Robust Stability for a Class of Linear Systems with Time-Varying Delay and Nonlinear Perturbations

QING-LONG HAN
Faculty of Informatics and Communication
Central Queensland University
Rockhampton, Qld 4702, Australia
q.han@cqu.edu.au

(Received November 2002; revised and accepted August 2003)

Abstract—The robust stability of uncertain linear systems with a single time-varying delay is investigated by employing a descriptor model transformation and a decomposition technique of the delay term matrix. The uncertainties under consideration are nonlinear perturbations and norm-bounded uncertainties, respectively. The proposed stability criteria are formulated in the form of a linear matrix inequality. Numerical examples are presented to indicate significant improvements over some existing results. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Stability, Time-varying delay, Uncertainty, Linear matrix inequality (LMI).

1. INTRODUCTION

In many practical control problems, there are a number of time-delay systems, such as chemical processes, hydraulic and rolling mill systems, due to measurement of system variables, physical properties of the equipment used in the systems' signal transmission, and so on [1]. The existence of delay in a practical system may induce instability, oscillation, and poor performance [2]. Therefore, the problem of stability of time-delay systems has been attracting the interest of many investigators for several decades.

In recent years, the problem of robust stability of time-delay systems with nonlinear perturbations has also received considerable attention. In [3], for example, some delay-independent and delay-dependent stability criteria are obtained by using the properties of matrix measure and a comparison theorem. In [4], the results in [3] are extended to the systems with time-varying delays. In [5], a sufficient stability condition is derived by employing the Razumikhin theorem. In [6], based on matrix measure, matrix norm, and a decomposition technique, two stability criteria are derived. In [7], a model transformation technique is used to transform the system with a discrete delay to a system with a distributed delay, and delay-dependent stability criteria are obtained by using a Lyapunov-Krasovskii functional approach. Although these results in [7] are less...
conservative than some existing ones, they are still conservative since the model transformation will introduce additional dynamics discussed in [8].

Recently, a new descriptor model transformation and a corresponding Lyapunov-Krasovskii functional have been introduced for stability analysis of systems with constant delays [9]. The advantage of this transformation is to transform the original system to an equivalent descriptor form representation and will not introduce additional dynamics in the sense defined in [10]. In [11], some results in [9] are extended to neutral systems with time-varying discrete delays. Although the result in [11] is less conservative than some existing ones, it can be improved by employing the decomposition technique to get a larger bound for time-varying discrete delays. Furthermore, nonlinear/norm-bounded uncertainties are not considered in [9,11].

In this paper, based on the descriptor model transformation and the decomposition technique of a discrete-delay term matrix, we investigate the robust stability of uncertain systems with a single time-varying discrete delay by applying an integral inequality that is introduced in this paper instead of applying bounding of the cross terms introduced in [12]. The robust stability problem of considered system is transformed into the existence of some symmetric positive-definite matrices. The stability criteria are formulated in the form of linear matrix inequalities (LMI). Numerical examples show that the results obtained in this paper are less conservative than some existing ones in the literature.

2. PROBLEM STATEMENT

Consider the following linear system with a single time-varying discrete delay

\[
\dot{x}(t) = Ax(t) + Bx(t - h(t)) + f(x(t), t) + g(x(t - h(t)), t),
\]

where \(x(t) \in \mathbb{R}^n\) is the state, \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times n}\) are constant matrices. The time-varying vector-valued functions \(f(x(t), t) \in \mathbb{R}^n\) and \(g(x(t - h(t)), t) \in \mathbb{R}^n\) are unknown and represent the parameter perturbations with respect to the current state \(x(t)\) and delayed state \(x(t - h(t))\) of the systems, respectively. They satisfy that \(f(0, t) = 0\) and \(g(0, t) = 0\). The discrete delay \(h(t)\) is a time-varying function which satisfies

\[
0 \leq h(t) \leq h_M, \quad \dot{h}(t) \leq h_d,
\]

where \(h_M\) and \(h_d\) are constants, and \(0 \leq h_d < 1\).

