Quantifiers and Partiality

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1 Varieties of Partiality

Quantification can involve partiality in several ways. Quantifiers loaded with presuppositions give rise to partiality by introducing truth value gaps (if there are no A, and the quantifier $\mathbf{all}^+$ carries an existence presupposition, then $\mathbf{all}^+$ *A are B* will lack a truth value). The study of quantifiers in contexts of incomplete information involves a quite different kind of partiality. This paper investigates the behaviour of binary quantifiers in settings of incomplete information, i.e., in partial models.

The framework to handle informational partiality that is presented in this paper is relevant for the behaviour of quantifiers in the natural language semantics of propositional attitudes and perception reports, for theories of vagueness in natural language and for semantic accounts of natural language fragments containing a truth predicate.

First, the constraints on binary relations on a universe that make these relations qualify for the honorific title *quantificational* are generalized to the partial case. Special consideration is given to quantifiers defined via supervaluation from quantifiers on total models. Next, properties of partial quantifiers are studied, with particular emphasis on behaviour under growth of information and growth of domain.

Quantifier interpretations for natural language determiners like *all*, *some*, *most*, *exactly two*, *at least three and at most five*, pick out binary relations on sets of individuals, on arbitrary universes E. Notation: $Q_E AB$. We call A the *restriction* of the quantifier and B its *body*. Many quantifier relations satisfy the familiar constraints EXT, CONS and ISOM (cf. the introduction to this volume).

We will put generalized quantifiers in a three valued setting. In such a setting, quantifiers themselves can introduce truth value gaps, but they can also have 'three valued sets' as arguments. Our framework deals with both these kinds of partiality. Below, in Section 2, new versions of EXT, CONS and ISOM will be proposed that also cover the cases where only partial information concerning the extensions of the body and the restriction of a quantifier is available.

The definite description *the n A are B* can be viewed as a partial quantifier that is undefined if the number of As is not equal to n, and that behaves like the universal quantifier otherwise (see ?). Consider the partial quantifier in (1).

(1) The four A are B.

In the numerical tree for this quantifier, * abbreviates the value 'undefined'.

$$
\begin{array}{lccccccccccc}
\#A = 0 & & & & & & * & & & & & \\
\#A = 1 & & & & & * & & * & & & & \\
\#A = 2 & & & & * & & * & & * & & & \\
\#A = 3 & & & * & & * & & * & & * & & \\
\#A = 4 & & - & & - & & - & & - & & + & \\
\#A = 5 & * & & * & & * & & * & & * & & * \\
\vdots & & & & & & \vdots & & & & &
\end{array}
$$

In the example case, the presupposition has the form of a unary quantifier, 'there are exactly four A'. Splitting out between truth and falsity conditions, the quantifier 'the n' is defined as follows:

- **the n** A *are* $B = 1$ iff there are exactly **n** A and all A are B;
- **the n** A *are* $B = 0$ iff there are exactly **n** A and not all A are B.

Similarly, we can express the 'existential import' of quantifiers. For instance, we can define $\mathbf{all}^+$ as follows:

- $\mathbf{all}^+$ A *are* $B = 1$ iff there are A and all A are B;
- $\mathbf{all}^+$ A *are* $B = 0$ iff there are A and not all A are B.

This gives the following numerical tree for $\mathbf{all}^+$:

$$
\begin{array}{lccccccccccc}
\#A = 0 & & & & & & * & & & & & \\
\#A = 1 & & & & & 0 & & + & & & & \\
\#A = 2 & & & & 0 & & 0 & & + & & & \\
\#A = 3 & & & 0 & & 0 & & 0 & & + & & \\
\#A = 4 & & 0 & & 0 & & 0 & & 0 & & + & \\
\#A = 5 & 0 & & 0 & & 0 & & 0 & & 0 & & + \\
\vdots & & & & & & \vdots & & & & &
\end{array}
$$

If we use $\mathbf{all}^+$ to paraphrase the sentence (2) then the paraphrase entails that John does have grandchildren.

(2) All John's grandchildren are boys.

The general pattern for describing the truth conditions of a quantifier $Q_P AB$ 'loaded' with a presupposition P is:

- $Q_P AB$ is true iff P and QAB.
- $Q_P AB$ is false iff P and not QAB.

Interestingly, in the case of quantifiers with presupposition, the presupposition itself generally seems to have the form of a quantitative statement. The following partitive noun phrases provide some more examples.

(3) At most two of John's ten grandchildren are boys.

(4) At least five of John's many grandchildren are boys.

(5) Less than half of the boys in Mary's class came to the party.

We can succinctly express the meaning of quantifiers with presupposition with a three valued connective for *interjunction*, as defined in ?:

$\otimes$	*1*	*	*0*
1	*1*	*	*
*	*	*	*
0	*	*	*0*

It is easy to check that the presuppositional quantifier $Q_P AB$ is defined by the following schema:

(6) $(P \wedge QAB) \otimes \neg(P \wedge \neg QAB)$.

Three valued truth tables for $\wedge$ and $\neg$ will be given below, but if one assumes that P and QAB are two valued, these are not needed to grasp the meaning of (6). It is obvious from the truth table for $\otimes$ that quantifiers defined by schema (6) do introduce truth value gaps. We will call such quantifiers *open*. Quantifiers that do not introduce truth value gaps wil be called *closed*. Formally:

CL–T For all $A, B \subseteq E$, either $Q_E AB = 1$ or $Q_E AB = 0$.

Partiality can also arise in connection with closed quantifiers. Consider cases where the extensions of certain predicates are only partially known. Assume we know how big the universe E is but have only partial information about the predicates $A, B \subseteq E$. Certain objects are known to be As, certain other objects are known to be non-As, but there can also be objects that are in neither class, and even objects that are in both classes (in this case our information is *incoherent*). Similarly with B. We call this kind of partiality *informational partiality*. Although informational and presuppositional partiality may co-occur, we have for simplicity assumed that the presuppositions themselves are two valued.

The importance of studying quantifiers in a three valued setting stems from the fact that the semantics of propositional attitudes—example (7) —and the semantics of perception reports—example (8) —need be stated in terms of partial models (?, ?).

(7) John believes that all grandparents are happy.

(8) John saw three children enter.

In examples (7) and (8), a propositional attitude or perception complement does contain a quantifier, so these quantifiers are to be evaluated in partial models. Note for instance that sentence (8) does not imply (9).

(9) John saw three children enter and smile or not smile.

A situation where John looks at the children from behind is most plausibly described as one where the predicate 'enter' does have a truth value for every individual in the scene, but 'enter and smile or not smile' does not. Still, by merely turning around the three children (so that John can view their faces), this situation can be changed into one where 'enter and smile or not smile' does have a truth value. See ? and ? for further details.

Another area where the theory of partial quantification is relevant is the account of vagueness in natural language. Here we have a kind of partiality that cannot directly be resolved by growth of information, because the vagueness is inherent in the truth and falsity conditions.

(10) There are many rich people in California.

When I assert (10), the truth value of this assertion may be impossible to specify because it is unclear what counts as 'being rich' (or what counts as 'many', for that matter). However, assuming that the truth of (10) becomes a topic of discussion, I may be asked to further specify my criteria for the satisfaction of the predicate 'rich' in this context. The further I specify these criteria, the more elements of the domain of discussion can be classified. One can view the situation as one in which a vague predicate $\langle P^+, P^- \rangle$ is first replaced by $\langle P'^+, P'^- \rangle$ with $P^+ \subseteq P'^+$, $P^- \subseteq P'^-$, then upon further questioning by $\langle P''^+, P''^- \rangle$ with $P'^+ \subseteq P''^+$, $P'^- \subseteq P''^-$, and so on. It is clear that this gradual replacement of partial predicates by more precise versions can be studied within a framework that accounts for growth of information about partial predicates.

Finally, partial quantifiers are relevant for the definition of natural language fragments containing their own truth predicate. In the wake of ? various proposals have been worked out for avoiding the semantic paradoxes that scared Tarski away from natural language, by starting out with truth value gaps for statements involving truth, and then gradually closing these gaps for the non paradoxical statements (see the accounts in ? and ?). In this context, 'quantified liars' and 'quantified samesayers' merit attention.

FIGURE 1 Quantified Samesayer.

1. At least one statement in this box is true.

An example of a quantified samesayer is given in Figure 1. Note that the example still works if the statement is replaced by *All statements in this box are true*, *Not all statements in this box are false*, or *No statements in this box are false*.

FIGURE 2 Quantified Liar.

1. At least one statement in this box is true.
2. At least one statement in this box is false.

A very simple example of a quantified liar is given in Figure 2. To see that this example is indeed paradoxical, assume that statement 1 is true.

Then the additional assumption that 2 is also true leads to a contradiction with what 2 says. From the assumption that 2 is false, on the other hand, it follows that all statements are true, i.e. both 1 and 2 are true, and contradiction with the assumption of 2's falsity. Now assume that statement 1 is false. Then none of the statements in the box is true, so it follows that 2 is false. But then both 1 and 2 are true, and contradiction. Note that replacement of both quantifiers by *all*, by *no*, or by *not all*, preserves the paradox.

FIGURE 3 Liar with Non Standard Quantifier.

1. Two plus two equals four.
2. Two plus two equals five.
3. More than half of the statements in this box are false.

An example of a quantified liar involving a non standard quantifier is given in Figure 3. Essentially the same example was given in ?.

FIGURE 4 Liar Involving Infinitely Many Different Quantifiers.

1. Not all statements in this box are true.
2. At least one statement in this box is true.
3. At least two statements in this box are true.

   ⋮

n. At least $n-1$ statements in this box are true.

   ⋮

A final example of a liar situation involving infinitely many different quantified statements is given in Figure 4. In this case the reasoning to establish the paradoxical nature of the example is slightly more involved. First assume that statement 1 in the box is false. Then, because of what 1 says, all statements in the box are true. This contradicts the falsity of statement 1. Now assume that statement 1 is true. Then at least one statement in the box is true. It follows that statement 2 is true as well, for this is what 2 states. From the truth of 1 and 2 it follows that at least two statements in the box are true. This is what statement 3 says, so 3 is true as well. In general, from the truth of statements 1 through n it follows that statement $n+1$ is true. Thus all statements in the box are true. But this situation is what statement 1 denied, so the assumption that 1 is true also leads to a contradiction.

We will leave the detailed study of these 'paradoxes of quantification' for another occasion. The examples cited above merely serve to illustrate one more case where the positive and negative extension of a predicate—

'is true' and 'is false', respectively—do not, indeed cannot, exhaust the domain of quantification.

2 General Framework

What is known about quantifier relations in situations of partial information can be expressed by describing both the states of affairs that verify the quantifier relation and the states of affairs that falsify it. For instance, a state of affairs in which $A^+ \cap B^- \neq \emptyset$ falsifies 'all A are B' (in its most natural reading), for finding a thing which is A and not B refutes the universal statement. Similarly, a state of affairs in which $E - A^- \subseteq B^+$ (where E is the domain of quantification) verifies 'all A are B' (again, under the most natural reading of 'all' in partial situations, which will be made more precise below). Note that these conditions are *precise* in that they describe the complete set of situations that verify or falsify a quantifier Q. As an example of an imprecise condition, $A^- = E$ verifies the quantifier *all* all right, but it describes a proper subset of the set of all situations that verify the quantifier. Verifying and falsifying conditions that are precise are formulated as biconditionals.

We will now introduce a language $\mathcal{L}_\mathcal{Q}$ for partial logic with binary quantifiers. More information about partial logic without quantifiers can be found in ? and ?. $\mathcal{L}_\mathcal{Q}$ will consist of predicate logic, plus a set of binary quantifiers $\mathcal{Q}$. The non-logical vocabulary of the language $\mathcal{L}_\mathcal{Q}$ consists of a set $\mathrm{C} = \{c_0, c_1, c_2, \ldots\}$ of *individual constants*, and for each $n > 0$ a set $\mathrm{P}^n = \{P_0^n, P_1^n, P_2^n, \ldots\}$ of *n-place predicate constants*. The language $\mathcal{L}_\mathcal{Q}$ is then given by the following BNF definition (assume c ranges over C, v over a set of individual variables V, $p \in \mathrm{P}^n$, and $q \in \mathcal{Q}$):

$$\begin{array}{lll} t & ::= & v \mid c \\ \varphi & ::= & t_1 \doteq t_2 \mid Pt_1 \cdots t_n \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid (\varphi_1 \vee \varphi_2) \mid \\ & & \forall v \varphi \mid \exists v \varphi \mid Qv(\varphi_1, \varphi_2). \end{array}$$

For convenience, we will assume the sets of individual constants and predicate constants fixed. Also, we will often omit outer parentheses and parentheses between conjuncts or disjuncts in cases where there is no danger of ambiguity.

We will also consider several extensions of $\mathcal{L}_\mathcal{Q}$ that result from adding new sentential operators. $\mathcal{L}_{\mathcal{Q},\sim}$ is the result of adding the unary sentential operator $\sim$ to the logical vocabulary of $\mathcal{L}_\mathcal{Q}$, $\mathcal{L}_{\mathcal{Q},\otimes}$ is the result of adding the binary sentential operator $\otimes$ to $\mathcal{L}_\mathcal{Q}$, and $\mathcal{L}_{\mathcal{Q},\sim,\otimes}$ is the result of adding both these operators to $\mathcal{L}_\mathcal{Q}$.

For the semantics of $\mathcal{L}_\mathcal{Q}$, we define partial models or *situations* for $\mathcal{L}_\mathcal{Q}$.

Definition 1 [Situations] A situation s for $\mathcal{L}_\mathcal{Q}$ is a triple $\langle E, I^+, I^- \rangle$ where E is a non-empty set (called the *domain* of s), and I^+, I^- are functions satisfying the following:

- I^+ maps every $c \in C$ to a member of E.
- For every $n > 0$, I^+, I^- map each member of P^n to a pair $\langle R^+, R^- \rangle$ of n-place relations on E, such that $R^+ \cap R^- = \emptyset$.

We will call s a situation on E. I^+, I^- are the positive and negative interpretation functions of s.

Our main concern in what follows will be the investigation of suitable constraints on the interpretation of the binary quantifiers in $\mathcal{Q}$. These quantifier interpretations are not part of situations proper, for they do not depend on the interpretation functions of the situations, but only on their domains.

For now, we only wish to stipulate that, given a domain E, every binary quantifier Q is interpreted as a partial binary relation on partial subsets of E. We use $\mathcal{P}[E]$ for the set of partial subsets of domain E, i.e., $\mathcal{P}[E] \stackrel{\text{def}}{=} \{\langle X, Y \rangle \mid X, Y \subseteq E \text{ and } X \cap Y = \emptyset\}$. The most convenient view on partial quantifiers will turn out to be to picture them as functions of pairs of partial sets given by their positive and gap parts, rather than their positive and negative parts. Note that if $\langle X, Y \rangle$ is a partial subset of E, then $\langle X, E - (X \cup Y) \rangle$ is also a partial subset of E. Thus, binary quantifiers defined in terms of the positive and gap parts of their arguments are functions that take two partial sets and deliver a value in $\{0, 1\}$. Partial binary quantifiers can be pictured as *pairs* $\mathbf{Q}_E^+, \mathbf{Q}_E^\circ$ of such functions.

