Jeffreys’ prior and the conjugate prior are discussed in section

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III along with the posterior distributions. In section IV, Bayes estimation of Zenga curve \( Z(p) \) is given under the squared error loss function (SELF). Further in section V, simulation techniques are applied for inter comparison of proposed Bayesian estimators in terms of squared error loss function, using three different priors. In section 5.1, robustness of hyper-parameters of conjugate prior is discussed and in section 5.2, a real life illustration is given, based on the data related to the gross domestic product of major G-20 countries. Finally, conclusion is given in section VI.

II. DEFINITIONS AND NOTATIONS

2.1 The Zenga curve in case of Pareto distribution

Let \( X \) be a non negative continuous random variable (r.v.) with distribution function \( F(x) \) and probability density function \( f(x) \) such that \( \int_0^b f(x) \, dx = 1 \), \((0 \leq a < b \leq \infty)\), where \( f(x) \) is strictly positive on the support \((a,b)\). The mean value \( \mu = E(X) \) is finite and positive.

The lower mean \( \mu^-_X \) and the upper mean \( \mu^+_X \) are respectively given by:

\[
\mu^-_X = \int_{a}^{b} [1 - F(x)]^{-1} f(t) \, dt, \quad (1)
\]

\[
\mu^+_X = \int_{a}^{b} [F(x)]^{-1} f(t) \, dt. \quad (2)
\]

The point inequality measure is obtained for, \( a \leq x \leq b \), by:

\[
A(x) = 1 - \frac{\mu^+_X}{\mu^-_X}. \quad (3)
\]

In order to represent the point inequality measure \( A(x) \) in the unit square, the transformations with \( p = F(x) \) are made in (1) & (2), such that

\[
\mu^-_{(p)} = \mu^-_{(F^{-1}(p))} = M^-_p = \frac{1}{p} \int_0^p F^{-1}(t) \, dt, \quad (4)
\]

\[
\mu^+_{(p)} = \mu^+_{(F^{-1}(p))} = M^+_p = \frac{1}{1-p} \int_0^1 F^{-1}(t) \, dt. \quad (5)
\]

The point index \( A(p) \) at \( p = F(x) \) is given by the relation

\[
Z(p) = 1 - \frac{M^+_p}{M^-_p}, \quad 0 < p < 1,
\]

where \( Z(p) \) is known as the Zenga curve for \( 0 < p < 1 \) and is also referred to as point inequality measure in the unit square (Zenga 2007). The Zenga curve \( Z(p) \) also lies between 0 and 1 (Figure 1). The Zenga curve \( Z(p) \) measures the inequality between the poorest \( 100 \) \( p\% \) of the population and the richer remaining \( 100 \) \( (1 - p)\% \) part of it by comparing the mean incomes of these two disjoint and exhaustive sub-populations.

The Zenga curve \( Z(p) \) is also related to the Lorenz curve \( L(p) \) by the relation:

\[
Z(p) = 1 - \frac{L(p)}{p} \frac{1-p}{1-L(p)}, \quad \text{for} \ 0 < p < 1, \quad (6)
\]

where \( L(p) = \frac{1}{\mu} \int_0^p F^{-1}(t) \, dt \), denotes the Lorenz curve which is a central tool of income inequality (Kovacevic and Binder 1997, Langell and Till’e 2009, Gastwirth 1971). Lorenz curve \( L(p) \) denotes the fraction of the total income received by the 100 \( p\% \) of the population which has the lowest income.

The area below the Zenga curve \( Z(p) \) representing the concentration area yields a new inequality index \( Z \) called the Zenga index (Zenga 2007) defined as:

\[
Z = \int_0^1 Z(p) \, dp, \quad 0 < p < 1.
\]

\( Z \) is related to Zenga curve as Gini index \( G \) is related to Lorenz curve \( L(p) \) and it varies between 0 and 1 in case of perfect equality and 1 in the case of perfect inequality.

Consider the Pareto \((m, \alpha)\) distribution with cumulative distribution function \( F(x) = 1 - \left(\frac{m}{x}\right)^\alpha \), and probability density function:

\[
f(x) = \frac{\alpha m^\alpha}{x^\alpha+1}, \quad m < x < \infty, \quad m > 0, \quad \alpha > 0, \quad (7)
\]

where the scale parameter is \( m \) and the shape parameter is \( \alpha \).