The initial condition of system (1) is given by

\[
x(\theta) = \varphi(\theta), \quad \forall \theta \in [-h_M, 0],
\]

where \(\varphi(.)\) is a continuous vector-valued initial function.

Rewrite system (1) in the following equivalent descriptor system

\[
\dot{x}(t) = y(t),
\]

\[
y(t) = Ax(t) + Bx(t - h(t)) + f(x(t), t) + g(x(t - h(t)), t).
\]

To derive delay-dependent stability conditions, which include the information of the time-delay \(h(t)\), one usually uses the fact

\[
x(t - h(t)) = \int_{t-h(t)}^t \dot{x}(\xi) \, d\xi = x(t) - \int_{t-h(t)}^t y(\xi) \, d\xi
\]

to transform the original system (1) to a system with a distributed delay. This transformation does not provide any additional dynamics because we do not expand \(\dot{x}(t)\) in terms of right-hand representation of (1).
In order to improve the bound of delay \( h(t) \), let us decompose matrix \( B \) as \( B = B_1 + B_2 \), where \( B_1 \) is a constant matrix. Then, system (4) can be represented in the form of descriptor system with discrete and distributed delays in the "fast variable" \( y \) [9]

\[
\dot{z}(t) = y(t),
\]

\[
0 = -y(t) + (A+B_1)x(t) + B_2x(t-h(t)) - B_1 \int_{t-h(t)}^{t} y(\xi) d\xi + f(x(t), t) + g(x(t-h(t)), t) + \beta \int_{t-h(t)}^{t} z(\xi) d\xi,
\]

(5b)

It is clear that system (5) is equivalent to system (1). In the following, we will employ system (5) to study the stability of system (1).

Now we introduce the S-procedure that will be used in this paper to handle the uncertainty.

**DEFINITION 1.** (See [13].) Denote the set \( \mathbb{Z} = \{z\} \) and let \( \mathcal{F}(z), \mathcal{Y}_1(z), \mathcal{Y}_2(z), \ldots, \mathcal{Y}_k(z) \) be some functionals or functions. Further define domain \( \mathbb{D} \) as

\[
\mathbb{D} = \{ z \in \mathbb{Z} : \mathcal{Y}_1(z) \geq 0, \mathcal{Y}_2(z) \geq 0, \ldots, \mathcal{Y}_k(z) \geq 0 \}
\]

and the two following conditions:

(A) \( \mathcal{F}(z) > 0, \forall z \in \mathbb{D} \);

(B) \( \exists \varepsilon_1 \geq 0, \varepsilon_2 \geq 0, \ldots, \varepsilon_k \geq 0 \) such that

\[
S(\varepsilon, z) = \mathcal{F}(z) - \sum_{j=1}^{k} \varepsilon_j \mathcal{Y}_j(z) > 0, \quad \forall z \in \mathbb{Z}.
\]

Then, (B) implies (A). The procedure of replacing (A) by (B) is called the S-procedure.

The purpose of this paper is to formulate some practically computable criteria to check the stability of system described by (1)–(3).

### 3. NONLINEAR TIME-VARYING PERTURBATIONS

In this section, we assume that \( f(x(t), t) \) and \( g(x(t-h(t)), t) \) represent the nonlinear time-varying perturbations of system (1) which satisfy that

\[
\|f(x(t), t)\| \leq \alpha \|x(t)\|, \quad \text{(6a)}
\]

\[
\|g(x(t-h(t)), t)\| \leq \beta \|x(t-h(t))\|, \quad \text{(6b)}
\]

where \( \alpha \geq 0 \) and \( \beta \geq 0 \) are given constants.

