Strong Convergence Theorems for Asymptotically Nonexpansive Nonself-Mappings

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Abstract: Suppose $C$ is a nonempty bounded closed convex retract of a real uniformly convex Banach space $X$ with uniformly Gâteaux differentiable norm and $P$ as a nonexpansive retraction of $X$ onto $C$. Let $T : C \to X$ be an asymptotically nonexpansive nonself-map with sequence $\{k_n\}_{n \geq 1} \subset [1, \infty)$, $\lim k_n = 1$, $F(T) = \{x \in C : Tx = x\}$, and let $u \in C$.

In this paper we study the convergence of the sequences $\{x_n\}$ and $\{y_n\}$ which defined by

$$x_n = \left(1 - \frac{t_n}{k_n}\right) u + \frac{t_n}{k_n} (PT)^n x_n$$

and

$$y_n = P \left( \left(1 - \frac{t_n}{k_n}\right) u + \frac{t_n}{k_n} T(PT)^{n-1} y_n \right),$$

where $t_n = \min \left\{ \frac{1}{2}, 1 - \frac{1}{n} \right\}$ for $n = 1, 2, \ldots$.

Keywords: Asymptotically nonexpansive nonself-maps, nonexpansive retraction, uniformly convex, uniformly Gâteaux differentiable norm.

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1 Introduction

Let $C$ be a nonempty closed convex subset of a Banach space $X$ and let $T : C \to X$ be a nonexpansive mapping (i.e. $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). For a given $u \in C$ and each $t \in (0, 1)$, we can define a contraction $T_t : C \to X$ by

$$T_t x = tTx + (1-t)u, \text{ for all } x \in C. \quad (1.1)$$

If $T$ is further assumed to be self-mapping i.e. $T(C) \subset C$, then $T_t$ maps $C$ into itself, hence the Banach's contraction principle yields a unique fixed point $x_t$ in $C$, that is, we have

$$x_t = tTx_t + (1-t)u. \quad (1.2)$$
Now a natural question give rise to whether \( \{x_n\} \) converges strong as \( t \to 1 \) to a fixed point of \( T \). The strong convergence of \( \{x_t\} \) as \( t \to 1 \) for a self-mapping \( T \) of a bounded set \( C \) was proved in a Hilbert space by Browder [3] in 1967 and in a uniformly smooth Banach space by Reich [12] in 1980.

Recently, Xu and Yin [18] proved that if \( C \) is a nonempty closed convex (not necessarily bounded) subset of a Hilbert space \( H \), if \( T : C \to H \) is a nonexpansive nonself-mapping, and if \( \{x_t\} \) is the sequence define by (2) which is bounded, then \( \{x_t\} \) converges strongly as \( t \to 1 \) to a fixed point of \( T \). They also studied other schemes involving the nearest point projection \( P \) from \( H \) onto \( C \), which were introduced by Marino and Trombetta [11].

In 1998, Takahashi and Kim [14] extended Xu and Yin’s result [18] to a reflexive Banach space with has uniform normal structure and a uniformly Gâteaux differentiable norm, using sunny nonexpansive retractions and Banach limits.

On the other hand, Lim and Xu [10] obtained the following : Let \( C \) be a bounded closed convex subset of a Banach space \( X \) and \( T : C \to C \) is an asymptotically nonexpansive mappings, that is, there is a sequence \( \{k_n\}_{n \geq 1} \subset [1,\infty) \), \( k_n \to 1 \) as \( n \to \infty \) such that
\[
\|T^n x - T^n y\| \leq k_n \|x - y\|, \forall x, y \in C, n = 1, 2, \ldots.
\]
(1.3)
Fix \( u \in C \) and for each integer \( n \geq 1 \) define the contraction \( S_n : C \to C \) by
\[
S_n x = \left(1 - \frac{t_n}{k_n}\right) u + \frac{t_n}{k_n} T^n x,
\]
(1.4)
where \( t_n = \min \left\{1 - (k_n - 1) \frac{1}{n}, 1 - \frac{1}{n}\right\} \). Then the Banach contraction principle yields a unique point \( x_n \) fixed by \( S_n \), that is, we have
\[
x_n = \left(1 - \frac{t_n}{k_n}\right) u + \frac{t_n}{k_n} T^n x_n.
\]
(1.5)
Further, if \( X \) is uniformly smooth \( \lim_{n \to \infty} \frac{t_n k_n}{k_n - t_n} = 0 \) and \( \lim_{n \to \infty} \|T^n x_n\| = 0 \), then the sequence \( \{x_n\} \) converges strongly to a fixed of \( T \).