The Zenga curve \( Z(p) \) in case of the above Pareto distribution is given by

\[
Z(p) = \left(\frac{1-(1-p)^\alpha}{p}\right), \quad 0 < p < 1. \quad (8)
\]

It should be noted that the Zenga curve \( Z(p) \) does not depend on the scale parameter \( m \).

Notations used:

- \( Z(p), 0 < p < 1 \) : Zenga curve.
- \( L(p), 0 < p < 1 \) : Lorenz curve.
- \( P(m, \alpha) \) : Pareto distribution with scale parameter \( m \) and shape parameter \( \alpha \).
- \( Z \) : Zenga index.
- \( G \) : Gini index.

2.2 Bayesian Inference

The word "Bayesian" refers to the influence of Reverend Thomas Bayes (1702-1761), who introduced the Bayes’ theorem which is the foundation of Bayesian inference.

The joint posterior distribution \( p(\theta|y) \), that expresses uncertainty about parameter set \( \theta \) after taking both the prior and data into account is given by:

\[
P(\theta|y) \propto p(y|\theta) \, p(\theta),
\]

where \( p(\theta) \) is the prior distribution for parameter set \( \theta \) and \( p(y|\theta) \) is the likelihood function.

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a Pareto distribution. For complete sample, the likelihood function \( L(x|\alpha) \) in case of Pareto distribution is (Arnold and Press 1983)

\[
L(x|\alpha) = \alpha^n m^n \alpha (\prod_{i=1}^n x_i)^{-(\alpha+1)}.
\]

The Log likelihood function for the complete sample is

\[
L = \alpha^n (\prod_{i=1}^n x_i) e^{-\alpha t}.
\]
\[ \log \mathcal{L}(x|\theta) \quad (10) \]
\[ \alpha > 0, \beta > 0 \]
\[ \frac{d\alpha}{\alpha} \int u^\alpha e^{-\beta u} du \]
\[ \mathcal{L}(x|\alpha) \quad (0 \leq \alpha \leq \infty), \quad (11) \]
\[ \int u^\alpha e^{-\beta u} du \]
\[ \log \mathcal{L}(\alpha) = \frac{\alpha}{\beta} \int u^\alpha e^{-\beta u} du \]
\[ \mathcal{L}(x|\alpha) \quad (0 \leq \alpha \leq \infty). \quad (12) \]

III. PRIOR AND POSTERIOR DENSITIES

In Bayesian inference, a prior probability distribution, often called simply the prior, of an uncertain parameter \( \theta \) or latent variable is a probability distribution that expresses uncertainty about \( \theta \) before the data are taken into account. An appropriate prior plays a crucial role in Bayesian analysis which can be both informative and non-informative.

3.1. Jeffreys’ prior

Jeffreys’ prior, also called Jeffreys rule, was introduced by Jeffreys (1961) in an attempt to establish a least informative prior that is invariant to transformations. The Jeffreys’ prior, named after Harold Jeffreys, is a non-informative (objective) prior that is invariant to transformations. The Jeffreys’ prior, also called Jeffreys rule, was introduced by Jeffreys (1961) in an attempt to establish a least informative prior that is invariant to transformations. The Jeffreys’ prior, named after Harold Jeffreys, is a non-informative (objective) prior that is invariant to transformations. The Jeffreys’ non-informative prior for the parameter \( \alpha \) in case of Pareto distribution is derived as follows:

The logarithm likelihood function in case of Pareto distribution is:

\[ \log \mathcal{L}(\alpha) = n \log \alpha + n \alpha \log m - (\alpha + 1) \sum_{i=1}^{n} x_i, \]

and

\[ I(\alpha) = -E \left[ \frac{\partial^2}{\partial \alpha^2} \log \mathcal{L}(\alpha) \right] = \frac{n}{\alpha^2}. \]

The prior density for \( \alpha \) is

\[ \pi(\alpha) \propto \frac{\sqrt{\alpha}}{\alpha}. \quad (10) \]