Definition 2 A coherent quantifier interpretation on E is a pair $\mathbf{Q}_E^+, \mathbf{Q}_E^\circ$ such that

- $\mathbf{Q}_E^+ \in (\mathcal{P}[E])^2 \to \{0, 1\}$,
- $\mathbf{Q}_E^\circ \in (\mathcal{P}[E])^2 \to \{0, 1\}$,
- $\mathbf{Q}_E^+(X, X^*, Y, Y^*) = 1$ implies $\mathbf{Q}_E^\circ(X, X^*, Y, Y^*) = 1$.

If $\mathbf{Q}_E^+, \mathbf{Q}_E^\circ$ form a coherent quantifier interpretation on E, then $\mathbf{Q}^+$ is its positive part or interior; $\mathbf{Q}^\circ$ its non-negative part or exterior. Proper constraints will be imposed on these quantifier interpretations later on.

To ensure that quantifiers are uniform over situations with the same domains, it is convenient to introduce the notion of an information system.

Definition 3 [Information Systems] An information system $\mathbf{S}$ is a tuple $\langle \mathrm{S}, \mathrm{E}, \mathrm{I}^+, \mathrm{I}^-, \mathrm{J}^+, \mathrm{J}^\circ \rangle$, where S is a set of situations with domains $\subseteq \mathrm{E}$, and I^+, I^-, J^+, J° are given by:

- For every $s \in S$, $\mathrm{I}^+(s) = I_s^+$ (the positive interpretation function of s), and $\mathrm{I}^-(s) = I_s^-$ (the negative interpretation function of s).
- For every $Q \in \mathcal{Q}$ and every $E \subseteq \mathrm{E}$, $\mathrm{J}^+(Q, E)$ and $\mathrm{J}^\circ(Q, E)$ form a coherent quantifier interpretation on E.

From now on, we consider a fixed information system S, and we will use $\mathbf{Q}_E^+$, $\mathbf{Q}_E^\circ$ for the positive and non-negative parts of the quantifier interpretation that get assigned to Q in S. Later on, our discussion of constraints on quantifiers will give rise to further conditions on J^+ and J°.

As usual, sentences involving quantification generally do not have sentences as parts but (open) formulae. As it is impossible to define truth for open formulae without making a decision about the interpretations of the free variables occurring in them, we employ infinite *assignments* of values to the variables of $\mathcal{L}_Q$, that is to say functions with domain V and range $\subseteq E$. As in the case of ordinary predicate logic, only the finite parts of the assignments that provide values for the free variables in a given formula are relevant.

The assignment function g enables us to define a function that assigns values in E to all terms of the language. Let $s = \langle E, I^+, I^- \rangle$ be a model for $\mathcal{L}_Q$ and g an assignment for $\mathcal{L}_Q$ in E. The function $w_{s,g}$ from the set of $\mathcal{L}_Q$ terms to E is given by the following clauses:

- If $t \in C$, then $w_{s,g}(t) = I^+(t)$.
- If $t \in V$, then $w_{s,g}(t) = g(t)$.

We explain what it means for an arbitrary formula φ of $\mathcal{L}_Q$ to be true, false or undefined in s relative to an assignment g, by recursively defining functions $[\![\cdot]\!]^+_{s,g}$ and $[\![\cdot]\!]^\circ_{s,g}$ from $\mathcal{L}_Q$ to $\{0,1\}$. $[\![\cdot]\!]^+_{s,g}$ and $[\![\cdot]\!]^\circ_{s,g}$, the positive and non-negative interpretation function respectively, are two valued, but they will later on be combined in the definition of a three valued function $[\![\cdot]\!]_{s,g}$. First we handle the basic case where φ is an atomic formula.

1. If φ has the form $t_1 \doteq t_2$, then $[\![\varphi]\!]^+_{s,g} = 1$ iff $[\![\varphi]\!]^\circ_{s,g} = 1$ iff $w_{s,g}(t_1) = w_{s,g}(t_2)$.
2. If φ has the form $Pt_1 \cdots t_n$, then $[\![\varphi]\!]^+_{s,g} = 1$ iff $\langle w_{s,g}(t_1), \ldots, w_{s,g}(t_n) \rangle \in I^+(P)$, and $[\![\varphi]\!]^\circ_{s,g} = 1$ iff $\langle w_{s,g}(t_1), \ldots, w_{s,g}(t_n) \rangle \notin I^-(P)$.

Note that the clause for $t_1 \doteq t_2$ reflects the choice to treat identity as a total relation, for we have: $[\![t_1 \doteq t_2]\!]^+_{s,g} = 1$ iff $[\![t_1 \doteq t_2]\!]^\circ_{s,g} = 1$.

A more radical perspective would partialize identity as well; this would involve a shift from individuals to proto-individuals. Proto-individuals are things that we have a handle on by means of a name ('Mr. Jones') or a functional relation ('Bobby's father'), but that may still fuse together with other proto-individuals as we learn more ('I had not realized that Mr. Jones is Bobby's father!'). Although this more radical approach to partiality seems necessary for getting to grips with famous identity puzzles of the Hesperus Phosphorus kind, we prefer a step by step approach and abstain from this further move in the present investigation.

The rules for the logical connectives run as follows:

3. If φ has the form $\neg\psi$, then $[\![\varphi]\!]^+_{s,g} = 1$ iff $[\![\psi]\!]^\circ_{s,g} = 0$, and $[\![\varphi]\!]^\circ_{s,g} = 0$ iff $[\![\psi]\!]^+_{s,g} = 1$.

4. If φ has the form $\psi \wedge \chi$, then $[\![\varphi]\!]^+_{s,g} = 1$ iff $[\![\psi]\!]^+_{s,g} = 1$ and $[\![\chi]\!]^+_{s,g} = 1$, and $[\![\varphi]\!]^\circ_{s,g} = 1$ iff $[\![\psi]\!]^\circ_{s,g} = 1$ and $[\![\chi]\!]^\circ_{s,g} = 1$.
5. If φ has the form $\psi \vee \chi$, then $[\![\varphi]\!]^+_{s,g} = 1$ iff $[\![\psi]\!]^+_{s,g} = 1$ or $[\![\chi]\!]^+_{s,g} = 1$, and $[\![\varphi]\!]^\circ_{s,g} = 1$ iff $[\![\psi]\!]^\circ_{s,g} = 1$ or $[\![\chi]\!]^\circ_{s,g} = 1$.

To treat the Fregean quantifiers $\forall, \exists$ and the binary quantifier Q we need the notion of an assignment g' that agrees with assignment g but for the fact that variable v gets value d. Formally:

$$g(v|d)(w) = \begin{cases} g(w) & \text{if } w \neq v \\ d & \text{if } w = v. \end{cases}$$

This allows us to dispose of the quantifier cases. We assume that s has domain E.

6. If φ has the form $\forall v \psi$, then $[\![\varphi]\!]^+_{s,g} = 1$ iff $[\![\psi]\!]^+_{s,g(v|d)} = 1$ for every $d \in E$, and $[\![\varphi]\!]^\circ_{s,g} = 1$ iff $[\![\psi]\!]^\circ_{s,g(v|d)} = 1$ for every $d \in E$.
7. If φ has the form $\exists v \psi$, then $[\![\varphi]\!]^+_{s,g} = 1$ iff $[\![\psi]\!]^+_{s,g(v|d)} = 1$ for some $d \in E$, and $[\![\varphi]\!]^\circ_{s,g} = 1$ iff $[\![\psi]\!]^\circ_{s,g(v|d)} = 1$ for some $d \in E$.
8. Suppose φ has the form $Qv(\psi, \chi)$.

   Let A^+ be $\{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 1\}$,
   let A^* be $\{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 0, [\![\psi]\!]^\circ_{s,g(v|d)} = 1\}$,
   let B^+ be $\{d \in E \mid [\![\chi]\!]^+_{s,g(v|d)} = 1\}$,
   let B^* be $\{d \in E \mid [\![\chi]\!]^+_{s,g(v|d)} = 0, [\![\chi]\!]^\circ_{s,g(v|d)} = 1\}$.

   Then $[\![\varphi]\!]^+_{s,g} = 1$ iff $\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$ and $[\![\varphi]\!]^\circ_{s,g} = 1$ iff $\mathbf{Q}^\circ_E(A^+, A^*, B^+, B^*) = 1$.

Note the recursion in terms of positive and non-negative conditions in the cases for the connectives $\wedge, \vee$, the quantifiers $\forall, \exists$ and the binary quantifiers. This shift from $[\![\cdot]\!]^+, [\![\cdot]\!]^-$ to $[\![\cdot]\!]^+, [\![\cdot]\!]^\circ$ will turn out to have some advantages for the conceptualization of partial binary quantifiers.

We still have to prove that the interpretation functions for the binary quantifier cases are well defined. The clause for binary quantifiers uses functions from $(\mathcal{P}[E])^2$ to $\{0, 1\}$, so this clause presupposes that the first and second arguments of the quantifier together represent a partial set, and similarly for the third and fourth arguments. This presupposition hinges on the coherence lemma, which will be stated shortly.

For the extended languages that also contain one or both of $\sim$ and $\otimes$, one or both of the following clauses should be added.

9. If φ has the form $\sim\psi$, then $[\![\varphi]\!]^+_{s,g} = 0$ iff $[\![\psi]\!]^+_{s,g} = 1$, and $[\![\varphi]\!]^\circ_{s,g} = 0$ iff $[\![\psi]\!]^+_{s,g} = 0$.

10. If φ has the form $\psi \otimes \chi$, then $[\![\varphi]\!]^+_{s,g} = 1$ iff $[\![\psi]\!]^+_{s,g} = 1$ and $[\![\chi]\!]^+_{s,g} = 1$, and $[\![\varphi]\!]^\circ_{s,g} = 0$ iff $[\![\psi]\!]^\circ_{s,g} = 0$ and $[\![\chi]\!]^\circ_{s,g} = 0$.

The reader is referred to **?** or **?** for a proof that every three valued truth function is expressible in $\mathcal{L}_{\sim,\otimes}$. It is useful to introduce some sentential operators by abbreviation. We will use $\top$ as an abbreviation of $\neg\exists x(Ax \wedge \neg Ax)$, $\varphi \rightarrow \psi$ as an abbreviation for $(\neg\varphi)\vee\psi$, and $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. It is easy to see that $\top$ always evaluates to 1.

In languages containing $\sim$, we will use $\varphi \rightsquigarrow \psi$ for $\sim\varphi \vee \psi$, $\varphi \Rightarrow \psi$ for $(\varphi \rightsquigarrow \psi) \wedge (\neg\psi \rightsquigarrow \neg\varphi)$, $\varphi \equiv \psi$ for $(\varphi \rightsquigarrow \psi) \wedge (\psi \rightsquigarrow \varphi)$, and $\varphi \Leftrightarrow \psi$ for $(\varphi \equiv \psi) \wedge (\neg\varphi \equiv \neg\psi)$.

Finally, in languages containing $\otimes$ we will use $*$ as an abbreviation for $\top \otimes \neg\top$. The reader should convince her- or himself that $\otimes$ does indeed get the truth table of Blamey's interjunction, and that the formula $*$ always evaluates to the value $*$.

A simple induction argument establishes the following:

Lemma 1 (Coherence) *If φ is a formula of $\mathcal{L}_Q$ (or one of its extensions), s a situation, and g an assignment for $\mathcal{L}_Q$ (or one of its extensions) in s, then*

1. $[\![\varphi]\!]^+_{s,g} = 1$ *entails* $[\![\varphi]\!]^\circ_{s,g} = 1$;
2. $[\![\varphi]\!]^\circ_{s,g} = 0$ *entails* $[\![\varphi]\!]^+_{s,g} = 0$.

Proof. First note that 2. follows from 1. by contraposition, because $[\![\cdot]\!]^+$ and $[\![\cdot]\!]^\circ$ are two valued.

For 1, all induction cases except for the binary quantifier case are left to the reader. For the case where φ has the form $Qv(\psi,\chi)$, note that $\{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 1\} \cap \{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 0, [\![\psi]\!]^\circ_{s,g(v|d)} = 1\} = \emptyset$, and $\{d \in E \mid [\![\chi]\!]^+_{s,g(v|d)} = 1\} \cap \{d \in E \mid [\![\chi]\!]^+_{s,g(v|d)} = 0, [\![\chi]\!]^\circ_{s,g(v|d)} = 1\} = \emptyset$, This guarantees that the first and second arguments A^+, A^*, respectively the third and fourth arguments B^+, B^*, of the quantifier interpretations $\mathbf{Q}^+_E$ and $\mathbf{Q}^\circ_E$ are partial sets, so $[\![\cdot]\!]^+$ and $[\![\cdot]\!]^\circ$ are indeed well defined for the binary quantifier case.

Assume $[\![Qv(\psi,\chi)]\!]^+_{s,g} = 1$. Then the fact that $[\![Qv(\psi,\chi)]\!]^\circ_{s,g} = 1$ follows from the semantic clause for binary quantifiers and the fact that $\mathbf{Q}^+_E(X, X^*, Y, Y^*) = 1$ entails $\mathbf{Q}^\circ_E(X, X^*, Y, Y^*) = 1$. ∎

The lemma justifies the lumping together of $[\![\cdot]\!]^+$ and $[\![\cdot]\!]^\circ$ in a definition of an interpretation function $[\![\cdot]\!]$ which takes values in $\{1, 0, *\}$, and guarantees that the three cases in the definition of $[\![\cdot]\!]$ are the only cases that can occur:

Definition 4 [Interpretation in a Situation]

- $[\![\varphi]\!]_{s,g} = 1$ if $[\![\varphi]\!]^{+}_{s,g} = 1$.
- $[\![\varphi]\!]_{s,g} = 0$ if $[\![\varphi]\!]^{\circ}_{s,g} = 0$.
- $[\![\varphi]\!]_{s,g} = *$ if $[\![\varphi]\!]^{+}_{s,g} = 0$ and $[\![\varphi]\!]^{\circ}_{s,g} = 1$.

Note that it follows from the clauses for $\forall, \exists$ and $\neg$ that:

$[\![\forall v\varphi]\!]^{+}_{s,g} = 1$
iff for all $d \in E$: $[\![\varphi]\!]^{+}_{s,g(v|d)} = 1$
iff for all $d \in E$: $[\![\neg\varphi]\!]^{\circ}_{s,g(v|d)} = 0$
iff $[\![\exists v\neg\varphi]\!]^{\circ}_{s,g} = 0$
iff $[\![\neg\exists v\neg\varphi]\!]^{+}_{s,g} = 1$.

Also:

$[\![\forall v\varphi]\!]^{\circ}_{s,g} = 1$
iff for all $d \in E$: $[\![\varphi]\!]^{\circ}_{s,g(v|d)} = 1$
iff for all $d \in E$: $[\![\neg\varphi]\!]^{+}_{s,g(v|d)} = 0$
iff for no $d \in E$: $[\![\neg\varphi]\!]^{+}_{s,g(v|d)} = 1$
iff $[\![\exists v\neg\varphi]\!]^{+}_{s,g} = 0$
iff $[\![\neg\exists v\neg\varphi]\!]^{\circ}_{s,g} = 1$.