Constraint (6) can be rewritten as

\[
f^T(x(t), t)f(x(t), t) \leq \alpha^2 x^T(t)x(t), \quad \text{(7a)}
\]

\[
g^T(x(t-h(t)), t)g(x(t-h(t)), t) \leq \beta^2 x^T(t-h(t))x(t-h(t)). \quad \text{(7b)}
\]

For the robust stability of system (1)–(3), with uncertainty (6), we have the following result.

**THEOREM 1.** The system described by (1)–(3), with uncertainty described by (6) is asymptotically stable if there exist real matrices \( P_2, P_3 \) and symmetric positive definite matrices \( P_1, Q, S, P_2, P_3 \), and scalars \( \varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \) such that the following LMI holds

\[
\Xi = \begin{pmatrix}
\Xi_{11} & \Xi_{12} & P_1^T B_2 & -h_M P_2^T B_1 & P_2^T & P_3^T \\
\Xi_{12} & \Xi_{22} & P_3^T B_2 & -h_M P_3^T B_1 & P_3^T & P_3^T \\
B_2^T P_2 & B_3^T P_3 & -(1-h_d)Q + \varepsilon_2 \beta^2 I & 0 & 0 & 0 \\
-h_M B_1^T P_2 & -h_M B_1^T P_3 & 0 & -(1-h_d)h_M S & 0 & 0 \\
P_2 & P_3 & 0 & 0 & -\varepsilon_1 I & 0 \\
P_3 & P_3 & 0 & 0 & 0 & -\varepsilon_2 I
\end{pmatrix} < 0, \quad \text{(8)}
\]
where

\[ \Xi_{11} = P_2^T (A + B_1) + (A + B_1)^T P_2 + Q + \varepsilon_1 \alpha^2 I, \]
\[ \Xi_{12} = P_1 - P_2^T + (A + B_1)^T P_3, \]
\[ \Xi_{22} = -P_3 - P_3^T + h_M S. \]

In order to prove Theorem 1, we need the following integral inequality.

**Lemma 1.** For any constant symmetric matrix \( M \in \mathbb{R}^{n \times n} \), \( M = M^T > 0 \), scalar \( \gamma > 0 \), vector function \( \omega : [0, \gamma] \rightarrow \mathbb{R}^n \) such that the integrations in the following are well defined, then

\[
\begin{align*}
    r(t) \int_{-r(t)}^0 \omega^T(\beta)Q\omega(\beta) d\beta &\geq \left( \int_{-r(t)}^0 \omega(\beta) d\beta \right)^T Q \left( \int_{-r(t)}^0 \omega(\beta) d\beta \right), \\
    &\quad \text{for any } -r(t) \leq \beta \leq 0. \tag{9}
\end{align*}
\]

**Proof.** It is easy to see, using Schur’s complement [14], that

\[
\left( \begin{array}{cc}
    \omega^T(\beta)Q\omega(\beta) & \omega(\beta) \\
    \omega(\beta) & Q^{-1} 
\end{array} \right) \geq 0,
\]

for any \(-r(t) \leq \beta \leq 0\). Integration of the above inequality from \(-r(t)\) to 0 yields

\[
\begin{align*}
    \int_{-r(t)}^0 \omega^T(\beta)Q\omega(\beta) d\beta &\geq \left( \int_{-r(t)}^0 \omega(\beta) d\beta \right)^T Q \left( \int_{-r(t)}^0 \omega(\beta) d\beta \right), \\
    &\quad \text{for any } -r(t) \leq \beta \leq 0. \tag{10}
\end{align*}
\]

Use Schur’s complement to reach

\[
\begin{align*}
    r(t) \int_{-r(t)}^0 \omega^T(\beta)Q\omega(\beta) d\beta &\geq \left( \int_{-r(t)}^0 \omega(\beta) d\beta \right)^T Q \left( \int_{-r(t)}^0 \omega(\beta) d\beta \right). \tag{10}
\end{align*}
\]

Noting that \( 0 \leq r(t) \leq r_M \), from (10) we get (9).