In this paper, we prove a strong convergence theorem for weakly asymptotically regular nonself-mappings in a real uniformly convex Banach space with uniformly Gâteaux differentiable norm, using sunny nonexpansive retractions and Banach limits. Our proof employ the methods of Lim and Xu [10] and Takahashi and Kim [14].

2 Preliminaries

Throughout this paper we denote by \( X \) and \( X^* \) a real Banach space and the dual space of \( X \), respectively. The value of \( x^* \in X^* \) at \( x \in X \) will be denote by \( \langle x, x^* \rangle \). We also denote by \( \mathbb{R} \) and \( \mathbb{R}^+ \) the set of real numbers and all nonnegative real numbers, respectively. When \( \{x_n\} \) is a sequence in \( X \), then \( x_n \to x \)(resp.
Let $x_n \to x$ will denote strong (resp. weak) convergence of the sequence $\{x_n\}$ to $x$. Let $C$ be a nonempty closed convex subset of $X$ and let $T$ be a mapping of $C$ into $X$. Then we denote by $F(T)$ the set of all fixed points of $T$, i.e. $F(T) = \{x \in C : Tx = x\}$. Let $D$ be a subset of $C$ and let $P$ be a mapping of $C$ in to $D$. Then $P$ is said to be sunny if

$$P(Px + t(x - Px)) = Px,$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping $P$ of $C$ into $C$ is said to be a retraction if $P^2 = P$. If a mapping $P$ of $C$ is a retraction, then $Pz = z$ for every $z \in R(P)$, where $R(P)$ is the range of $P$. A subset $D$ is said to be a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction of $C$ onto $D$; for more details, see [8, 15].

Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of $X$ is said to be Gâteaux differentiable (and $X$ is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x$ and $y$ in $S(X)$. It is also said to be uniformly Gâteaux differentiable if for each $y \in S(X)$, the limit (2.1) attained uniformly for $x$ in $S(X)$. With each $x \in X$, we associate the set

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Then $J : X \to X^*$ is said to be the duality mapping. It is well know if $X$ is smooth, then the duality mapping $J$ is single-valued and strong-weak* continuous. It is also know that if $X$ has a uniformly Gâteaux differentiable norm, $J$ is uniformly continuous on bounded sets when $X$ has its strong topology while $X^*$ has its weak star topology; see Day [5] or Diestel [6].

Let $\mu$ be a mean on positive integers $N$, i.e. a continuous linear functional on $l^\infty$ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that $\mu$ is a mean on $N$ if and only if

$$\inf \left\{a_n : n \in \mathbb{N} \right\} \leq \mu(a) \leq \sup \left\{a_n : n \in \mathbb{N} \right\}$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. According to time and circumstance, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean $\mu$ on $\mathbb{N}$ is called a Banach limit if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the existence of a Banach limit. We know that if $\mu$ is a Banach limit, then

$$\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. So, if $a = (a_1, a_2, \ldots) \in l^\infty$ and $a_n \to c$, as $n \to \infty$ we have $\mu_n(a_n) = \mu(a) = c$. Further, we know the following results [13].
Lemma 2.1 Let $C$ be a nonempty closed convex subset of a Banach space $X$ with a uniformly Gâteaux differentiable norm, let $\{x_n\}$ be a bounded sequence of $X$ and let $\mu$ be a mean on $\mathbb{N}$. Let $z \in C$. Then

$$
\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2
$$

if and only if $\mu_n \langle y - z, J(x_n - z) \rangle \leq 0$ for all $y \in C$, where $J$ is the duality mapping of $X$.

Lemma 2.2 Let $C$ be a closed convex subset of a smooth Banach space $X$ and let $P : X \rightarrow C$ be a retraction, then the following are equivalent:

(i) $\langle x - Px, J(y - Px) \rangle \leq 0$ for all $x \in X$ and $y \in C$;

(ii) $\|Pz - Pw\|^2 \leq \langle z - w, J(Pz - Pw) \rangle$ for all $z$ and $w$ in $X$;

(iii) $P$ is both sunny any nonexpansive.