This means that $\forall$ and $\exists$ are duals. It is convenient to also have a notion of duality for binary quantifiers.

Definition 5 $\mathbf{Q}_E$ and $\tilde{\mathbf{Q}}_E$ are dual quantifier interpretations if the following holds:

$$\mathbf{Q}^{+}_{E}(X, X^*, Y, Y^*) = 1 \text{ iff } \tilde{\mathbf{Q}}^{\circ}_{E}(X, X^*, E - (Y \cup Y^*), Y^*) = 0.$$

The next proposition shows that this definition does indeed have the desired effect.

Proposition 2 *If $Q, \tilde{Q}$ are interpreted as dual quantifiers, then for all formulae φ, ψ, for every situation s, and for every g for s:*

$$[\![Qv(\varphi, \psi) \Leftrightarrow \neg\tilde{Q}v(\varphi, \neg\psi)]\!]_{s,g} = 1.$$

Proof. The reader is invited to check that it follows from the definition of the sentential operator $\Leftrightarrow$ given above that the statement of the proposition is equivalent to the conjunction of the following:

1. $[\![Qv(\varphi, \psi)]\!]^{+}_{s,g} = 1$ iff $[\![\neg\tilde{Q}v(\varphi, \neg\psi)]\!]^{+}_{s,g} = 1$,
2. $[\![Qv(\varphi, \psi)]\!]^{\circ}_{s,g} = 1$ iff $[\![\neg\tilde{Q}v(\varphi, \neg\psi)]\!]^{\circ}_{s,g} = 1$.

Suppose s has domain E.

Let A^+ be $\{d \in E \mid [\![\varphi]\!]^+_{s,g(v|d)} = 1\}$,
let A^* be $\{d \in E \mid [\![\varphi]\!]^+_{s,g(v|d)} = 0, [\![\varphi]\!]^\circ_{s,g(v|d)} = 1\}$,
let B^+ be $\{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 1\}$,
let B^* be $\{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 0, [\![\psi]\!]^\circ_{s,g(v|d)} = 1\}$,
let C^+ be $\{d \in E \mid [\![\neg\psi]\!]^+_{s,g(v|d)} = 1\}$,
let C^* be $\{d \in E \mid [\![\neg\psi]\!]^+_{s,g(v|d)} = 0, [\![\neg\psi]\!]^\circ_{s,g(v|d)} = 1\}$.

To be able to use the fact that $\mathbf{Q}_E$ and $\tilde{\mathbf{Q}}_E$ are duals, we have to establish that $B^* = C^*$ and that $C^+ = E - (B^+ \cup B^*)$. The first of these follows from the fact that

$[\![\neg\psi]\!]^+_{s,g(v|d)} = 0$ and $[\![\neg\psi]\!]^\circ_{s,g(v|d)} = 1$
iff (clause for $\neg$) $[\![\psi]\!]^\circ_{s,g(v|d)} = 1$ and $[\![\psi]\!]^+_{s,g(v|d)} = 0$.

The second of these follows from the fact that

$[\![\neg\psi]\!]^+_{s,g(v|d)} = 1$
iff (clause for $\neg$) $[\![\psi]\!]^\circ_{s,g(v|d)} = 0$
iff (coherence) $[\![\psi]\!]^+_{s,g(v|d)} = 0$ and $[\![\psi]\!]^\circ_{s,g(v|d)} = 0$.

Now from the fact that $\mathbf{Q}_E$ and $\tilde{\mathbf{Q}}_E$ are duals:

$$\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1 \text{ iff } \tilde{\mathbf{Q}}^\circ_E(A^+, A^*, C^+, C^*) = 0,$$

so by the semantic clause for binary quantifiers,

$$[\![Qv(\varphi,\psi)]\!]^+_{s,g} = 1 \text{ iff } [\![\tilde{Q}v(\varphi,\neg\psi)]\!]^\circ_{s,g} = 0.$$

By the semantic clause for $\neg$:

$$[\![Qv(\varphi,\psi)]\!]^+_{s,g} = 1 \text{ iff } [\![\neg\tilde{Q}v(\varphi,\neg\psi)]\!]^+_{s,g} = 1.$$

Similarly: $[\![Qv(\varphi,\psi)]\!]^\circ_{s,g} = 1$ iff $[\![\neg\tilde{Q}v(\varphi,\neg\psi)]\!]^\circ_{s,g} = 1$. ∎

The preceding text illustrates that in discussing the behaviour of a formula $Qv(\varphi,\psi)$ in a situation s on domain E under an assignment g in E, it is often convenient to abbreviate

$\{d \in E \mid [\![\varphi]\!]^+_{s,g(v|d)} = 1\}$ as A^+,

$\{d \in E \mid [\![\varphi]\!]^\circ_{s,g(v|d)} = 0\}$ as A^-,

$\{d \in E \mid [\![\varphi]\!]^+_{s,g(v|d)} = 0, [\![\varphi]\!]^\circ_{s,g(v|d)} = 1\}$ as A^*,

$\{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 1\}$ as B^+,

$\{d \in E \mid [\![\psi]\!]^\circ_{s,g(v|d)} = 0\}$ as B^-,

and

$\{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 0, [\![\psi]\!]^\circ_{s,g(v|d)} = 1\}$ as B^*.

Thus, from now on we adopt the convention that once we have fixed the situation and assignment parameters, we represent the positive, negative, and gap extensions of the first argument of a quantifier (the restriction argument) by A^+, A^-, A^*, respectively, and the positive, negative and gap extensions of the second argument (the body argument) by B^+, B^-, B^*, respectively. By the coherence lemma, we have for any domain E, any situation s on E, any assignment g in E and any $Qv(\varphi, \psi)$ that $A^+ \cap A^- = A^+ \cap A^* = A^- \cap A^* = \emptyset$, and $B^+ \cap B^- = B^+ \cap B^* = B^- \cap B^* = \emptyset$. For convenience we will tacitly assume from now on that in all cases where a partial set is introduced in terms of its positive and negative extensions C^+, C^- or, relative to some universe E, in terms of its positive and gap extensions C^+, C^*, then $C^+ \cap C^- = \emptyset$, respectively $C^+ \cap C^* = \emptyset$.

If we consider a quantified formula $Qv(\varphi, \psi)$ in a situation s with domain E, given an assignment g, then the amount of available information about the extension of φ is reflected in the gap A^*. As long as $A^* \neq \emptyset$, only partial information is available about the extension of φ. Growth of information about the extension of φ means that elements of A^* get inspected and classified. An $x \in A^*$ can turn out to be an A^+ or an A^-, which means that each new act of classification makes A^+ increase or A^- increase, thus narrowing the gap between A^+ and $E - A^-$. In the limit case of $A^+ \cup A^- = E$, i.e., $A^* = \emptyset$, full information about the extension of φ is available. Similarly for the extension of the second argument ψ.

Definition 6 A situation $s = \langle E, I^+, I^- \rangle$ is *total* if for every $n > 0$, I^+, I^- map each member of P^n to n-place relations R^+ and R^- on E for which $R^+ \cup R^- = E^n$. Otherwise s is *partial.*

Note that the definition of the quantifier relations on a total situation need not be total. It is convenient to keep the notions of totality for the interpretation of the nonlogical vocabulary and for the interpretation of the quantifiers separate.

Lemma 3 (Totality) *If $\varphi \in \mathcal{L}_{\mathcal{Q},\sim}$ does not contain occurrences of quantifiers from $\mathcal{Q}$, then for any total s and any g for s, $[\![\varphi]\!]^+_{s,g} = 1$ iff $[\![\varphi]\!]^\circ_{s,g} = 1$.*

Proof. Use induction on the complexity of φ. ■

Note that this result does not hold for the languages $\mathcal{L}_{\mathcal{Q},\otimes}$ or $\mathcal{L}_{\mathcal{Q},\sim,\otimes}$. In Section 5, total situations will be used to define supervaluation quantifier relations.

A quantifier interpretation $\mathbf{Q}^+, \mathbf{Q}^\circ$ on a universe E can equivalently be viewed as a function $\mathcal{Q}$ with domain $(\mathcal{P}[E])^2$ and range $\subseteq \{1, 0, *\}$. In other words, a quantifier interpretation takes four set-arguments and ranges over the values true, false and undefined (neither true nor false). Relative to a situation s and an assignment g, the three valued quantifier function that interprets $Qv(\varphi, \psi)$ on a universe E is defined as follows.

Definition 7 [Quantifier Interpretations as Functions of A^+, A^*, B^+, B^*]

- $\mathbf{Q}_E(A^+, A^*, B^+, B^*) = 1$ if $\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$.

- $\mathbf{Q}_E(A^+, A^*, B^+, B^*) = 0$ if $\mathbf{Q}_E^\circ(A^+, A^*, B^+, B^*) = 0$.
- $\mathbf{Q}_E(A^+, A^*, B^+, B^*) = *$ if $\mathbf{Q}_E^+(A^+, A^*, B^+, B^*) = 0$ and $\mathbf{Q}_E^\circ(A^+, A^*, B^+, B^*) = 1$.

Again, it follows from the coherence lemma that if A^+, A^*, B^+, B^* interpret φ, ψ relative to some s and g, then the case with $\mathbf{Q}_E^+(A^+, A^*, B^+, B^*) = 1$ and $\mathbf{Q}_E^\circ(A^+, A^*, B^+, B^*) = 0$ cannot occur.

The relation of consequence $\models$ for language $\mathcal{L}_Q$ (and its extensions) of partial logic with binary quantifiers is defined as follows (let Γ and Δ be sets of sentences of $\mathcal{L}_Q$ or one of its extensions):

Definition 8 [Logical Consequence]

- $\Gamma \models^+ \Delta$ if for all situations s: if $[\![\varphi]\!]_s^+ = 1$ for all $\varphi \in \Gamma$, then $[\![\psi]\!]_s^+ = 1$ for some $\psi \in \Delta$.
- $\Gamma \models^- \Delta$ if for all situations s: if $[\![\psi]\!]_s^- = 1$ for all $\psi \in \Delta$, then $[\![\varphi]\!]_s^- = 1$ for some $\varphi \in \Gamma$.
- $\Gamma \models \Delta$ if $\Gamma \models^+ \Delta$ and $\Gamma \models^- \Delta$.

This notion of logical consequence is the so-called *double-barrelled* consequence from ?. Recall the definitions of $\varphi \rightsquigarrow \psi$ as $\sim\varphi \vee \psi$, of $\varphi \Rightarrow \psi$ as $(\varphi \rightsquigarrow \psi) \wedge (\neg\psi \rightsquigarrow \neg\varphi)$, of $\varphi \equiv \psi$ as $(\varphi \rightsquigarrow \psi) \wedge (\psi \rightsquigarrow \varphi)$, and of $\varphi \Leftrightarrow \psi$ as $(\varphi \equiv \psi) \wedge (\neg\varphi \equiv \neg\psi)$. It is easy to see that we have the following (set brackets for premisses and conclusion are omitted for readability):

(11) $\varphi \models^+ \psi$ iff $\models \varphi \rightsquigarrow \psi$.

(12) $\varphi \models^- \psi$ iff $\models \neg\psi \rightsquigarrow \neg\varphi$.

(13) $\varphi \models \psi$ iff $\models \varphi \Rightarrow \psi$.

(14) Both $\varphi \models^+ \psi$ and $\psi \models^+ \varphi$ iff $\models \varphi \equiv \psi$.

(15) Both $\varphi \models \psi$ and $\psi \models \varphi$ iff $\models \varphi \Leftrightarrow \psi$.

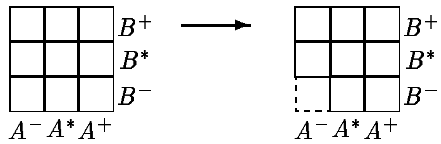
Note that at the meta level three valued equivalence (the counterpart of $\Leftrightarrow$) is expressed by the identity symbol. We will use $\mathbf{Q}_E^+(A^+, A^*, B^+, B^*) = \mathbf{Q}_{E'}^+(C^+, C^*, D^+, D^*)$ to express that the values of $\mathbf{Q}_E^+(A^+, A^*, B^+, B^*)$ and $\mathbf{Q}_{E'}^+(C^+, C^*, D^+, D^*)$ are the same: both 1, both 0 or both $*$.

3 Constraints on Partial Quantifiers

After these preliminaries we are ready to focus on the quantifier interpretations. We need versions of **EXT**, **CONS** and **ISOM** tailored to our partial perspective. We assume an information system $\mathbf{S}$, a situation $s \in S$, and an assignment g for s fixed, so we can talk about the positive and gap extensions of the first and second arguments of a quantifier as A^+, A^*, B^+, B^* without ambiguity.

For *extension*, we want to say that adding individuals known to be neither A nor B to the universe does not matter. Suppose $A^+, A^*, B^+, B^* \subseteq E$. Suppose we add a set X of individuals which are neither A nor B to

FIGURE 5 The Effect of EXT.



the universe E of a situation. Then A^+, A^*, B^+, B^* remain unchanged, and the truth or falsity of the quantifier is not affected. In other words, extension says that there is no need to look outside $A^+ \cup A^* \cup B^+ \cup B^*$.

A formal rendering of **EXT** runs like this.

EXT For all $E, E' \supseteq A^+ \cup A^* \cup B^+ \cup B^*$,

$$\mathbf{Q}_E(A^+, A^*, B^+, B^*) = \mathbf{Q}_{E'}(A^+, A^*, B^+, B^*).$$

Note that the constraint is very easy to state, thanks to the fact that we have chosen to define partial quantifiers in terms of positive extensions and gaps rather than positive and negative extensions. If one views a binary quantifier on partial sets as a function on two partial sets given in terms positive and negative extensions, then the **EXT** constraint would have to relate a quantifier on domain E with given arguments to a quantifier on domain E' with different arguments, because if $E \neq E'$ and $E, E' \supseteq A^+ \cup A^* \cup B^+ \cup B^*$, the sets $E - (A^+ \cup A^*)$ and $E' - (A^+ \cup A^*)$ will be different, and similarly for the negative part of the second argument.

EXT allows us to restrict any domain $E \supseteq A^+ \cup A^* \cup B^+ \cup B^*$ for $\mathbf{Q}_E$ to $A^+ \cup A^* \cup B^+ \cup B^*$:

(16) $\mathbf{Q}_E(A^+, A^*, B^+, B^*) = \mathbf{Q}_{A^+ \cup A^* \cup B^+ \cup B^*}(A^+, A^*, B^+, B^*)$.

It follows that for quantifier interpretations satisfying **EXT** the parameter E can be dropped altogether. This hinges on the fact that quantifiers are defined as functions of four arguments A^+, A^*, B^+, B^*.

It is convenient to use pictures to illustrate the effects of the various quantifier constraints. The pictures show which of the subsets of the domain are relevant for the interpretation $\mathbf{Q}_E(A^+, A^*, B^+, B^*)$ of $Qv(\varphi, \psi)$ in s, given g. The boxes $\square$ in the first parts of the pictures indicate on which sets the quantifier interpretation depends when the relevant constraints are not imposed, the boxes $\square$ in the part of the picture following the $\longrightarrow$ indicate on which sets the quantifier interpretation depends when they are imposed. A pictorial representation of the effect of **EXT** is given in Figure 5. The picture shows that Qs observing **EXT** do not depend on the set $A^- \cap B^-$.