**Proof of Theorem 1.** Choose the Lyapunov-Krasovskii functional candidate for system (5) as

\[ V(t) = V_1(t) + V_2(t) + V_3(t), \]

where

\[ V_1(t) = \left( x^T(t) \quad y^T(t) \right) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \]
\[ V_2(t) = \int_{-h(t)}^t x^T(\xi)Qx(\xi) d\xi, \]
\[ V_3(t) = \int_{-h(t)}^t (h(t) - t + \xi) y^T(\xi)Sy(\xi) d\xi, \]

where \( P_2, P_3 \) and symmetric positive definite matrices \( P_1, Q, \) and \( S \) are the solutions of (8). This functional is degenerated as it is usual for descriptor systems [15].

The derivative of \( V(t) \) along the trajectory of system (5) is given by \( \dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \). Since

\[ \dot{V}_1(t) = \frac{d}{dt} \left[ \begin{bmatrix} x^T(t) \\ y^T(t) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \right], \]
\[ = 2x^T(t)P_1 \dot{x}(t), \]
\[ = 2 \begin{bmatrix} x^T(t) \\ y^T(t) \end{bmatrix} \begin{bmatrix} P_1 & P_2^T \\ 0 & P_3^T \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} \]
from (5), we have

\[
\dot{V}_1(t) = 2(x^T(t) y^T(t)) \begin{bmatrix} P_1 & P_2^T \\ 0 & P_3^T \end{bmatrix} \begin{bmatrix} y(t) \\ 0 \end{bmatrix} = 2(x^T(t) y^T(t)) \begin{bmatrix} P_1 & P_2^T \\ 0 & P_3^T \end{bmatrix} \begin{bmatrix} y(t) \\ -y(t) + (A + B_1) x(t) + B_2 x(t-h(t)) - B_1 \int_{t-h(t)}^t y(\xi) d\xi \end{bmatrix} 
\]

\[
+ 2(x^T(t) y^T(t)) \begin{bmatrix} P_1 & P_2^T \\ 0 & P_3^T \end{bmatrix} \begin{bmatrix} f(x(t), t) + g(x(t-h(t)), t) \\ 0 \end{bmatrix} 
\]

\[
= x^T(t) [P_2^T (A + B_1) + (A + B_1)^T P_3] x(t) + 2x^T(t) [P_1 - P_2^T + (A + B_1)^T P_3] y(t) 
\]

\[
+ 2x^T(t) [P_2^T B_2] x(t-h(t)) + 2x^T(t) [-h_M P_2^T B_1] \left( \frac{1}{h_M} \int_{t-h(t)}^t y(\xi) d\xi \right) 
\]

\[
+ 2y^T(t) [-h_M P_3^T B_1] \left( \frac{1}{h_M} \int_{t-h(t)}^t y(\xi) d\xi \right) 
\]

\[
+ 2y^T(t) P_2 f(x(t), t) + 2y^T(t) P_3 g(x(t-h(t)), t). 
\]

\[\dot{V}_2(t)\] is computed as follows

\[
\dot{V}_2(t) = x^T(t) Q x(t) - (1 - \dot{h}(t)) x^T(t-h(t)) Q x(t-h(t)) 
\]

\[
\leq x^T(t) Q x(t) - (1 - \dot{h}_d) x^T(t-h(t)) Q x(t-h(t)). 
\]

Use Lemma 1 to obtain

\[
\dot{V}_3(t) = y^T(t) [h(t)S] y(t) - (1 - \dot{h}(t)) \int_{t-h(t)}^t y^T(\xi) S y(\xi) d\xi 
\]

\[
\leq y^T(t) [h_M S] y(t) - \left( \frac{1}{h_M} \int_{t-h(t)}^t y(\xi) d\xi \right)^T [(1 - \dot{h}_d) h_M S] \left( \frac{1}{h_M} \int_{t-h(t)}^t y(\xi) d\xi \right). 
\]