Combining likelihood function (9) with prior density (10) by using Bayes theorem, the posterior density is

\[ \pi^*(\alpha) = \frac{L(x|\alpha) \cdot \pi(\alpha)}{\int_0^\infty L(x|\alpha) \cdot \pi(\alpha) d\alpha} = \frac{\alpha^{-n}(\prod_{i=1}^{n} x_i)^{e^{-\alpha t}} \cdot \frac{\sqrt{n}}{\alpha}}{\int_0^\infty \alpha^{-n}(\prod_{i=1}^{n} x_i)^{e^{-\alpha t}} \cdot \frac{\sqrt{n}}{\alpha} d\alpha} \]

\[ = \frac{1}{\Gamma(n)} \int_0^\infty u^{n-1} e^{-u} du \]

\[ = \frac{1}{\Gamma(n)} \alpha^{n-1} e^{-\alpha t}, \quad (0 \leq \alpha \leq \infty), \quad (11) \]

where \( \Gamma(n) = \int_0^\infty u^{n-1} e^{-u} du \), is the complete gamma function.

Remark: Extension of Jeffreys’ prior

Another prior used in the literature proposed by Al-Kutubi et al. (2009) is the extension of Jeffreys’ prior and is given by

\[ \pi(\alpha) \propto [I(\alpha)]^c, \quad 0.5 \leq c \leq \infty, \]

where \( c \) is a constant. Jeffreys’ prior is a special case of extension of Jeffreys’ prior when \( c = 1/2 \).

The prior density for \( \alpha \) in case of Pareto distribution is

\[ \pi_\epsilon(\alpha) \propto \left[ \frac{n}{\alpha^2} \right]^c. \]

On similar lines, the posterior distribution for \( \alpha \) under the assumption of extension of Jeffreys’ prior is obtained as:

\[ \pi^*_\epsilon(\alpha) = \frac{t^{-2c+1}}{\Gamma(n-2c+1)} \alpha^{n-2c} e^{-\alpha t}, \quad (0 \leq \alpha \leq \infty). \quad (12) \]

3.2. Conjugate prior

Quite often a prior distribution is chosen which satisfies specified summaries. It is usually advocated that in the absence of the correct prior, we may pick up the most convenient distribution to which the summaries may fit. Conjugate prior is used in such cases. A conjugate prior is constructed by first factoring the likelihood function into two parts. One factor must be independent of the parameter(s) of interest but may be dependent on the data. The second factor is a function dependent on the parameter(s) of interest and dependent on the data only through the sufficient statistics. The conjugate prior family is defined to be proportional to this second factor. Raiffa and Schlaifer (1961) also show that the posterior distribution arising from the conjugate prior is itself a member of the same family as the conjugate prior.

Given a likelihood, the conjugate prior is the prior distribution such that the prior and posterior are in the same family of distributions.

If \( f(x|\alpha) \) is a Pareto distribution with shape parameter \( \alpha \) and scale parameter \( m \), then its conjugate prior is Gamma distribution with density \( f(\alpha) = \frac{\alpha^n}{\Gamma(n)} \alpha^{n-1} e^{-\alpha m} \), \( \alpha > 0, m > 0 \).

The prior of \( \alpha \) is

\[ \pi(\alpha) \propto \alpha^{n-1} e^{-\alpha m}. \]

Similarly, the posterior distribution is obtained as

\[ \pi^*_\epsilon(\alpha) = \frac{L(x|\alpha) \cdot \pi_\epsilon(\alpha)}{\int_0^\infty L(x|\alpha) \cdot \pi_\epsilon(\alpha) d\alpha} \]

\[ = \frac{\alpha^n(-\prod_{i=1}^{n} x_i)^{e^{-\alpha t}} \cdot \alpha^{n-1} e^{-\alpha m}}{\int_0^\infty \alpha^n(-\prod_{i=1}^{n} x_i)^{e^{-\alpha t}} \cdot \alpha^{n-1} e^{-\alpha m} d\alpha} \]

\[ = \frac{1}{(t+b)^{n+a}} \int_0^\infty u^{n+a-1} e^{-u} du \]

\[ = \frac{(t+b)^{n+a}}{\Gamma(n+a)} \alpha^{n+a-1} e^{-\alpha(t+b)}, \quad (0 \leq \alpha \leq \infty). \quad (13) \]