For conservativity the situation is less straighforward. *Prima facie* there are several options. The weakest possible variant seems to be to demand that the quantifier is only sensitive to the effect of those entities in the domain that *may end up* in the positive extension of the first argument.

FIGURE 6 The Effect of W-CONS.

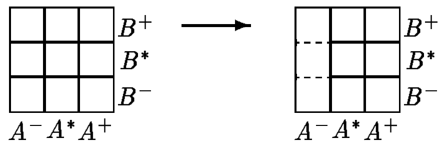
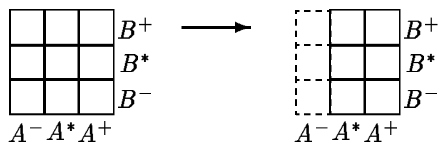


FIGURE 7 The Combined Effect of EXT and W-CONS.



In other words: the set $A^+ \cup A^*$ sets the stage. We will call this *weak conservativity*, abbreviation **W-CONS**. Here is the formal version.

W-CONS $Q_E(A^+, A^*, B^+, B^*) = Q_E(A^+, A^*, B^+ \cap (A^+ \cup A^*), B^* \cap (A^+ \cup A^*))$.

The effect of **W-CONS** is pictured in Figure 6. The picture shows that quantifiers observing **W-CONS** are invariant under borderline crossings between $A^- \cap B^-$, $A^- \cap B^*$, and $A^- \cap B^+$. Figure 7 pictures the combined effect of **EXT** and **W-CONS**. This picture shows that quantifiers observing these constraints are insensitive to changes in A^-.

Several strenghtenings of weak conservativity can be considered. A strong requirement is the following. The set of entities that definitely are in the positive extension of the first argument sets the stage. In other words: the quantifier is only sensitive in its B^+, B^* arguments to what happens inside A^+. There are two variants of this: in the first variant (very strong conservativity), the A^+ set also sets the stage for the first argument, so the quantifier is completely insensitive to the contents of A^*. In the second variant (strong conservativity), A^+ only sets the stage for the B argument. Formally:

VS-CONS $Q_E(A^+, A^*, B^+, B^*) = Q_E(A^+, \emptyset, B^+ \cap A^+, B^* \cap A^+)$.
S-CONS $Q_E(A^+, A^*, B^+, B^*) = Q_E(A^+, A^*, B^+ \cap A^+, B^* \cap A^+)$.

Note that **VS-CONS** implies **S-CONS**, and **S-CONS** in its turn implies **W-CONS**. The combined effect of **EXT** and **VS-CONS** is pictured in Figure 8. The combined effect of **EXT** and **S-CONS** is pictured in Figure 9. We will see in the next section that the constraint **VS-CONS** can be ruled out immediately as being too strong.

Another version of conservativity is considered in ?. Van Benthem discusses the merits of the version of conservativity that results from generalizing 'total intersection' $A \cap B$ to 'partial intersection' $A^+ \cap B^+, A^- \cup B^-$.

FIGURE 8 The Combined Effect of EXT and VS-CONS.

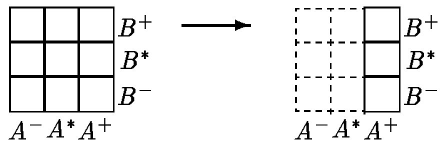
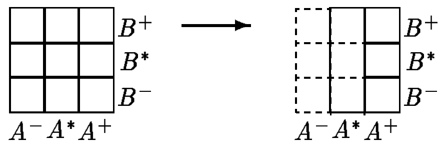


FIGURE 9 The Combined Effect of EXT and S-CONS.



This version of conservativity holds if the quantifier is indifferent to a substitution of $A^+ \cap B^+$ for B^+ and a substitution of $A^- \cup B^-$ for B^-. Van Benthem's version of conservativity can be formulated as the following principle of *mixed conservativity*. The constraint is easy to state for quantifiers defined in terms of positive and negative extensions, but it becomes awkward for quantifiers defined in terms of positive extensions and gaps. We give both formulations, using $\check{\mathbf{Q}}$ for the quantifier corresponding to $\mathbf{Q}$ but defined in terms of positive and negative extensions rather than positive extensions and gaps.

M-CONS (in terms of A^+, A^*, B^+, B^*)

$$\mathbf{Q}_E(A^+, A^*, B^+, B^*) = \mathbf{Q}_E(A^+, A^*, B^+ \cap A^+, (A^+ \cap B^*) \cup (A^* \cap B^+) \cup (A^* \cap B^*)).$$

M-CONS (in terms of A^+, A^-, B^+, B^-)

$$\check{\mathbf{Q}}_E(A^+, A^-, B^+, B^-) = \check{\mathbf{Q}}_E(A^+, A^-, B^+ \cap A^+, B^- \cup A^-).$$

The combined effect of **EXT** and **M-CONS** is pictured in Figure 10.

Looking at the pictural effects of the various conservativity restrictions, we see that there must be a fourth possibility, namely the constraint that blurs the distinction between $A^* \cap B^-$ and $A^* \cap B^*$, while leaving the borderline between $A^* \cap B^+$ and $A^* \cap B^*$ intact. This turns out to be the following principle (again we use $\check{\mathbf{Q}}$ for the quantifier corresponding to $\mathbf{Q}$ but defined

FIGURE 10 The Combined Effect of EXT and M-CONS.

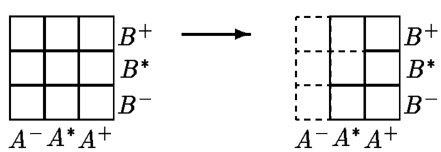
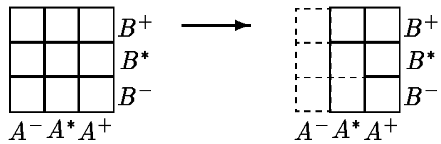


FIGURE 11 The Combined Effect of EXT and M'-CONS.



in terms of positive and negative extensions rather than positive extensions and gaps):

M'-CONS (in terms of A^+, A^*, B^+, B^*)

$$\mathbf{Q}_E(A^+, A^*, B^+, B^*) = \mathbf{Q}_E(A^+, A^*, B^+ \cap (A^+ \cup A^*), B^* \cap A^+).$$

M'-CONS (in terms of A^+, A^-, B^+, B^-)

$$\check{\mathbf{Q}}_E(A^+, A^-, B^+, B^-) = \check{\mathbf{Q}}_E(A^+, A^-, B^+ \cap (A^+ \cup A^*), B^- \cup A^- \cup A^*).$$

The combined effect of **EXT** and **M'-CONS** is shown in Figure 11. Note that **S-CONS** implies both **M-CONS** and **M'-CONS**. The choice between **W-CONS**, **S-CONS**, **M-CONS** and **M'-CONS** will not be made until later. The following table shows that all these versions of conservativity are systematically related, and that the list of possibilities that we have given is exhaustive. The relationships are clearest if the principles are formulated in terms of positive and negative extensions. In the arguments of the quantifiers, $\overline{S}$ is used for the complement of S with respect to the domain of the quantifier.

VS CONS	$\check{Q}_E(A^+, A^-, B^+, B^-)$	iff	$\check{Q}_E(A^+, \overline{A^+}, B^+ \cap A^+, B^- \cup \overline{A^+})$.
S CONS	$\check{Q}_E(A^+, A^-, B^+, B^-)$	iff	$\check{Q}_E(A^+, A^-, B^+ \cap A^+, B^- \cup \overline{A^+})$.
M CONS	$\check{Q}_E(A^+, A^-, B^+, B^-)$	iff	$\check{Q}_E(A^+, A^-, B^+ \cap A^+, B^- \cup A^-)$.
M' CONS	$\check{Q}_E(A^+, A^-, B^+, B^-)$	iff	$\check{Q}_E(A^+, A^-, B^+ \cap \overline{A^-}, B^- \cup \overline{A^+})$.
W CONS	$\check{Q}_E(A^+, A^-, B^+, B^-)$	iff	$\check{Q}_E(A^+, A^-, B^+ \cap \overline{A^-}, B^- \cup A^-)$.

The new version of *isomorphy* says that only the sizes of the sets A^+, A^*, B^+, B^* matter:

ISOM If f is a bijection from E to E', then: $Q_E(A^+, A^*, B^+, B^*) = Q_{E'}(f[A^+], f[A^*], f[B^+], f[B^*])$.

Note that **EXT**, **ISOM** and the various versions of **CONS** state both verification and falsification conditions. The statements used to formulate these principles say that two truth values are equal: both 1, both 0, or both * (undefined).

In what follows, we restrict attention to quantifier interpretations that observe **EXT** and **ISOM**, so that we can drop the parameters for the universes that the quantifier relations range over.

4 Further Properties

In this section we will look at three basic properties that partial quantifiers can have, namely closedness, persistence under growth of information and predictiveness. It will turn out that if one knows that the quantifiers that interpret the binary quantifier symbols $\mathcal{Q}$ in a language have one or more of these properties, one can say quite a lot about the expressive power of the language.

In Section 1 a property of quantifiers called *closedness* was mentioned. In a total setting, a quantifier is closed if it does not itself introduce truth value gaps. In the present partial setting this requirement takes the following natural shape.

CL For all $X, Y \subseteq E$, $\mathbf{Q}_E^+(X, \emptyset, Y, \emptyset) = 1$ iff $\mathbf{Q}_E^\circ(X, \emptyset, Y, \emptyset) = 1$.

We can also consider the property of closedness for certain given arguments A^+, B^+. A quantifier $\mathbf{Q}_E$ is closed for those arguments if $\mathbf{Q}_E^+(A^+, \emptyset, B^+, \emptyset) = \mathbf{Q}_E^\circ(A^+, \emptyset, B^+, \emptyset)$. Quantifiers not satisfying **CL** (for arguments A^+, B^+) are *open* (for arguments A^+, B^+). For a pictorial rendering of **CL**, observe that a closed quantifier must yield either true or false in all situations satisfying Figure 12.

We can now relate the **CL** property of quantifier interpretations to the following property of formulae.

Definition 9 A formula φ is determinable if for any total situation s and any g for s, $[\![\varphi]\!]^+_{s,g} = 1$ iff $[\![\varphi]\!]^\circ_{s,g} = 1$.

If we know that all quantifiers in $\mathcal{L}_\mathcal{Q}$ are closed, then we can strengthen the totality lemma.

Theorem 4 (Determinability) *If all quantifiers in $\mathcal{Q}$ are interpreted as closed quantifiers, then any $\varphi \in \mathcal{L}_{\mathcal{Q},\sim}$ is determinable.*

Proof. Induction on the complexity of φ. For the case where φ has the form $Qv(\psi, \chi)$, the induction hypothesis yields that for total s and arbitrary $g(v|d)$ it holds that $[\![\psi]\!]^+_{s,g(v|d)} = [\![\psi]\!]^\circ_{s,g(v|d)}$ and $[\![\chi]\!]^+_{s,g(v|d)} = [\![\chi]\!]^\circ_{s,g(v|d)}$, and therefore $A^* = B^* = \emptyset$. But then it follows from the fact that $\mathbf{Q}_E^+(A^+, \emptyset, B^+, \emptyset) = 1$ iff $\mathbf{Q}_E^\circ(A^+, \emptyset, B^+, \emptyset) = 1$ that $[\![Qv(\varphi, \psi)]\!]^+_{s,g} = 1$ iff $[\![Qv(\varphi, \psi)]\!]^\circ_{s,g} = 1$. ∎

It follows immediately from the theorem that all quantifiers definable in $\mathcal{L}_{\mathcal{Q},\sim}$ in terms of closed quantifiers will be closed. Also, we have the following corollary.

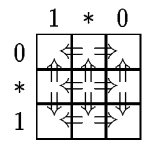
Corollary 5 *All quantifiers definable in $\mathcal{L}_\sim$ observe* **CL**.

Note that Theorem 4 does not generalize to the languages $\mathcal{L}_{\mathcal{Q},\otimes}$ or $\mathcal{L}_{\mathcal{Q},\sim,\otimes}$. If Q and Q' are interpreted as two *different* closed quantifiers, then $Qx(Ax, Bx) \otimes Q'x(Ax, Bx)$ does not define a closed quantifier, as can be

FIGURE 12 Situations Where Closed Quantifiers Yield 1 or 0.

	A^-	A^*	A^+
B^+		$\emptyset$	
B^*	$\emptyset$	$\emptyset$	$\emptyset$
B^-		$\emptyset$	

FIGURE 13 Climbing Up on the $\leq$ Ladder in a Three Valued Truth Table



easily seen, as follows. From the fact that Q, Q' are interpreted as different closed quantifiers we have that there are interpretations for A, B for which there is a total situation t with $[\![Qx(Ax, Bx)]\!]_t = 1$ and $[\![Q'x(Ax, Bx)]\!]_t = 0$. By the semantic clause for $\otimes$, $[\![Qx(Ax, Bx) \otimes Q'x(Ax, Bx)]\!]_t = *$, so $Qv(\varphi, \psi) \otimes Q'v(\varphi, \psi)$ is not determinable, i.e., the sentence does not define a closed quantifier.

The second property we are interested in is persistence under growth of information about the positive and/or negative extensions of the quantifier arguments. First we define a relation $\leq$ between situations.

Definition 10 $s \leq u$ if the following hold:

- $E_s = E_u$,
- for all predicate symbols P in the language: $I_s^+(P) \subseteq I_u^+(P)$ and $I_s^-(P) \subseteq I_u^-(P)$.

It is easily checked that $\leq$ is a partial order.

Definition 11 [$\leq$-persistence] A formula φ is $\leq$ persistent if for any $s \leq u$ and for any assignment g for s, u the following hold:

- $[\![\varphi]\!]_{s,g} = 1$ implies $[\![\varphi]\!]_{u,g} = 1$.
- $[\![\varphi]\!]_{s,g} = 0$ implies $[\![\varphi]\!]_{u,g} = 0$.

This notion carries over to truth functions, as follows. Call a three valued truth function $f(x_1, \ldots, x_n)$ $\leq$ persistent if it interprets a $\leq$ persistent formula $\varphi(p_1, \ldots, p_n)$, where the $p_1, \ldots, p_n$ are the atomic parts of φ. Then it is not difficult to see that $f(x_1, \ldots, x_n)$ is $\leq$ persistent iff for no argument sequences yielding the function value 1 or 0 is it possible to change that value to something else by a series of changes of arguments x_i from $*$ to 0 or 1.

We can simply inspect the truth tables of the sentential connectives for this property. Just checking if none of the steps in Figure 13 get one from a

FIGURE 14 Three-valued Truth Tables.