Therefore, we have

\[
\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \leq q^T(t) \Xi_0 q(t), 
\]

where

\[
q^T(t) = \begin{pmatrix} x^T(t) y^T(t) x^T(t-h(t)) \end{pmatrix} \left( \frac{1}{h_M} \int_{t-h(t)}^t y(\xi) d\xi \right)^T f^T(x(t), t) g^T(x(t-h(t)), t) 
\]

and

\[
\Xi_0 = \begin{pmatrix} \Xi_{011} & \Xi_{012} & P_1^T B_2 & -h_M P_2^T B_1 & P_2^T P_2^T & P_2^T P_3^T \\ \Xi_{012} & \Xi_{022} & P_2^T B_2 & -h_M P_3^T B_1 & P_3^T & P_3^T \\ B_2^T P_2 & B_2^T P_3 & -(1 - \dot{h}_d)Q & 0 & 0 & 0 \\ -h_M B_1^T P_2 & -h_M B_1^T P_3 & 0 & -(1 - \dot{h}_d) h_M S & 0 & 0 \\ P_2 & P_3 & 0 & 0 & 0 & 0 \\ P_2 & P_3 & 0 & 0 & 0 & 0 \end{pmatrix}, 
\]

with

\[
\Xi_{011} = P_2^T (A + B_1) + (A + B_1)^T P_2 + Q, 
\]

\[
\Xi_{012} = \Xi_{12} = P_1 - P_2^T + (A + B_1)^T P_3, 
\]

\[
\Xi_{022} = \Xi_{22} = -P_3 - P_3^T + h_M S. 
\]
A sufficient condition for asymptotic stability of system (1) is that there exist real matrices $P_2, P_3$ and symmetric positive definite matrices $P_1, Q, S$ such that

$$
\dot{V}(t) \leq q(t)^T \Xi q(t) < 0,
$$

for all $q(t) \neq 0$ (which is equivalent to $(x^T(t) \ x^T(t-h(t)))^T \neq 0$, see Remark 1 below) satisfying (7), where (12) means that $V(t)$ is negative definite whenever neither $x(t)$ nor $x(t-h(t))$ is zero. Using the $S$-procedure, we see that this condition is implied by the existence of nonnegative scalars $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ such that

$$
\begin{align*}
q^T(t) \Xi q(t) &+ \varepsilon_1 (\alpha^2 x^T(t)x(t) - f^T(x(t),t)f(x(t),t)) \\
&+ \varepsilon_2 (\beta^2 x^T(t-h(t))x(t-h(t)) - g^T(x(t-h(t)),t)g(x(t-h(t)),t)) < 0,
\end{align*}
$$

for all $q(t) \neq 0$. Therefore, if there exist real matrices $P_2, P_3$ and symmetric positive definite matrices $P_1, Q, S$, and scalars $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ such that LMI (8) is satisfied, then system (1)–(3), with uncertainty (6), is asymptotically stable.

**Remark 1.** It is easy to see that $q(t) \neq 0$ is equivalent to $(x^T(t) \ x^T(t-h(t)))^T \neq 0$. In fact, if $(x^T(t) \ x^T(t-h(t)))^T = 0$, then $f(x(t),t) = 0$ and $g(x(t-h(t)),t) = 0$. From (1), one obtains that $\dot{x}(t) = 0$. Therefore, $y(t) = x(t) = 0$ and $(1/h_M) \int_{t-h(t)}^{t} y(\xi) \, d\xi = 0$. Thus, $q(t) = 0$.

**Remark 2.** When $f$ and $g$ are linear time-varying perturbations, i.e., they are of the form

$$
\begin{align*}
f(x(t),t) &= \Delta A(t)x(t), \\
g(x(t-h(t)),t) &= \Delta B(t)x(t-h(t)),
\end{align*}
$$

with the assumption that $\|\Delta A(t)\| \leq \alpha$ and $\|\Delta B(t)\| \leq \beta$. The result of Theorem 1 is also true.