IV. BAYES ESTIMATORS OF ZENGA CURVE (POINT MEASURE) USING BOTH INFORMATIVE AND NON-INFORMATIVE PRIORS.

4.1. Jeffreys’ non-informative prior

Using the Bayesian posterior density, the Bayes estimator of point measure \( Z(p) \), \( 0 < p < 1 \) under the assumption of squared error loss function (SELF) is
\[
\mathcal{Z} = \int_0^\infty Z(p) * \pi^*(\alpha) d\alpha \\
= \int_0^\infty \left(1 - (1 - p)\frac{\alpha^2}{p}\right) t^n \frac{n-1}{\Gamma(n)} \alpha^{n-1} e^{-\alpha t} d\alpha, \quad 0 < p < 1 \\
= \frac{t^n}{p \Gamma(n)} \left[ \int_0^\infty \alpha^{n-1} e^{-\alpha t} d\alpha - \int_0^\infty (1 - p)\frac{\alpha^2}{p} \alpha^{n-1} e^{-\alpha t} d\alpha \right] \\
= \frac{t^n}{p \Gamma(n)} \left[ \int_0^\infty \alpha^{n-1} e^{-\alpha t} d\alpha - \int_0^\infty \alpha^{n-1} e^{-\frac{1}{\alpha} \ln(\frac{1}{1-p})} d\alpha \right] \\
= \frac{1}{p^2} + 2 \frac{t^n}{p^2 \Gamma(n)} \left(2 \log(1 \frac{1}{1-p})\right)^2 K_n \left(2 \sqrt{t \log(1 \frac{1}{1-p})} - \frac{(\mathcal{Z}_e)^2}{2}\right), \quad 0 < p < 1,
\]

where

\[
\int_0^\infty x^{\gamma-1} e^{-\frac{x^\beta}{\gamma}} dx = 2 \left(\frac{\beta^\gamma}{\gamma}\right) K_v(2\sqrt{\beta \gamma}) \text{ real } \beta > 0, \text{real } \gamma > 0
\]

and \(K_v(.)\) is the modified Bessel function of IIIrd kind. As is evident, the Bayesian estimator \(\mathcal{Z}\) does not depend on the shape parameter \(\alpha\).

The expected loss of \(\mathcal{Z}\) is

\[
\text{ELJ}_\mathcal{Z} = \int_\infty^\infty (Z - \mathcal{Z})^2 * \pi^*(\alpha) d\alpha \\
= \int_\infty^\infty \left(Z^2 - \pi^*(\alpha)\right) d\alpha - \mathcal{Z}^2 \\
= \int_0^\infty \left(2 \pi^*(\alpha)\right) d\alpha - \mathcal{Z}^2 \\
= \frac{t^n}{p \Gamma(n)} \left[ \int_0^\infty \alpha^{n-1} e^{-\alpha t} d\alpha - \int_0^\infty (1 - p)\frac{\alpha^2}{p} \alpha^{n-1} e^{-\alpha t} d\alpha \right] - (\mathcal{Z}_e)^2 \\
= \frac{1}{p^2} + 2 \frac{t^n}{p^2 \Gamma(n)} \left(2 \log(1 \frac{1}{1-p})\right)^2 K_n \left(2 \sqrt{t \log(1 \frac{1}{1-p})} - (\mathcal{Z}_e)^2\right), \quad 0 < p < 1,
\]

for \(0 < p < 1\).

4.2. Extension of Jeffreys’ prior

The Bayes estimator of point measure \(Z(p)\) under the assumption of squared error loss function (SELF) is also obtained in case of extension of Jeffreys’ prior as

\[
\mathcal{Z}_e = \int_0^\infty Z(p) * \pi^*_e(\alpha) d\alpha \\
= \int_0^\infty \left(1 - (1 - p)\frac{\alpha^2}{p}\right) t^{n-2c+1} \frac{n-1}{\Gamma(n - 2c + 1)} \alpha^{n-2c} e^{-\alpha t} d\alpha \\
= \frac{1}{p^2} + 2 \frac{t^{n-2c+1}}{p^2 \Gamma(n - 2c + 1)} \left(2 \sqrt{t \log(1 \frac{1}{1-p})}\right)^2 K_{n-2c+1}(2 \sqrt{t \log(1 \frac{1}{1-p})} - (\mathcal{Z}_e)^2), \quad 0 < p < 1,
\]

where \(K_v(.)\) is the modified Bessel function of IIIrd kind. In this case also, Bayes estimator \(\mathcal{Z}_e\) does not depend on the shape parameter \(\alpha\).