	$\neg$
1	*0*
*	*
0	*1*

	$\sim$
1	*0*
*	*1*
0	*1*

$\wedge$	*1*	*	*0*
1	*1*	*	*0*
*	*	*	*0*
0	*0*	*0*	*0*

$\vee$	*1*	*	*0*
1	*1*	*1*	*1*
	1	*	*
0	*1*	*	*0*

$\otimes$	*1*	*	*0*
1	*1*	*	*
*	*	*	*
0	*	*	*0*

1 position to a non 1 position or from a 0 position to a non 0 position is all there is to it. This truth table inspection (see Figure 4) yields immediately that $\neg$, $\wedge$, $\vee$ and $\otimes$ are $\leq$ persistent, but that $\sim$ is not.

It is useful to be able to impose $\leq$ persistence directly as a constraint on quantifier interpretations.

$\leq$-PERSIST If $A^+ \subseteq C^+ \subseteq E$, $A^- \subseteq C^- \subseteq E$ and $B^+ \subseteq D^+ \subseteq E$, $B^- \subseteq D^- \subseteq E$ then:

- $\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$ implies $\mathbf{Q}^+_E(C^+, C^*, D^+, D^*) = 1$.
- $\mathbf{Q}^\circ_E(A^+, A^*, B^+, B^*) = 0$ implies $\mathbf{Q}^\circ_E(C^+, C^*, D^+, D^*) = 0$.

We now turn to some expressibility issues for languages with $\leq$ persistent quantifiers.

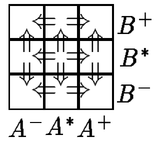
Theorem 6 *If all quantifier symbols in $\mathcal{Q}$ are interpreted as quantifiers satisfying* **$\leq$-PERSIST**, *then any $\varphi \in \mathcal{L}_{\mathcal{Q},\otimes}$ is $\leq$-persistent.*

Proof. Induction on the complexity of φ. The atomic formulae are $\leq$-persistent by the definition of of the evaluation function. Formulae of the forms $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, and $\psi \otimes \chi$ are $\leq$ persistent if their components are, by the preceding truth table argument. The cases of $\forall v\psi$ and $\exists v\psi$ are left to the reader. For the binary quantifier case, suppose φ has the form $Qv(\psi, \chi)$ and let s, u be situations with $s \leq u$.

Let A^+ be $\{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 1\}$,
let A^* be $\{d \in E \mid [\![\psi]\!]^+_{s,g(v|d)} = 0, [\![\psi]\!]^\circ_{s,g(v|d)} = 1\}$,
let B^+ be $\{d \in E \mid [\![\chi]\!]^+_{s,g(v|d)} = 1\}$,
let B^* be $\{d \in E \mid [\![\chi]\!]^+_{s,g(v|d)} = 0, [\![\chi]\!]^\circ_{s,g(v|d)} = 1\}$,
let C^+ be $\{d \in E \mid [\![\psi]\!]^+_{u,g(v|d)} = 1\}$,
let C^* be $\{d \in E \mid [\![\psi]\!]^+_{u,g(v|d)} = 0, [\![\psi]\!]^\circ_{u,g(v|d)} = 1\}$,
let D^+ be $\{d \in E \mid [\![\chi]\!]^+_{u,g(v|d)} = 1\}$,
let D^* be $\{d \in E \mid [\![\chi]\!]^+_{u,g(v|d)} = 0, [\![\chi]\!]^\circ_{s,g(v|d)} = 1\}$.

In order to be able to use the fact that $\mathbf{Q}_E$ satisfies **$\leq$-PERSIST**, we have

FIGURE 15 1,0-Preserving Transitions for $\leq$-Persistent Quantifiers



to establish that $A^+ \subseteq C^+ \subseteq E$, $A^- \subseteq C^- \subseteq E$, and that $B^+ \subseteq D^+ \subseteq E$, $B^- \subseteq D^- \subseteq E$.

The induction hypothesis yields that $[\![\psi]\!]^+_{s,g(v|d)} = 1$ implies that $[\![\psi]\!]^+_{u,g(v|d)} = 1$, and therefore $A^+ \subseteq C^+ \subseteq E$. The induction hypothesis also yields that $[\![\psi]\!]^\circ_{s,g(v|d)} = 0$ implies that $[\![\psi]\!]^\circ_{u,g(v|d)} = 0$, and therefore that $A^- \subseteq C^- \subseteq E$. Similarly, the induction hypothesis that $B^+ \subseteq D^+ \subseteq E$ and $B^- \subseteq D^- \subseteq E$.

Assume $[\![Qv(\varphi,\psi)]\!]_{s,g} = 1$.
Then $[\![Qv(\varphi,\psi)]\!]^+_{s,g} = 1$, i.e., $\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$.
By $\leq$-**PERSIST** of $\mathbf{Q}_E$, it follows that $\mathbf{Q}^+_E(C^+, C^*, D^+, D^*) = 1$.

Assume $[\![Qv(\varphi,\psi)]\!]_{s,g} = 0$.
Then $[\![Qv(\varphi,\psi)]\!]^\circ_{s,g} = 0$, i.e., $\mathbf{Q}^\circ_E(A^+, A^*, B^+, B^*) = 0$.
By $\leq$-**PERSIST** of $\mathbf{Q}_E$, it follows that $\mathbf{Q}^\circ_E(C^+, C^*, D^+, D^*) = 0$. ∎

The theorem shows that provided all quantifier symbols in $\mathcal{Q}$ are interpreted as quantifiers satisfying $\leq$-**PERSIST**, any quantifier definable in $\mathcal{L}_{\mathcal{Q},\otimes}$ will satisfy $\leq$-**PERSIST** as well. Also, we have the following corollary.

Corollary 7 *All quantifiers definable in $\mathcal{L}_\otimes$ satisfy* $\leq$-**PERSIST**.

The pictorial effect of $\leq$-persistence for binary quantifiers is given in Figure 15. We can look at the process of finding out more about a situation s as a shift from s to a situation u with $s \leq u$. Moreover, we can picture this process of information growth in a step-by-step way. Finding out that an object in $A^* \cap B^*$ is in fact in A^+ can be pictured as a transition $\Rightarrow$ from the $A^* \cap B^*$ region to the $A^+ \cap B^*$ region in the diagram. It is clear that in the case of two partial predicates there are twelve possible transitions of information growth. Thus, for $\leq$-persistent quantifiers, all twelve $\Rightarrow$ transitions in Figure 15 must preserve truth as well as falsity.

The third property we are interested in is predictiveness. We first define this property for formulae.

Definition 12 A formula φ is predictive if for any pair of situations s, u with $s < u$ and any assignment g for s and u the following holds:

If $[\![\varphi]\!]_{s,g} = *$ and $[\![\varphi]\!]_{u,g} = 1(0)$, then there is a w with $s < w$ and $[\![\varphi]\!]_{w,g} = 0(1)$.

What predictiveness of φ says is that if there are situations where φ is still neither true nor false, and it is possible by acquiring more information to arrive at a situation where φ is true (false), then things might have turned out differently, and we might have ended up in a situation where φ was false (true).

This notion carries over to truth functions, as follows. Call a three valued truth function $f(x_1, \ldots, x_n)$ predictive if it interprets a predictive formula $\varphi(p_1, \ldots, p_n)$, where the $p_1, \ldots, p_n$ are the atomic parts of φ. It is not difficult to see that $f(x_1, \ldots, x_n)$ is predictive iff for any change of an input argument x_i from $*$ to 1 that results in a function value change from $*$ to 1(0), it is the case that after a change of x_i from $*$ to 0 there exists a number of zero or more changes of other input arguments from $*$ to 1 or 0 with a function value of 0(1) as a result.

To check whether a given sentential connective expresses a predictive truth function we can again simply inspect the truth table for this property. To see if a truth table pictures a predictive truth function, simply refer to Figure 13 and check whether for any $\Rightarrow$ step from a $*$ position to a 1(0) position there also is a path of $\Rightarrow$ steps to a 0(1) position. It is easily seen from the truth tables that $\neg$, $\sim$, $\wedge$, and $\vee$ are predictive, but that $\otimes$ is not.

Again, there is a corresponding notion for quantifier interpretations (this notion was defined already in **?**). A quantifier interpretation observes prediction in its first argument if the following holds. If, in a given situation s, the discovery about some set of objects X that $X \subseteq A^+$ makes the quantifier true (false), and the discovery that $X \subseteq A^-$ also makes the quantifier true (false), then the quantifier is already true (false) in s. Similarly, a quantifier observes prediction in its second argument if the same holds for the B predicate. In the formal version of these prediction properties we lump prediction for the first and second argument together.

PREDICT For all $A^+, A^-, B^+, B^- \subseteq E, X \subseteq A^*, Y \subseteq B^*$:

if $\mathbf{Q}_E(A^+, A^*, B^+, B^*) = *$,

then $\mathbf{Q}_E(A^+ \cup X, A^* - X, B^+ \cup Y, B^* - Y) = 1\ (0)$

iff $\mathbf{Q}_E(A^+, A^* - X, B^+, B^* - Y) = 0\ (1)$.

The first thing to be noted is that **PREDICT** does not follow from $\leq$-persistence: the quantifier which is true in all situations which are total with respect to A, B and undefined otherwise is $\leq$-persistent, but does not observe **PREDICT**. This quantifier is defined by the following formula of $\mathcal{L}_\otimes$:

(17) $\forall x((Ax \vee \neg Ax) \wedge (Bx \vee \neg Bx)) \vee *$.

Note, by the way, that the quantifier of (17) does satisfy **VS–CONS**.

Also, $\leq$-**PERSIST** does not follow from **PREDICT**, witness the quantifier that is true in all partial situations (with respect to either A or B) and false in all situations which are total with respect to these predicates. This quantifier observes **PREDICT** without satisfying $\leq$-**PERSIST**. It is defined by the following formula of $\mathcal{L}_{\sim}$:

(18) $\exists x((\sim Ax \wedge \sim\neg Ax) \vee (\sim Bx \wedge \sim\neg Bx))$.

Theorem 8 *If all quantifier symbols in $\mathcal{Q}$ are interpreted as quantifiers observing* **PREDICT**, *then any formula φ of $\mathcal{L}_{\mathcal{Q},\sim}$ is predictive.*

Proof. Induction on the complexity of φ. Atomic formulae of $\mathcal{L}_{\mathcal{Q},\sim}$ are surely predictive. To see that predictiveness is preserved for φ of the form $\neg\psi$, $\sim\psi$, $\psi\wedge\chi$, $\psi\vee\chi$, check the truth tables in the manner explained above.

Suppose φ has the form $\forall v\psi$ and assume ψ is predictive. Assume there is an s, g with $[\![\psi]\!]_{s,g} = *$, and there is an $s' > s$ with $[\![\forall\psi]\!]_{s',g} = 1(0)$. For definiteness, let us assume $[\![\forall\psi]\!]_{s',g} = 1$. Let E be the universe of s and s'. We have to show that there is a $w > s$ with $[\![\forall v\psi]\!]_{w,g} = 0$. Let D be $\{d \in E \mid [\![\psi]\!]_{s,g(v|d)} = *\}$. By the semantic clause for $\forall$, this set is nonempty. If $d \in D$, then by the fact that $[\![\forall\psi]\!]_{s',g} = 1$, we have $[\![\psi]\!]_{s',g(v|d)} = 1$, so by the predictiveness of ψ there is some situation u for which $[\![\psi]\!]_{u,g(v|d)} = 0$. Let U be the set of all such u and let u_d be the situation $\langle E, \bigcap_{u\in U} I_u^+, \bigcap_{u\in U} I_u^-\rangle$. Let w be the situation $\langle E, \bigcup_{d\in D} I_{u_d}^+, \bigcup_{d\in D} I_{u_d}^-\rangle$. It is then easy to check that $w > s$ and $[\![\forall v\psi]\!]_{w,g} = 0$, so φ is predictive.

Finally, the binary quantifier case. Suppose φ has the form $Qv(\psi,\chi)$, where ψ and χ are predictive, and Q is interpreted as a quantifier $\mathbf{Q}$ which observes **PREDICT**. Assume there is some situation s and assignment g where $[\![Qv(\psi,\chi)]\!]_{s,g} = *$. Then by the semantic clause for binary quantifiers, $\mathbf{Q}(A^+, A^*, B^+, B^*) = *$ (here the sets A^+, A^*, B^+, B^* depend on the parameters s, g). Assume that for some $u > s$, $[\![Qv(\varphi,\psi)]\!]_{u,g} = 1(0)$. Then by the semantic clause for binary quantifiers, $\mathbf{Q}(C^+, C^*, D^+, D^*) = 1(0)$ (here the sets C^+, C^*, D^+, D^* depend on the parameters u, g). For the sake of definiteness we assume $\mathbf{Q}(C^+, C^*, D^+, D^*) = 1$. It is not difficult to see that the sets are related as follows: there are $X \subseteq Y \subseteq A^*$ and $U \subseteq V \subseteq B^*$ such that $C^+ = A^+ \cup X$, $C^* = A^* - Y$, $D^+ = B^+ \cup U$, $D^* = B^* - V$. Thus, $\mathbf{Q}(C^+, C^*, D^+, D^*) = 1$ can be rewritten as (19).

(19) $\mathbf{Q}(A^+ \cup X, A^* - Y, B^+ \cup U, B^* - V) = 1$.

Now consider the value of $\mathbf{Q}(A^+ \cup X, A^* - X, B^+ \cup U, B^* - U)$. There are three possibilities.

Suppose the value is 0. This gives a $w > s$ with $[\![\varphi]\!]_{w,g} = 0$, so φ is predictive, and we are done.

Suppose the value is 1. Then use the **PREDICT** property of $\mathbf{Q}$ to derive that $Q(A^+, A^* - X, B^+, B^* - U) = 0$. So again we have a $w > s$ with $[\![\varphi]\!]_{w,g} = 0$, which shows φ is predictive.

Finally, suppose the value is $*$. Using Z for $Y - X$ and W for $V - U$, we see that (19) can be rewritten as (20).

(20) $\mathbf{Q}(A^+ \cup X, (A^* - X) - Z, B^+ \cup U, (B^* - U) - W) = 1.$

Now use the **PREDICT** property of **Q** in the other direction to see that (21).

(21) $\mathbf{Q}(A^+ \cup Y, A^* - X, B^+ \cup V, B^* - U) = 0.$

So in this case as well there is a $w > s$ with $[\![\varphi]\!]_{w,g} = 0$, and again φ is predictive. ∎

It follows from the theorem that all quantifiers built from predictive quantifiers in the language $\mathcal{L}_{\mathcal{Q},\sim}$ will themselves be predictive. Finally, we have the usual corollary:

Corollary 9 *All quantifiers definable in* $\mathcal{L}_{\sim}$ *satisfy* **PREDICT**.

5 Supervaluation Quantifiers

Supervaluation quantifiers are the ordinary generalized quantifiers, transposed in a three valued setting via a 'supervaluation' definition. There is good reason of being interested in supervaluation quantifiers, for by virtue of their respectable origins they can be expected to be well-behaved. By studying them we can find out more about how well-established intuitions for total quantifiers generalize in the present partial setting.

Supervaluation quantifiers are defined in terms of quantifiers for total situations. However, these quantifiers for total situations need not themselves be two valued. We therefore represent a binary quantifier on total situations over a domain E as a pair of functions $\dot{\mathbf{Q}}^+, \dot{\mathbf{Q}}^\circ$, both in $(\mathcal{P}E)^2 \to \{0, 1\}$, and satisfying the condition that $\dot{\mathbf{Q}}^+(X, Y) = 1$ implies $\dot{\mathbf{Q}}^\circ(X, Y) = 1$.