**Remark 3.** Theorem 1 provides a delay-dependent stability criterion for time-delay systems with nonlinear time-varying perturbations in terms of the solvability of an LMI. For each fixed $h_d$, it is also interesting to note that $h_M$ appears linearly. Therefore, a generalized eigenvalue problem as defined in [14] can be formulated to solve the minimum acceptable $1/h_M$, and therefore, the maximum $h_M^{\text{max}}$ to maintain asymptotic stability as judged by the criterion.

In the case when there is no nonlinear time-varying parameter perturbation in systems (1)–(3), i.e., $f(x(t),t) \equiv 0$ and $g(x(t-h(t)),t) \equiv 0$, we have the following result.

**Corollary 1.** The system described by (1)–(3), with $f(x(t),t) \equiv 0$ and $g(x(t-h(t)),t) \equiv 0$ is asymptotically stable if there exist real matrices $P_2, P_3$ and symmetric positive definite matrices $P_1, Q, S$ such that the following LMI holds

$$
\left[
\begin{array}{cccc}
P_2^T (A+B_1) + (A+B_1)^T P_2 + Q & P_2 - P_3^T + (A+B_1) & -h_M P_3 & B_1 \\
P_2 - P_3^T + (A+B_1) & -P_3 + h_M S & P_3 & -h_M P_3 \\
B_1^T P_2 & B_1^T P_3 & -(1-h_d)Q & 0 \\
-h_M B_1^T P_2 & -h_M B_1^T P_3 & 0 & -(1-h_d)h_M S
\end{array}
\right] < 0.
$$

**Remark 4.** The efficiency of Theorem 1 and Corollary 1 depends on the decomposition of matrix $B$. Matrix $B_1$ is chosen such that $A+B_1$ is “more stable” than matrix $A$. The decomposition idea was first introduced by Goubet-Batholomeus et al. [6]. Now, we consider how to decompose matrix $B$. For the sake of simplicity, we take Corollary 1 as an example and restrict $Q, S$, and to a special case of $Q = S = P_1$. From (15), one can see that $-P_2 - P_3 < 0$. So the matrix $[P_3 \ 0]$ is nonsingular. Define

$$
[ P_1 \ 0 ]^{-1} = \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix} = X \quad \text{and} \quad K = \text{diag}(X,I,I).
$$
Then, we multiply (15) by $K^T$ and $K$, on the left and right, respectively. Using Schur’s complement to the quadratic term in $K$, and introducing a new variable $W = B_1 X_1$ to yield the following LMI

$$
\begin{bmatrix}
X_2 + X_2^T & -X_2^T + X_1 A^T + W^T & 0 & 0 & X_1 & h_M X_2^T \\
-X_2 + AX_1 + W & -X_2^T + X_2 A^T W & B X_1 - W & -h_M W & 0 & h_M X_3^T \\
0 & X_1 B^T - W & -(1 - h_d) X_1 & 0 & 0 & 0 \\
0 & -h_M W^T & 0 & -(1 - h_d) h_M X_1 & 0 & 0 \\
X_1 & 0 & 0 & 0 & X_1 & 0 \\
h_M X_2 & h_M X_3 & 0 & 0 & 0 & -(1 - h_d) h_M X_1
\end{bmatrix}
\leq 0. \quad (16)
$$

We can conclude that the system described by (1)-(3), with $f(x(t), t) \equiv 0$ and $g(x(t - h(t)), t) \equiv 0$ is asymptotically stable if there exist real matrices $X_2$, $X_3$, $W$, and a symmetric positive definite matrix $X_1$ such that the LMI (16) holds. Furthermore, matrix $B_1$ is given by $B_1 = WX_1^{-1}$. For the general case, the idea is the same. It is omitted.