The expected loss of \(\mathcal{Z}_e\) under extension of Jeffreys’ prior is

\[
\text{ELEJ}_\mathcal{Z} = \int_\infty^\infty (Z - \mathcal{Z}_e)^2 * \pi^*_e(\alpha) d\alpha \\
= \frac{1}{p^2} + 2 \frac{t^{n-2c+1}}{p^2 \Gamma(n - 2c + 1)} \left(2 \sqrt{t \log(1 \frac{1}{1-p})}\right)^2 K_{n-2c+1}(2 \sqrt{t \log(1 \frac{1}{1-p})} - (\mathcal{Z}_e)^2), \quad 0 < p < 1.
\]

4.3. Conjugate prior

The Bayes estimator of the Zenga curve \(Z(p)\) under the assumption of squared error loss function (SELF) for the conjugate prior is obtained as

\[
\mathcal{Z}_c = \int_0^\infty Z(p) * \pi^*_c(\alpha) d\alpha \\
= \int_0^\infty \left(1 - (1 - p)\frac{\alpha^2}{p}\right) t^n \frac{n+a}{\Gamma(n + a)} \alpha^{n+a-1} e^{-\alpha t} d\alpha \\
= (t + b)^{n+a} \frac{p}{\Gamma(n + a)} \int_0^\infty \alpha^{n+a-1} e^{-\alpha(t+b)} d\alpha \\
= (t + b)^{n+a} \frac{p}{\Gamma(n + a)} \left[ \Gamma(n + a) \alpha^{n+a-1} e^{-\alpha(t+b)} d\alpha \right] \\
= \frac{1}{p} + 2 \frac{t^n}{p^2 \Gamma(n + a)} \left(2 \sqrt{t \log(1 \frac{1}{1-p})}\right)^2 K_{n+a}(2 \sqrt{t \log(1 \frac{1}{1-p})} - (\mathcal{Z}_c)^2), \quad 0 < p < 1.
\]

for \(0 < p < 1\).

where \(K_v(.)\) is the modified Bessel function of IIIrd kind.
Expected Loss of $\hat{Z}_c$ is obtained as
\[
\text{ELC}_n = \int_0^\infty (Z - \hat{Z}_c)^2 \pi^*(c) \, d\alpha \\
= \int_0^\infty (Z^2 \pi^*(c) \, d\alpha - (\hat{Z}_c)^2 \\
= \int_0^\infty \left( \frac{1}{p^2} \right) (t + b)^{n+a/2} \pi^*(c) \, d\alpha - (\hat{Z}_c)^2 \\
\begin{align*}
&= \frac{1}{p^2} + 2 \frac{(b + t)^{n+a/2}}{p^2 \Gamma(n + a)} \left( 2 \log \left( \frac{1}{1-p} \right) \right)^{n+a/2} \\
&\quad - 4 \frac{(b + t)^{n+a/2}}{p^2 \Gamma(n + a)} \left( \log \left( \frac{1}{1-p} \right) \right)^{n+a/2} \\
&\quad + K_{n+a} \left( 2 \frac{(b + t)^{n+a/2}}{p^2 \Gamma(n + a)} \left( 2 \log \left( \frac{1}{1-p} \right) \right)^{n+a/2} \\
&\quad - 4 \frac{(b + t)^{n+a/2}}{p^2 \Gamma(n + a)} \left( \log \left( \frac{1}{1-p} \right) \right)^{n+a/2} \\
&\quad + K_{n+a} \left( 2 \frac{(b + t)^{n+a/2}}{p^2 \Gamma(n + a)} \left( 2 \log \left( \frac{1}{1-p} \right) \right)^{n+a/2} \\
&\quad - 4 \frac{(b + t)^{n+a/2}}{p^2 \Gamma(n + a)} \left( \log \left( \frac{1}{1-p} \right) \right)^{n+a/2} \right) \left( \hat{Z}_c \right)^2 \text{, for } 0 < p < 1.
\end{align*}
\]

5.1. Robustness of Hyper-parameters

In order to study the robustness of the hyper-parameters on the Bayes estimates, the procedure given by Sinha (1980) is followed, wherein it is suggested that a Bayes estimate is robust with respect to its hyper-parameters if it leads to a high (min \( \frac{\text{max}}{\text{max}} \)) index of the estimate for the varying values of those hyper-parameters. For this, the (min \( \frac{\text{max}}{\text{max}} \)) index of the estimates of Zenga curve for the conjugate prior when \( m = 1.75 \), and \( a = 2.5 \) for various values of the hyper-parameters as \( a, b = 0, 1, 2, 3 \).