Definition 13 A quantifier interpretation **Q** is a **supervaluation** interpretation if its truth and falsity conditions are given in terms of binary quantifiers $\dot{\mathbf{Q}}$ for total situations, as follows:

- $\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$ if for all X, Y with $A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$, it holds that $\dot{\mathbf{Q}}^+_E(X, Y) = 1$.
- $\mathbf{Q}^\circ_E(A^+, A^*, B^+, B^*) = 0$ if for all X, Y with $A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$, it holds that $\dot{\mathbf{Q}}^\circ_E(X, Y) = 0$.

It is instructive to look at some examples of supervaluation quantifiers. Consider the supervaluation quantifier **all** based on the total binary quantifier $X \subseteq Y$. Its truth conditions are given by:

$\mathbf{all}^+(A^+, A^*, B^+, B^*) = 1$ iff for all X, Y with $A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$: $X \subseteq Y$.

Equivalently:

$\mathbf{all}^+(A^+, A^*, B^+, B^*) = 1$ iff $A^+ \cup A^* \subseteq B^+$.

Its falsity conditions are given by:

> $\mathbf{all}^{\circ}(A^{+}, A^{*}, B^{+}, B^{*}) = 0$ iff for all X, Y with $A^{+} \subseteq X \subseteq A^{+} \cup A^{*}$ and $B^{+} \subseteq Y \subseteq B^{+} \cup B^{*}$: $X \not\subseteq Y$.

Equivalently:

> $\mathbf{all}^{\circ}(A^{+}, A^{*}, B^{+}, B^{*}) = 0$ iff $A^{+} \not\subseteq B^{+} \cup B^{*}$.

The reader is invited to check that this quantifier is defined in $\mathcal{L}$ by the following formula.

(22) $\forall x(Ax \rightarrow Bx)$.

Next, consider the supervaluation quantifier **at least 2** based on the total binary quantifier $\#(X \cap Y) \geq 2$. Its truth conditions are given by:

> $\mathbf{atleast2}^{+}(A^{+}, A^{*}, B^{+}, B^{*}) = 1$ iff for all X, Y with $A^{+} \subseteq X \subseteq A^{+} \cup A^{*}$ and $B^{+} \subseteq Y \subseteq B^{+} \cup B^{*}$: $\#(X \cap Y) \geq 2$.

Equivalently:

> $\mathbf{atleast2}^{+}(A^{+}, A^{*}, B^{+}, B^{*}) = 1$ iff $\#(A^{+} \cap B^{+}) \geq 2$.

Its falsity conditions are given by:

> $\mathbf{atleast2}^{\circ}(A^{+}, A^{*}, B^{+}, B^{*}) = 0$ iff for all X, Y with $A^{+} \subseteq X \subseteq A^{+} \cup A^{*}$ and $B^{+} \subseteq Y \subseteq B^{+} \cup B^{*}$: $\#(X \cap Y) < 2$.

Equivalently:

> $\mathbf{atleast2}^{\circ}(A^{+}, A^{*}, B^{+}, B^{*}) = 0$ iff $\#((A^{+} \cup A^{*}) \cap (B^{+} \cup B^{*})) < 2$.

The reader is invited to check that the following $\mathcal{L}$ formula defines this quantifier:

(23) $\exists x \exists y((\neg x \doteq y) \wedge Ax \wedge Ay \wedge Bx \wedge By)$.

Consider supervaluation **most**, based on the total binary quantifier $\#(X \cap Y) > \#(X - Y)$. Its truth conditions are given by:

> $\mathbf{most}^{+}(A^{+}, A^{*}, B^{+}, B^{*}) = 1$ iff for all X, Y with $A^{+} \subseteq X \subseteq A^{+} \cup A^{*}$ and $B^{+} \subseteq Y \subseteq B^{+} \cup B^{*}$: $\#(X \cap Y) > \#(X - Y)$.

Equivalently:

> $\mathbf{most}^{+}(A^{+}, A^{*}, B^{+}, B^{*}) = 1$ iff $\#(A^{+} \cap B^{+}) \geq \#((A^{+} \cup A^{*}) - B^{+})$.

Its falsity conditions are given by:

> $\mathbf{most}^{\circ}(A^{+}, A^{*}, B^{+}, B^{*}) = 0$ iff for all X, Y with $A^{+} \subseteq X \subseteq A^{+} \cup A^{*}$ and $B^{+} \subseteq Y \subseteq B^{+} \cup B^{*}$: $\#(X \cap Y) \leq \#(X - Y)$.

Equivalently:

> $\mathbf{most}^{\circ}(A^{+}, A^{*}, B^{+}, B^{*}) = 0$ iff $\#((A^{+} \cup A^{*}) \cap (B^{+} \cup B^{*})) \leq \#(A^{+} - (B^{+} \cup B^{*}))$.

In order to express this in our partial language we will have to interpret some binary quantifier symbol M as the quantifier **most** with precisely the behaviour described above. Under these conditions, the formula $Mx(Ax, Bx)$ of $\mathcal{L}_M$ defines this quantifier.

We conclude with an example of an open quantifier, the supervaluation quantifier **the 2** based on the total quantifier $X \subseteq Y$ with presupposition $\#(X) = 2$. Its truth conditions are given by:

> $\mathbf{the2}^+(A^+, A^*, B^+, B^*) = 1$ iff for all X, Y with $A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$: $\#(X) = 2$ and $X \subseteq Y$.

Equivalently:

> $\mathbf{the2}^+(A^+, A^*, B^+, B^*) = 1$ iff $\#(A^+) = 2$, $A^* = \emptyset$, and $A^+ \subseteq B^+$.

Its falsity conditions are given by:

> $\mathbf{the2}^\circ(A^+, A^*, B^+, B^*) = 0$ iff for all X, Y with $A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$: $\#(X) = 2$ and $X \not\subseteq Y$.

Equivalently:

> $\mathbf{the2}^\circ(A^+, A^*, B^+, B^*) = 0$ iff $\#(A^+) = 2$, $A^* = \emptyset$, and $A^+ \not\subseteq B^+ \cup B^*$.

The reader is invited to check that this quantifier is defined in $\mathcal{L}_\otimes$ by the following formula.

$$\begin{array}{ll} & \exists x \exists y \forall z(Az \leftrightarrow (z \doteq x \vee z \doteq y)) \wedge \forall x(Ax \rightarrow Bx) \\ (24) & \otimes \\ & \exists x \exists y \forall z(Az \leftrightarrow (z \doteq x \vee z \doteq y)) \wedge \neg\forall x(Ax \rightarrow Bx). \end{array}$$

To demonstrate that supervaluation quantifiers in a partial setting provide a litmus test for extensions of notions defined for quantifiers in a total setting, we will now look at the generalisations of the notions of extension, isomorphy and conservativity. First we show that the notions of extension and isomorphy that were given in Section 2 are indeed the correct generalizations from the total to the partial case for the supervaluation quantifiers.

Proposition 10 *The supervaluation quantifier* $\mathbf{Q}$ *based on the total quantifier* $\dot{\mathbf{Q}}$ *satisfies* **EXT** *iff* $\dot{\mathbf{Q}}$ *satisfies* **EXT–T**.

Proof. Suppose $\mathbf{Q}$ satisfies **EXT**. Then:

> $\dot{\mathbf{Q}}^+_E(X, Y) = 1$
> iff (definition of $\mathbf{Q}$) $\mathbf{Q}^+_E(X, \emptyset, Y, \emptyset) = 1$
> iff ($\mathbf{Q}_E$ satisfies **EXT**) $\mathbf{Q}^+_{E'}(X, \emptyset, Y, \emptyset) = 1$ for all $E' \subseteq X \cup Y$
> iff (definition of $\mathbf{Q}$) $\dot{\mathbf{Q}}^+_{E'}(X, Y) = 1$ for all $E' \subseteq X \cup Y$.

Similarly: $\dot{\mathbf{Q}}^\circ_E(X, Y) = 0$ iff $\dot{\mathbf{Q}}^\circ_{E'}(X, Y) = 0$ for all $E' \subseteq X \cup Y$.

Conversely, suppose $\dot{\mathbf{Q}}$ satisfies **EXT-T**. Then:

$\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$
iff (definition of $\mathbf{Q}$) $\dot{\mathbf{Q}}^+_E(X, Y) = 1$ for all X, Y with
$A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$
iff ($\dot{\mathbf{Q}}$ satisfies **EXT-T**) $\dot{\mathbf{Q}}^+_{E'}(X, Y) = 1$ for all X, Y with
$A^+ \subseteq X \subseteq A^+ \cup A^* \subseteq E'$ and $B^+ \subseteq Y \subseteq B^+ \cup B^* \subseteq E'$
iff (set theoretic reasoning) $\dot{\mathbf{Q}}^+_{E'}(X, Y) = 1$ for all X, Y with
$A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$
and all E' with $E' \supseteq A^+ \cup A^* \cup B^+ \cup B^*$
iff (definition of supervaluation quantifiers) $\mathbf{Q}^+_{E'}(A^+, A^*, B^+, B^*) = 1$
for all E' with $E' \supseteq A^+ \cup A^* \cup B^+ \cup B^*$.

Similarly: $\mathbf{Q}^\circ_E(A^+, A^*, B^+, B^*) = 0$ iff $\mathbf{Q}^\circ_{E'}(A^+, A^*, B^+, B^*) = 0$
for all E' with $E' \supseteq A^+ \cup A^* \cup B^+ \cup B^*$. ∎

Proposition 11 *The supervaluation quantifier* $\mathbf{Q}$ *based on the total quantifier* $\dot{\mathbf{Q}}$ *satisfies* **ISOM** *iff* $\dot{\mathbf{Q}}$ *satisfies* **ISOM–T**.

Proof. Assume $\mathbf{Q}_E$ satisfies **ISOM** and let f be a bijection of E to E'. Then:

$\dot{\mathbf{Q}}^+_E(X, Y) = 1$
iff (definition of $\mathbf{Q}$) $\mathbf{Q}^+_E(X, \emptyset, Y, \emptyset) = 1$
iff ($\mathbf{Q}$ satisfies **ISOM**) $\mathbf{Q}^+_{E'}(f[X], \emptyset, f[Y], \emptyset) = 1$
iff (definition of $\mathbf{Q}$) $\dot{\mathbf{Q}}^+_{E'}(f[X], f[Y]) = 1$for all $E' \subseteq X \cup Y$.

Similarly: $\dot{\mathbf{Q}}^\circ_E(X, Y) = 0$ iff $\dot{\mathbf{Q}}^\circ_E(f[X], f[Y]) = 0$.

Conversely, assume $\dot{\mathbf{Q}}$ satisfies **ISOM–T**, and let f be a bijection of E to E'. Then:

$\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$
iff (definition of $\mathbf{Q}$) $\dot{\mathbf{Q}}^+_E(X, Y) = 1$ for all X, Y with
$A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$
iff (**ISOM-T** of $\dot{\mathbf{Q}}$) $\dot{\mathbf{Q}}^+_{E'}(f[X], f[Y]) = 1$ for all $f[X], f[Y]$ with
$f[A^+] \subseteq f[X] \subseteq f[A^+ \cup A^*]$ and $f[B^+] \subseteq f[Y] \subseteq f[B^+ \cup B^*]$
iff (definition of $\mathbf{Q}$) $\mathbf{Q}^+_{E'}(f[A^+], f[A^*], f[B^+], f[B^*]) = 1$.

Similarly: $\mathbf{Q}^\circ_E(A^+, A^*, B^+, B^*) = 0$ iff $\mathbf{Q}^\circ_{E'}(f[A^+], f[A^*], f[B^+], f[B^*]) = 0$. ∎

The following proposition tells us which of the varieties of conservativity that were distinguished in Section 2 is the proper generalization for the case of supervaluation quantifiers.

Proposition 12 *The supervaluation quantifier* $\mathbf{Q}$ *based on the total quantifier* $\dot{\mathbf{Q}}$ *satisfies* **W–CONS** *iff* $\dot{\mathbf{Q}}$ *satisfies* **CONS–T**.

Proof. Assume $\dot{\mathbf{Q}}$ satisfies **CONS–T**. Then:

$\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$
iff (Q supervaluation quantifier based on $\dot{\mathbf{Q}}$) $\dot{\mathbf{Q}}^+_E(X, Y) = 1$ for all X, Y with
$A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$
iff (**CONS–T** of $\dot{\mathbf{Q}}$) $\dot{\mathbf{Q}}^+_E(X, X \cap Y) = 1$ for all X, Y with
$A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$
iff (set theoretic reasoning) $\dot{\mathbf{Q}}^+_E(X, X \cap Y) = 1$ for all X, Y with
$A^+ \subseteq X \subseteq A^+ \cup A^*$ and $A^+ \cap B^+ \subseteq X \cap Y \subseteq (B^+ \cap (A^+ \cup A^*)) \cup (B^* \cap (A^+ \cup A^*))$
iff (Q supervaluation quantifier based on $\dot{\mathbf{Q}}$) $\mathbf{Q}^+_E(A^+, A^*, B^+ \cap (A^+ \cup A^*), B^* \cap (A^+ \cup A^*)) = 1$.

Similarly: $\mathbf{Q}^\circ_E(A^+, A^*, B^+, B^*) = 0$ iff $\mathbf{Q}^\circ_E(A^+, A^*, B^+ \cap (A^+ \cup A^*), B^* \cap (A^+ \cup A^*)) = 0$.

Conversely, assume $\mathbf{Q}$ is a supervaluation quantifier based on $\dot{\mathbf{Q}}$, and $\mathbf{Q}$ satisfies **W–CONS**. Then:

$\dot{\mathbf{Q}}^+_E(X, Y) = 1$
iff ($\mathbf{Q}$ supervaluation quantifier based on $\dot{\mathbf{Q}}$) $\mathbf{Q}^+_E(X, \emptyset, Y, \emptyset) = 1$
iff ($\mathbf{Q}$ satisfies **W–CONS**) $\mathbf{Q}^+_E(X, \emptyset, X \cap Y, \emptyset) = 1$
iff ($\mathbf{Q}$ supervaluation quantifier based on $\dot{\mathbf{Q}}$) $\dot{\mathbf{Q}}^+_E(X, X \cap Y) = 1$.

Similarly, $\dot{\mathbf{Q}}^\circ_E(X, Y) = 0$ iff $\dot{\mathbf{Q}}^\circ_E(X, X \cap Y) = 0$. ∎

Finally, there is a simple result about closedness.

Proposition 13 *The supervaluation quantifier* $\mathbf{Q}$ *based on the total quantifier* $\dot{\mathbf{Q}}$ *satisfies* **CL** *iff* $\dot{\mathbf{Q}}$ *satisfies* **CL–T**.

Proof. Immediate from the definitions. ∎

It is useful to define the supervaluation property for formulae in general:

Definition 14 A formula φ is a supervaluation formula if for any situation s and any assignment g for s:

$[\![\varphi]\!]_{s,g} = 1\ (0)$ iff for all total $t \geq s$, $[\![\varphi]\!]_{t,g} = 1\ (0)$.