### 4. NORM-BOUNDED UNCERTAINTY

In this section, we will handle the case that $f(x(t), t)$ and $g(x(t - h(t)), t)$ are norm-bounded uncertainties. Then, system (1) becomes the following system

$$
\dot{x}(t) = (A + LF(t)E_a) x(t) + (B + LF(t)E_b) x(t - h(t)), \quad (17)
$$

where $F(t) \in \mathbb{R}^{p \times q}$ is an unknown real and possibly time-varying matrix with Lebesgue measurable elements satisfying

$$
\sigma_{\text{max}}(F(t)) \leq 1 \quad (18)
$$

and $L$, $E_a$, $E_b$, and $E_a$ are known real constant matrices which characterize how the uncertainty enters the nominal matrices $A$ and $B$.

System (15) can be written as

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + B x(t - h(t)) + Lu, \\
y &= E_a x(t) + E_b x(t - h(t)),
\end{align*}
$$

with the constraint

$$
u = F(t) y. \quad (20)$$

We further rewrite (17), (18) as

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + B x(t - h(t)) + Lu, \\
u^T u &\leq (E_a x(t) + E_b x(t - h(t)))^T (E_a x(t) + E_b x(t - h(t))).
\end{align*}
$$

We now state and establish the following result.

**Theorem 2.** The system described by (17), (18), (2), (3) is asymptotically stable if there exist real matrices $\hat{P}_2, \hat{P}_3$ and symmetric positive definite matrices $\hat{P}_1, \hat{Q},$ and $\hat{S}$, the following LMI

$$
\begin{bmatrix}
\Xi_{11} & \Xi_{12} & \hat{P}_2^T B_2 & -h_M \hat{P}_2^T B_1 & \hat{P}_2^T L & E_a^T \\
\Xi_{12} & \Xi_{22} & \hat{P}_3^T B_2 & -h_M \hat{P}_3^T B_1 & \hat{P}_3^T L & 0 \\
B_2^T \hat{P}_2 & B_3^T \hat{P}_3 & -(1 - h_d) \hat{Q} & 0 & 0 & E_b^T \\
-h_M B_2^T \hat{P}_2 & -h_M B_3^T \hat{P}_3 & 0 & -(1 - h_d) h_M \hat{S} & 0 & 0 \\
L^T \hat{P}_2 & L^T \hat{P}_3 & 0 & 0 & -I & 0 \\
E_a & 0 & E_b & 0 & 0 & -I
\end{bmatrix}
\leq 0, \quad (22)
$$

holds.
where
\[
\Xi_{11}^0 = \dot{P}_2^T (A + B_1) + (A + B_1)^T \dot{P}_2 + \dot{Q},
\]
\[
\Xi_{12}^0 = \dot{P}_2 - \dot{P}_2^T (A + B_1) \dot{P}_3,
\]
\[
\Xi_{22}^0 = -\dot{P}_3 - \dot{P}_3^T + h_M \dot{S}.
\]

**PROOF.** Similar to the proof of Theorem 1, we can conclude that the system described by (17), (18), (2), (3) is asymptotically stable if there exist real matrices \( P_2, P_3 \) and symmetric positive definite matrices \( P_1, Q, S, \) and a scalar \( \varepsilon \geq 0 \) such that the following LMI holds
\[
\begin{bmatrix}
\Xi_{011} & \Xi_{012} & P_2^T B_2 + \varepsilon E_a^T E_b & -h_M P_2^T B_1 & P_2^T L \\
\Xi_{012} & \Xi_{022} & P_3^T B_2 & -h_M P_3^T B_1 & P_3^T L \\
P_2^T P_2 + \varepsilon E_b^T E_b & P_3^T P_3 & -(1 - h_d)Q + \varepsilon E_b^T E_b & 0 & 0 \\
-h_M B_2^T P_2 & -h_M B_3^T P_3 & 0 & -(1 - h_d)h_M S & 0 \\
L^T P_2 & L^T P_3 & 0 & 0 & -\varepsilon I
\end{bmatrix} < 0,
\]
(23)

where \( \Xi_{011}, \Xi_{012}, \) and \( \Xi_{022} \) are the same as (11). Noting that (23) implies \( \varepsilon > 0 \) and introducing new variables \( \dot{P}_i = \varepsilon^{-1} P_i \) \((i = 1, 2, 3), \) \( \dot{Q} = \varepsilon^{-1} Q, \) and \( \dot{S} = \varepsilon^{-1} S \) and using Schur complement yields (22).