As seen from Table 2, it is observed that the (min \( \frac{\text{max}}{\text{max}} \)) ratio is close to 1 suggesting thereby that the Bayes estimates are robust with respect to the hyper-parameters \( a \) and \( b \).

5.2. Real life example based on Gross domestic product (GDP) of G-20 Countries.

A real life example of the data comprising of GDP based on purchasing-power-parity of the major G-20 countries (in Billion $) across the globe is taken up for inter-comparison of the three proposed Bayesian estimators of Zenga curve \( Z(p) \), \( 0 < p < 1 \) [20]. Using Easyfit, the dataset is checked to fit the Pareto distribution with shape parameter \( \alpha = 0.77 \), scale parameter \( m = 577 \) and the p value according to the Kolmogorov-Smirnov goodness of fit test is 0.39233 at 5% level of significance. The expected loss resulting from Bayes estimation of Zenga curve \( Z(p) \) are obtained for this data of G-20 countries in case of all the three priors proposed in sections 4.1, 4.2 and 4.3. The values of the expected loss is reported in table 3 with sample size \( n = 19 \), \( p = 0.20 \) and the values of hyper parameters (in case of conjugate prior) are taken as \( a = 3 \) and \( b = 0 \).

The result shows that the conjugate prior results in smaller expected loss in comparison with both the other priors-Jeffreys’ and extension of Jeffreys’ prior. The results are in conformity with the results obtained earlier using simulation techniques.

VI. CONCLUSION

In this paper, Bayesian estimators of Zenga curve, a new curve of inequality, are derived in case of the Pareto distribution using squared error loss function (SELF) for three different priors viz. Jeffreys’ prior, extension of Jeffreys’ prior and conjugate prior. A simulation study is conducted to compare the relative efficiency of the proposed Bayesian estimators using different configurations of sample sizes and shape parameter. The results suggest that the conjugate prior performs better than other two priors in terms of expected loss function. The results are further reaffirmed by the real life illustration pertaining to the dataset concerning gross domestic product (GDP) of major G-20 countries.
Table 1: Values of Expected loss for Zenga curve for Jeffreys’ prior, Extension of Jeffreys’ prior and Conjugate Prior

<table>
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<th>α</th>
<th>p</th>
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<th>$E_{LEJ}$</th>
<th>$E_{LC}$</th>
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<td>$c=4$</td>
</tr>
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<td>0.02754944</td>
<td>0.03420166</td>
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</tbody>
</table>

Table 2: Bayes Estimate of Zenga curve in case of Conjugate prior

$\alpha = 2.5$ and $\alpha = 0.77$

<table>
<thead>
<tr>
<th>a</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\frac{\min}{\max}/b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
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<td>0.9669808</td>
<td>0.929075</td>
<td>0.9418371</td>
<td>0.9139892</td>
</tr>
<tr>
<td>1</td>
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<td>0.9071645</td>
<td>0.8848718</td>
<td>0.9184278</td>
<td>0.90897901</td>
</tr>
<tr>
<td>2</td>
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<td>0.938738</td>
<td>0.9339138</td>
<td>0.9149879</td>
<td>0.90437595</td>
</tr>
<tr>
<td>3</td>
<td>0.9654612</td>
<td>0.9651935</td>
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<td>0.8819993</td>
<td>0.91355230</td>
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</table>

Table 3: Values of the Expected loss for Zenga curve in case of GDP of major G-20 countries

$n = 19, p = 0.20, \alpha = 0.77$

<table>
<thead>
<tr>
<th>Priors</th>
<th>Expected Loss for Zenga curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jeffreys’ prior</td>
<td>0.0700046</td>
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<tr>
<td>Extension Of Jeffreys’ prior $c=1$</td>
<td>0.06083901</td>
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<td>Extension Of Jeffreys’ prior $c=4$</td>
<td>0.1960747</td>
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<tr>
<td>Conjugate prior</td>
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</tr>
</tbody>
</table>
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REFERENCES


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