Now the following useful theorem is easy to prove:

Theorem 14 *A formula* φ *is a supervaluation formula iff* φ *is both predictive and* $\leq$ *persistent.*

Proof. Assume φ is a supervaluation formula. Then it is easy to check that φ is both predictive and $\leq$ persistent.

Assume φ is not a supervaluation formula. Then for some situation s and assignment g for s, either of the following must be the case.

1. $[\![\varphi]\!]_{s,g} = 1(0)$ and for some total $t \geq s$, $[\![\varphi]\!]_{t,g} \neq 1\ (0)$.

2. $[\![\varphi]\!]_{s,g} = *$ and for all total $t \geq s$, $[\![\varphi]\!]_{t,g} = 1(0)$.

In case 1 φ is not $\leq$ persistent, in case 2 φ is not predictive. ■

We can immediately derive the following:

Theorem 15 $\mathbf{Q}$ *is a supervaluation quantifier iff* $\mathbf{Q}$ *observes both* $\leq$-**PERSIST** *and* **PREDICT**.

Theorem 16 *If all quantifier symbols in* $\mathcal{Q}$ *are interpreted as supervaluation quantifiers, then every* φ *in* $\mathcal{L}_\mathcal{Q}$ *is a supervaluation formula.*

Theorem 17 *If all quantifier symbols in* $\mathcal{Q}$ *are interpreted as closed supervaluation quantifiers, then every* φ *in* $\mathcal{L}_\mathcal{Q}$ *is a determinable supervaluation formula.*

Finally, the characterization of supervaluation quantifiers as quantifiers satisfying $\leq$ **PERSIST** and **PREDICT** makes it possible to rule out some of the conservativity notions that were distinguished in Section 3.

For a given universe E, T_E^1, T_E^0 and T_E^* are the quantifiers on E which are respectively always true, always false or always undefined on E. We call these quantifiers *trivial* on E. Also, we use T^1, T^0 and T^* for the quantifiers which are always true, always false or always undefined, on any universe. These are the *trivial* quantifiers. Note that a quantifier $\mathbf{Q}$ can be trivial on any universe without being identical to any of T^1, T^0 or T^* ($\mathbf{Q}$ might equal T_E^1 on E and $T_{E'}^*$ on E', say.)

Theorem 18 *The only supervaluation quantifiers satisfying* **VS-CONS** *are the quantifiers which are trivial on any universe* E.

Proof. Let E be a universe, and let s be the situation on E with $A^* = B^* = E$. Let $\mathbf{Q}$ be a supervaluation quantifier satisfying **VS-CONS**. If $\mathbf{Q}$ is false on s, then because of $\leq$ persistence, $\mathbf{Q}$ will always be false on E, so $\mathbf{Q} = T_E^0$, i.e.,$\mathbf{Q}$ is trivial on E. Similarly, if $\mathbf{Q}$ is true on s, $\mathbf{Q} = T_E^1$.

Assume $\mathbf{Q}$ has value $*$ on s. The one step transitions from $A^* \cap B^*$ in the directions B^+ and B^- cannot change this value because of **VS-CONS**. Suppose some one step transition from $A^* \cap B^+$ to $A^+ \cap B^+$ changes the value $*$ to 1 (0). Then by **PREDICT**, the transition from $A^* \cap B^+$ to $A^- \cap B^+$ changes the value $*$ to 0 (1), thus leading to a contradiction with **VS-CONS**. Thus, no one step transition from $A^* \cap B^+$ to $A^+ \cap B^+$ does change the value $*$. It follows from $\leq$ **PERSIST** that no one step transitions from $A^* \cap B^*$ to $A^+ \cap B^*$ and from $A^+ \cap B^*$ to $A^+ \cap B^+$ can change the value $*$. For information growth in the direction towards $A^+ \cap B^-$ the reasoning is similar. Thus, in this third case $\mathbf{Q} = T_E^*$. ■

To also rule out the constraint of strong conservativity **S-CONS** we need the property of *variety*, well known from standard generalized quantifier theory:

VAR–T If $A \neq \emptyset$ then there are $B, B' \subseteq E$ with $Q_E AB = 1$ and $Q_E AB' = 0$.

Here is the variant we need for the present partial setting.

VAR If $A^+ \neq \emptyset$ then there are $B^+, B'^+ \subseteq E$ with $\mathbf{Q}_E(A^+, \emptyset, B^+, \emptyset) = 1$ and $\mathbf{Q}_E(A^+, \emptyset, B'^+, \emptyset) = 0$.

The following proposition is immediate.

Proposition 19 *The supervaluation quantifier* $\mathbf{Q}$ *based on the total quantifier* $\dot{\mathbf{Q}}$ *satisfies* **VAR** *iff* $\dot{\mathbf{Q}}$ *satisfies* **VAR–T**.

Theorem 20 *If a supervaluation quantifier satisfies* **VAR** *then it will not satisfy* **S–CONS**.

Let $\mathbf{Q}$ be a supervaluation quantifier satisfying **S-CONS**. We show that $\mathbf{Q}$ will not satisfy **VAR**. Let E be a universe, and let s be the situation on E with $A^* = B^* = E$. If $\mathbf{Q}$ is false on s, then because of $\leq$ persistence, $\mathbf{Q}$ will always be false on E, so $\mathbf{Q} = T^0_E$, i.e.,$\mathbf{Q}$ is trivial on E, and so $\mathbf{Q}$ does not satisfy **VAR**. Similarly, if $\mathbf{Q}$ is true on s, $\mathbf{Q} = T^1_E$, and $\mathbf{Q}$ does not satisfy **VAR**.

Assume $\mathbf{Q}$ has value $*$ on s. The one step transitions from $A^* \cap B^*$ in the directions B^+ and B^- cannot change this value because of **S-CONS**. Suppose that some one step transition from $A^* \cap B^+$ to $A^+ \cap B^+$ changes the value $*$ to 1 (0). Then by **PREDICT**, in that situation the transition from $A^* \cap B^+$ to $A^- \cap B^+$ changes the value $*$ to 0 (1). But then by **S-CONS**, the transition from $A^* \cap B^-$ to $A^- \cap B^-$ also changes the value $*$ to 0 (1), so, again by **PREDICT**, the transition from $A^* \cap B^-$ to $A^+ \cap B^-$ will change the value $*$ to 1 (0). It follows from $\leq$ **PERSIST** that in the same situation the two step transition from $A^* \cap B^*$ to $A^+ \cap B^+$ via $A^+ \cap B^*$ and the two step transition from $A^* \cap B^*$ to $A^+ \cap B^-$ via $A^+ \cap B^*$ will also change the value $*$ to 1 (0). This shows that in this case too, $\mathbf{Q}$ does not satisfy **VAR**. *Proof.* end

6 Domain Persistence and Information Persistence

The inclusion relation $\subseteq$ between models is defined in the usual way:

Definition 15 $s \subseteq u$ if $E_s \subseteq E_u$ and for all n, for all $P \in \mathrm{P}^n$;

- $I^+_u(P) \cap E^n_s = I^+_s(P)$.
- $I^-_u(P) \cap E^n_s = I^-_s(P)$.

The definition engenders the following notion of domain persistence.

Definition 16 A sentence φ is $\subseteq$ persistent if for all $s \subseteq u$ the following holds: $[\![\varphi]\!]_s = 1$ implies $[\![\varphi]\!]_u = 1$ and $[\![\varphi]\!]_u = 0$ implies $[\![\varphi]\!]_s = 0$.

Domain antipersistence is the converse of domain persistence. Truth domain (anti)persistence ($\subseteq^+$) and falsity domain (anti)persistence ($\subseteq^-$) are the two halves of domain (anti)persistence. There are corresponding properties for quantifier interpretations.

$\subseteq$ **PERSIST** If $X^+, X^-, Y^+, Y^- \subseteq E'$, then:

- $\mathbf{Q}_E(A^+, A^*, B^+, B^*) = 1$ implies
  $\mathbf{Q}_{E'}(A^+ \cup X^+, A^* \cup X^*, B^+ \cup Y^+, B^* \cup Y^*) = 1$,
- $\mathbf{Q}_{E'}(A^+ \cup X^+, A^* \cup X^*, B^+ \cup Y^+, B^* \cup Y^*) = 0$ implies
  $\mathbf{Q}_E(A^+, A^*, B^+, B^*) = 0$.

It is sometimes useful to exclude cases where a formula φ is $\leq$ persistent for the trivial reason that it is always undefined in partial situations. This can be accomplished by means of the following definition.

Definition 17 A formula φ of $\mathcal{L}_{\mathcal{Q},\sim,\otimes}$ is *informative* if there is a partial situation s and an assignment g for s such that $[\![\varphi]\!]_s = 0$ or $[\![\varphi]\!]_s = 1$.

Again, there is a corresponding property for quantifier interpretations.

INFORM There are X^*, Y^* with $X^* \cup Y^* \neq \emptyset$, such that

$$\mathbf{Q}_E(X, X^*, Y, Y^*) = 1 \text{ or } \mathbf{Q}_E(X, X^*, Y, Y^*) = 0.$$

An example of a quantifier which is not informative is the supervaluation quantifier **an even number of**. It is easily seen that $\leq$ persistence is only an interesting notion for informative quantifiers: quantifiers that are not informative are always $\leq$–persistent. The following proposition is immediate.

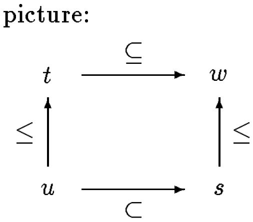
Proposition 21 *If* $\mathbf{Q}$ *satisfies* **CL** *and* **PREDICT***, then* $\mathbf{Q}$ *satisfies* **INFORM***.*

In what follows we will sometimes abbreviate $\mathbf{Q}_{E_s}(A_s^+, A_s^*, B_s^+, B_s^*) = 1$ as $s \models \mathbf{Q}$ and $\mathbf{Q}_{E_s}(A_s^+, A_s^*, B_s^+, B_s^*) = 0$ as $s \dashv \mathbf{Q}$.

Proposition 22 *For all closed supervaluation quantifiers* $\mathbf{Q}$*:*

$$\mathbf{Q} \textit{ is truth } \subseteq \textit{ (anti)persistent iff } \mathbf{Q} \textit{ is falsity } \subseteq \textit{ (anti)persistent.}$$

Proof. Assume $\mathbf{Q}$ is a closed and truth $\subseteq$–persistent supervaluation quantifier, and consider a situation s with $s \dashv \mathbf{Q}$. Assume for some $u \subseteq s$, $u \not\dashv \mathbf{Q}$. Now $u \models \mathbf{Q}$ would contradict the truth $\subseteq$–persistence of $\mathbf{Q}$, so we have $u \not\models \mathbf{Q}$ and $u \not\dashv \mathbf{Q}$. Because $\mathbf{Q}$ is a closed supervaluation quantifier there is a total $t \geq u$ with $t \models Q$. Let w be the result of adding to t what must be added to u to get s. We then have $t \subseteq w$ and $s \leq w$, i.e., we have the situation in the following picture:



Because **Q** is truth $\subseteq$–persistent, $w \models \mathbf{Q}$. But this contradicts the fact that **Q** is a supervaluation quantifier: every situation $\geq s$ must falsify **Q**. This proves that $u \dashv \mathbf{Q}$ for all $u \subseteq s$, i.e., **Q** is falsity $\subseteq$–persistent. The reasoning for the converse and for the case of antipersistence is similar. ∎

Proposition 23 *Q is trivial iff Q is* $\subseteq$*–persistent and* $\subseteq$*–antipersistent.*

Proof. It is easily checked that the trivial quantifiers T^1, T^0 and T^* (always true, always false or always undefined, on any universe) are both $\subseteq$–persistent and $\subseteq$–antipersistent.

Conversely, assume **Q** is both $\subseteq$–persistent and $\subseteq$–antipersistent. We show that if there is some s with $s \models \mathbf{Q}$ then $u \models \mathbf{Q}$ for any situation u. Take an arbitrary situation u and let u' be the result of deleting the part of u outside $A^+_u \cup A^*_u$. Transform s into s' by adding (deleting) elements to (from) $A^+ \cap B^+$, $A^+ \cap B^*$, $A^+ \cap B^-$, $A^* \cap B^+$, $A^* \cap B^*$ and $A^* \cap B^-$, until there is a one–one map from the domain of s' onto that of u'. Now $s' \models Q$, by $\subseteq$–persistence and $\subseteq$–antipersistence of Q. By **ISOM**, $u' \models Q$, and by conservativity it follows that $u \models Q$. Similarly, if there is some s with $s \dashv Q$ then Q is false in any situation. ∎

The precise connection between $\leq$–persistence and being a supervaluation quantifier was given in Theorem 15. We have seen some examples already of quantifiers lacking $\leq$–persistence. Example (25) gives another such case:

(25) All entities known to be A are B.

The quantifier in (25) might be true in some situation s but become false at a later stage of knowledge acquisition, in a situation s' with $s \leq s'$. The non $\leq$–persistent quantifier of this example is the result of 'relativising' a supervaluation quantifier to local knowledge. In fact, this localising process can take place in an 'existential' and a 'universal' sense, in both arguments. (25) gives a 'universal' local for **all**.

There also is a kind of converse to localisation, namely globalisation: adding the information that all is known about the first and/or second argument of a quantifier. A quantified formula $Qv(\varphi, \psi)$ is globalized in its first argument by taking the following interjunction:

(26) $(Qv(\varphi, \psi) \wedge \forall v(\varphi \vee \neg\varphi)) \otimes (Qv(\varphi, \psi) \vee \neg\forall v(\varphi \vee \neg\varphi))$.

7 Monotonicity

To extend the monotonicity properties from the total case to the case of partial information, verification and falsification must again be considered separately. For the verification part of **MON** in the left argument, ↑-direction, we want to say that if the quantifier holds for given A^+, A^*, then it will continue to hold in a situation in which things which were A^- or A^* have changed into things which are A^+. Formally: if $X \subseteq E$, and the quantifier holds for A^+, A^*, then it must also hold for $A^+ \cup X, A^* - X$.

Conversely, if the quantifier yields falsity for given A^+, A^*, and things which were A^+ or A^* have changed into things which are A^-, then the quantifier must again yield falsity. Here are the two parts of upward left monotonicity:

$\uparrow$**MON**$^+$ If $X \subseteq E$ and $\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$ then

$$\mathbf{Q}^+_E(A^+ \cup X, A^* - X, B^+, B^*) = 1.$$

$\uparrow$**MON**$^\circ$ If $X \subseteq E$ and $\mathbf{Q}^\circ_E(A^+, A^*, B^+, B^*) = 0$ then

$$\mathbf{Q}^\circ_E(A^+ - X, A^* - X, B^+, B^*) = 0.$$

Note that there is a clear difference between upward monotonicity and $\leq$–persistence: all supervaluation quantifiers are $\leq$–persistent, but not all supervaluation quantifiers are upward monotone in their first arguments.