5. EXAMPLES

**EXAMPLE 1.** Consider system (1) with
\[
A = \begin{bmatrix}
-1.2 & 0.1 \\
-0.1 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
-0.6 & 0.7 \\
-1 & -0.8
\end{bmatrix},
\]
\[
\|f(x(t), t)\| \leq \alpha\|x(t)\|, \quad \|g(x(t - h(t)), t)\| \leq \beta\|x(t - h(t))\|
\]
where \( \alpha \geq 0, \beta \geq 0. \)

Decompose the matrix \( B \) as \( B = B_1 + B_2, \) where
\[
B_1 = \begin{bmatrix}
-0.6 & 0 \\
0 & -0.6
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 & 0.7 \\
-1 & -0.2
\end{bmatrix}.
\]

Applying the criteria in [7] and in this paper, the maximum value of \( h_M \) for stability of system under consideration is listed in the following table. It is easy to see that the stability criterion in this paper gives a much less conservative result than one in [7]. Other results surveyed in [7] are even more conservative.

<table>
<thead>
<tr>
<th>( h_d = 0 )</th>
<th>( h_d = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cao and Lam [7]</td>
<td>0.6811</td>
</tr>
<tr>
<td>This Paper</td>
<td>1.3279</td>
</tr>
</tbody>
</table>

**EXAMPLE 2.** Consider the following uncertain system with time-varying delay [16]
\[
\dot{x}(t) = [A_0 + \Delta A(t)]x(t) + [B_0 + \Delta B(t)]x(t - h(t)),
\]
(24)

with
\[
A = \begin{bmatrix}
-2 & 0 \\
0 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
-1 & 0 \\
-1 & -1
\end{bmatrix},
\]
\[
\Delta A(t) = \begin{bmatrix}
\delta_1 & 0 \\
0 & \delta_2
\end{bmatrix}, \quad \Delta B(t) = \begin{bmatrix}
\gamma_1 & 0 \\
0 & \gamma_2
\end{bmatrix},
\]
where $\delta_1$, $\delta_2$, $\gamma_1$, and $\gamma_2$ are unknown parameters satisfying

$$|\delta_1| \leq 1.6, \quad |\delta_2| \leq 0.05, \quad |\gamma_1| \leq 0.1, \quad |\gamma_2| \leq 0.3.$$

For $h_d = 0$, the maximum value of $h_M$ for stability of system (24) was reported as $h_M = 0.2412$ and $h_M = 1.0$ in [16] and [11], respectively. Now we use the criterion in this paper to study the same case. Let us decompose matrix $B$ as $B = B_1 + B_2$, where

$$B_1 = \begin{bmatrix} -0.47 & 0 \\ -0.01 & -0.58 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.53 & 0 \\ -0.99 & -0.42 \end{bmatrix},$$

the maximum value of $h_{\text{max}}$ for the system to be asymptotically stable is $h_M = 1.1285$. This example shows that the stability criterion in this paper gives a much less conservative result than these in [16] and [11].

6. CONCLUSION

The robust stability problem for a class of uncertain linear systems with time-varying delay has been investigated. Stability criteria have been obtained. Numerical examples have shown significant improvements over some existing results.

REFERENCES

11. E. Fridman and U. Shaked, Stability and $H_{\infty}$ control of systems with time-varying delays, In Proc. of 15th IFAC World Congress (CD), Barcelona, Spain, (July 2002).