The formulation of left monotonicity in the downward direction is completely analogous:

$\downarrow$**MON**$^+$ If $X \subseteq E$ and $\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$ then

$$\mathbf{Q}^+_E(A^+ - X, A^* - X, B^+, B^*) = 1.$$

$\downarrow$**MON**$^\circ$ If $X \subseteq E$ and $\mathbf{Q}^\circ_E(A^+, A^*, B^+, B^*) = 0$ then

$$\mathbf{Q}^\circ_E(A^+ \cup X, A^* - X, B^+, B^*) = 0.$$

Similarly, principles **MON**$\uparrow$ and **MON**$\downarrow$ for monotonicity in the second argument can be formulated.

The first thing we must show is that these principles are the correct generalizations of the monotonicity principles for the total case. As before, we use the supervaluation quantifiers as a litmus test.

Proposition 24 *The supervaluation quantifier* $\mathbf{Q}$ *based on the total quantifier* $\dot{\mathbf{Q}}$ *satisfies* $\uparrow$**MON** *(*$\downarrow$**MON***,* **MON**$\uparrow$*,* **MON**$\downarrow$*) iff* $\dot{\mathbf{Q}}$ *satisfies* $\uparrow$**MON-T** *(*$\downarrow$**MON-T***,* **MON**$\uparrow$**-T***,* **MON**$\downarrow$**-T***).*

Proof. We will just prove the case of $\uparrow$**MON**. Suppose $\mathbf{Q}$ satisfies $\uparrow$**MON**. Then:

$\dot{\mathbf{Q}}^+_E(X, Y) = 1$
iff (definition of $\mathbf{Q}_E$) $\mathbf{Q}^+_E(X, \emptyset, Y, \emptyset) = 1$
only if ($\mathbf{Q}_E$ satisfies $\uparrow$**MON**) $\mathbf{Q}^+_E(X \cup Z, \emptyset, Y, \emptyset) = 1$ for all $Z \subseteq E$
iff (definition of $\mathbf{Q}$) $\dot{\mathbf{Q}}^+_E(X', Y) = 1$ for all X' with $X \subseteq X' \subseteq E$.

Similarly: if $\dot{\mathbf{Q}}^\circ_E(X, Y) = 0$ then $\dot{\mathbf{Q}}^\circ_{E'}(X', Y) = 0$ for all $X' \subseteq X$.

Conversely, suppose $\dot{\mathbf{Q}}$ satisfies $\uparrow$**MON-T**. Then:

$\mathbf{Q}^+_E(A^+, A^*, B^+, B^*) = 1$
iff (definition of $\mathbf{Q}$) $\dot{\mathbf{Q}}^+_E(X, Y) = 1$ for all X, Y with $A^+ \subseteq X \subseteq A^+ \cup A^*$ and $B^+ \subseteq Y \subseteq B^+ \cup B^*$
only if ($\dot{\mathbf{Q}}$ satisfies $\uparrow$**MON-T**) $\dot{\mathbf{Q}}^+_{E'}(X', Y) = 1$ for all X' with

$X \subseteq X' \subseteq E$

and all X, Y with $A^+ \subseteq X \subseteq E'$ and $B^+ \subseteq Y \subseteq E'$

iff (definition of supervaluation quantifiers) $\mathbf{Q}_E^{+}(A^+ \cup Z, A^* - Z, B^+, B^*) = 1$.

Similarly: if $\mathbf{Q}_E^{\circ}(A^+, A^*, B^+, B*) = 0$ then $\mathbf{Q}_E^{\circ}(A^+ - Z, A^* - Z, B^+, B^*) = 0$. ∎

Next, we chart the connection between monotonicity and persistence properties. The first connection is immediate.

Proposition 25

1. *If* $\mathbf{Q}$ *is* $\uparrow$**MON** *then* $\mathbf{Q}$ *is truth* $\leq$ *persistent and falsity* $\leq$ *antipersistent in its first argument.*
2. *If* $\mathbf{Q}$ *is* $\downarrow$**MON** *then* $\mathbf{Q}$ *is truth* $\leq$ *antipersistent and falsity* $\leq$ *persistent in its first argument.*
3. *If* $\mathbf{Q}$ *is* **MON**$\uparrow$ *then* $\mathbf{Q}$ *is truth* $\leq$ *persistent and falsity* $\leq$ *antipersistent in its second argument.*
4. *If* $\mathbf{Q}$ *is* **MON**$\downarrow$ *then* $\mathbf{Q}$ *is truth* $\leq$ *antipersistent and falsity* $\leq$ *persistent in its second argument.*

Proposition 26

1. *Quantifiers which are* $\subseteq$*–persistent and* $\leq$*–persistent are* $\uparrow$**MON**.
2. *Quantifiers which are* $\subseteq$*–antipersistent and* $\leq$*–persistent are* $\downarrow$**MON**.

Proof. 1. Suppose $\mathbf{Q}$ is $\subseteq$–persistent and $\leq$–persistent. Because of

$$\langle E, A^+, A^-, B^+, B^- \rangle \subseteq \langle E \cup X, A^+ \cup X, A^-, B^+, B^- \cup X \rangle.$$

it follows from $\subseteq$ persistence of $\mathbf{Q}$ that

$$\mathbf{Q}(A^+, A^*, B^+, B^*) = 1 \Rightarrow \mathbf{Q}(A^+ \cup X, A^*, B^+, B^*) = 1.$$

Because of $\leq$ persistence of $\mathbf{Q}$:

$$\mathbf{Q}(A^+ \cup X, A^*, B^+, B^*) = 1 \Rightarrow \mathbf{Q}(A^+ \cup X, A^* - X, B^+, B^*) = 1.$$

This establishes the positive part of the $\uparrow$**MON** property for $\mathbf{Q}$.

Because of

$$\langle E - X, A^+ - X, A^-, B^+, B^- - X \rangle \subseteq \langle E, A^+, A^-, B^+, B^- \rangle,$$

it follows from $\subseteq$ persistence of $\mathbf{Q}$ that:

$$\mathbf{Q}(A^+, A^*, B^+, B^*) = 0 \Rightarrow \mathbf{Q}(A^+ - X, A^*, B^+, B^*) = 0.$$

Because of $\leq$ persistence of $\mathbf{Q}$:

$$\mathbf{Q}(A^+ - X, A^*, B^+, B^*) = 0 \Rightarrow \mathbf{Q}(A^+ - X, A^* - X, B^+, B^*) = 0.$$

This establishes the negative part of the $\uparrow$**MON** property for $\mathbf{Q}$. It follows that $\mathbf{Q}$ is $\uparrow$**MON**. The reasoning for 26.2 is similar. ∎

Proposition 27

1. $\uparrow$**MON** *and* $\leq$*–persistent quantifiers are* $\subseteq$*–persistent.*
2. $\downarrow$**MON** *and* $\leq$*–persistent quantifiers are* $\subseteq$*–antipersistent.*

Proof. We only prove 1, the proof of 2 being analogous. Assume $\mathbf{Q}$ is $\uparrow$**MON** and $\leq$–persistent, but not not $\subseteq$–persistent. Then there are s, u with $s \subseteq u$ and either $\mathbf{Q}_{E_s}(A_s^+, A_s^*, B_s^+, B_s^*) = 1$ and $\mathbf{Q}_{E_u}(A_u^+, A_u^*, B_u^+, B_u^*) \neq 1$, or $\mathbf{Q}_{E_u}(A_u^+, A_u^*, B_u^+, B_u^*) = 0$ and $\mathbf{Q}_{E_s}(A_s^+, A_s^*, B_s^+, B_s^*) \neq 0$. Suppose the former is the case. Let u' be the situation $\langle E_u, A_u^+, A_s^-, B_s^+, B_s^- \rangle$. Then $u' \leq u$, and because $\mathbf{Q}$ is $\leq$–persistent, (27) holds.

(27) $\mathbf{Q}_{E_u}(A_u^+, A_s^*, B_s^+, E_u - (B_s^+ \cup B_s^-)) \neq 1.$

Put $X = A_u^+ - A_s^+$. Then (27) can be rewritten as (28).

(28) $\mathbf{Q}_{E_s \cup X}(A_s^+ \cup X, A_s^*, B_s^+, B_s^* \cup X) \neq 1.$

It follows from $\mathbf{Q}_{E_s}(A_s^+, A_s^*, B_s^+, B_s^*) = 1$ and **EXT**, **W-CONS** that $\mathbf{Q}_{E_s \cup X}(A_s^+, A_s^*, B_s^+, B_s^* \cup X) = 1$. By $\uparrow$**MON** of $\mathbf{Q}$, it follows from this that $\mathbf{Q}_{E_s \cup X}(A_s^+ \cup X, A_s^*, B_s^+, B_s^* \cup X) = 1$, and contradiction with (28).

The assumption of $\mathbf{Q}_{E_u}(A_u^+, A_u^*, B_u^+, B_u^*) = 0$ and $\mathbf{Q}_{E_s}(A_s^+, A_s^*, B_s^+, B_s^*) \neq 0$ leads to a contradiction with $\uparrow$**MON** in a similar way. ∎

The next proposition combines the previous two.

Proposition 28 *For all* $\leq$ *persistent* $\mathbf{Q}$ *(and* a fortiori *for all supervaluation quantifiers* $\mathbf{Q}$*):*

1. $\mathbf{Q}$ *is* $\uparrow$**MON** *iff* $\mathbf{Q}$ *is* $\subseteq$*–persistent.*
2. $\mathbf{Q}$ *is* $\downarrow$**MON** *iff* $\mathbf{Q}$ *is* $\subseteq$*–antipersistent.*

The proposition is relevant for the semantics of perception reports.

(29) I saw John prepare a sandwich.

Perception verbs such as 'see' in (29) can be interpreted as relations R between individuals (perceivers) and situations (perceived scenes) for which principle **S–INCL** holds.

S–INCL For all R denoting relations between perceivers p and perceived situations s: if pRs holds in situation s' then $s \subseteq s'$.

In other words: if in a given situation some scene is perceived by someone in that situation, then the scene is included in the situation.

The fact that proper names, viewed as properties of properties, are $\subseteq$ persistent explains the entailment relation between (29) and (30).

(30) John prepared a sandwich.

Similarly, Proposition 28 explains the entailment between (31) (from a Dutch children's song) and (32).

(31) I saw two bears prepare a sandwich.

(32) Two bears prepared a sandwich.

Pace Alice's White King, (34) does not follow from (33). Again, this non-entailment is explained by Proposition 28.

(33) I see nobody on the road.

(34) Nobody is on the road.

In case the reader wonders why the reasoning from Proposition 26 cannot be used to establish a connection between $\subseteq$-persistence and monotonicity in the second argument, the answer is that in this case the connection is spoilt by (weak) conservativity. If $X \subseteq A^-$, then $(B^+ \cup X) \cap (A^+ \cup A^*) = B^+$, so adding a set X of individuals from A^- to B^+ does not change the truth or falsity conditions of a (weakly) conservative quantifier.

There is a nice connection between monotonicity properties and a property attesting to the possibility of knowledge acquisition in the face of basic ignorance. We can ask ourselves which quantifiers remain true or false when individuals are added to the domain that are in the gap extensions of both the first and the second quantifier argument. Here is a formal version of this requirement.

IGNOR (in terms of A^+, A^*, B^+, B^*)

$$Q(A^+, A^*, B^+, B^*) = 1\ (0)\ \textit{and}\ X \cap A^+ \cap B^+ = \emptyset \Rightarrow$$
$$Q(A^+, A^* \cup X, B^+, B^* \cup X) = 1\ (0).$$

IGNOR (in terms of A^+, A^-, B^+, B^-)

$$\check{Q}_E(A^+, A^-, B^+, B^-) = 1\ (0)\ \textit{and}\ X \cap E = \emptyset \Rightarrow$$
$$\check{Q}_{E \cup X}(A^+, A^-, B^+, B^-) = 1\ (0).$$

The converse of **IGNOR** follows immediately from $\leq$ **PERSIST**. The connection with monotonicity is given by the next proposition.

Proposition 29 $\uparrow$**MON** *implies positive* **IGNOR**. $\downarrow$**MON** *implies negative* **IGNOR**.

Proof.

$$Q^+(A^+, A^*, B^+, B^*) = 1$$
$$\Rightarrow (\uparrow\mathbf{MON})\ Q^+(A^+, A^* \cup X, B^+, B^*) = 1$$
$$\Leftrightarrow (\mathbf{W\text{-}CONS})\ Q^+(A^+, A^* \cup X, B^+, B^* \cap X) = 1.$$

The proof of the second claim is similar. ∎

It is not difficult to show that $\uparrow$**MON** does not imply negative **IGNOR**, nor does $\downarrow$**MON** imply positive **IGNOR**. It follows from these facts and the above proposition that it is possible to verify *Some A are B* or *Not all A are B* in a world which is only partially known, but impossible to falsify such claims in such a world. By the same token, it is possible to falsify *All A are B*, *At most* **n** *A are B*, and *no A are B* in a fathomless world, but impossible to verify such claims. Contemporary philosophy of science bears witness to the fact that one may spin long yarns of philosophical argument starting from such simple logical observations.

We end with a connection between monotonicity and the notion **M-CONS** from Section 3.

Proposition 30 *$\uparrow$**MON**, $\leq$ **PERSIST** and **IGNOR** imply the positive part of **M-CONS**;*
*$\downarrow$**MON**, $\leq$ **PERSIST** and **IGNOR** imply the negative part of **M-CONS**.*

Proof.

$\check{Q}^+_E(A^+, A^-, B^+, B^-) = 1$
iff ($\Rightarrow$: $\leq$ **PERSIST**, $\Leftarrow$: $\uparrow$**MON**) $\check{Q}^+_E(A^+, A^- \cup \overline{A^+}, B^+, B^-) = 1$
iff (**W-CONS**) $\check{Q}^+_E(A^+, A^- \cup \overline{A^+}, B^+ \cap \overline{\overline{A^- \cup \overline{A^+}}}, B^- \cup A^- \cup \overline{A^+}) = 1$
iff $\check{Q}^+_E(A^+, A^- \cup \overline{A^+}, B^+ \cap A^+, B^- \cup A^- \cup \overline{A^+}) = 1$
iff (**EXT**) $\check{Q}^+_{E - \overline{A^+}}(A^+, A^-, B^+ \cap A^+, B^- \cup A^-) = 1$
iff ($\Rightarrow$: **IGNOR**, $\Leftarrow$: $\leq$ **PERSIST**) $\check{Q}^+_E(A^+, A^-, B^+ \cap A^+, B^- \cup A^-) = 1$.

The second claim is proved similarly. ∎

8 Conclusion

We have sketched the relational theory of three valued generalized quantifiers. It emerged that the supervaluation approach generates a class of well behaved three valued quantifiers. The supervaluation perspective made it also possible to generalize a number of quantifier intuitions from the total to the partial case. But the partial perspective also gave rise to a number of new intuitions, with information persistence as the most important.

The theory of total quantifiers makes use of tree representations and semantic automata to represent quantifiers. Extensions of these notions to cover the partial case can easily be given. On the application side, one would like to know more about the ways in which logical properties of quantified expressions influence linguistic usage. Also, a link should be established between partial generalized quantifier theory and theories of vagueness for natural language expressions, e.g. the theory of vague adjectives.

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