Lemma 2.3 (See [17]) Let $C$ be a nonempty subset of a Banach space $X$ and let $T : C \rightarrow C$ be a mapping of asymptotically nonexpansive type on $C$. Suppose there exists a nonempty bounded closed convex subset $M$ of $C$ with the property $(w)$. For each $x$ in $K$, define the function

$$
\rho_x(y) = \limsup_{n \rightarrow \infty} \|T^n x - y\|, y \in X.
$$

Then the functional $\rho_x$ is a constant on $M$ and this constant is independent of $x$ in $M$.

Lemma 2.4 (See [9] and [10]) Suppose that $X$ is a Banach space with uniformly normal structure, $C$ is a nonempty bounded subset of $X$ and $T : C \rightarrow C$ is a asymptotically nonexpansive mapping. Further, suppose that there exists nonempty bounded closed convex subset $E$ of $C$ with the property $(w)$:

$$
x \in E \text{ implies } W_w(x) \subset E,
$$

where $W_w(x)$ is the weak-limit set of $T$ at $x$; that is, the set

$$
\left\{ y \in X : y = \text{ weak } \lim_j T^{n_j} x \text{ for some } n_j \rightarrow \infty \right\}.
$$

Then $T$ has a fixed point in $E$.

3 Strong convergence theorems

In this section, we give a definitions and prove our main theorems.
**Strong Convergence Theorems for Asymptotically Nonexpansive...**

**Definition 3.1** Let $X$ be a real normed linear space, $C$ a nonempty subset of $X$. Let $P : X \to C$ be the nonexpansive retraction of $X$ onto $C$. A mapping $T : C \to X$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}_{n \geq 1} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in C$, the following inequality holds:

$$\|T(PT)^n x - T(PT)^n y\| \leq k_n \|x - y\|,$$

for all $n \geq 1$. \hfill (3.1)

$T$ is called weakly asymptotically regular on $C$ if

$$T(PT)^n x - T(PT)^n y \text{ weakly converges to } 0 \text{ for all } x \in C.$$ \hfill (3.2)

**Remark 3.2** If $T$ is a self-map, then $PT = T$ so that (3.1) coincide with (1.3)

**Theorem 3.3** Suppose that $X$ is a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. Let $C$ be a nonempty bounded closed convex subset of $X$ and $T : C \to C$ be an asymptotically nonexpansive mapping. Let

$$t_n = \min \left\{ 1 - \left( k_n - 1 \right) \frac{1}{2}, 1 - \frac{1}{n} \right\} \text{ for } n = 1, 2, \ldots, \text{ and } u \in C.$$

Then, a mapping $S_n$ on $C$ given by

$$S_n(x) = \left( 1 - \frac{t_n}{k_n} \right) u + \frac{t_n}{k_n} T^n x \text{ for all } x \in C$$ \hfill (3.3)

has a unique fixed point $x_n$ in $C$. Further, if $T$ is weakly asymptotically regular and completely continuous, then $\{x_n\}$ converges strongly to a fixed point of $T$.

**Proof.** Suppose that the contraction $S_n$ defined by (3.3) and $u \in C$. Then the Banach contraction principle yields a unique point $x_n \in C$ that is fixed by $S_n$, that is, we have

$$x_n = \left( 1 - \frac{t_n}{k_n} \right) u + \frac{t_n}{k_n} T^n x_n.$$

Then we show that $F(T) \neq \emptyset$ and the sequence $\{x_n\}$ converges strongly to a fixed point of $T$. Now let $\mu$ be a Banach limit and define $f : C \to [0, \infty)$ by

$$f(z) = \mu_n \|x_n - z\| \text{ for every } z \in C.$$

Then, since the function $f$ on $C$ is convex and continuous, $f(z) \to \infty$ as $\|z\| \to \infty$, and $X$ is reflexive it follows from [2] that there exists $v \in C$ with $f(v) = \inf_{z \in C} f(z)$. Define the set

$$M = \left\{ v \in C : f(v) = \inf_{z \in C} f(z) \right\}.$$

Then $M$ is a nonempty, bounded closed and convex subset of $C$ see [15, 16]. We further claim that $M$ has the property $(w)$. We must show that $\{T^m x\}$ weak
converges to a fixed point of $T$. Let $y \in W_n(x)$ then $y = \text{weak } \lim_n T^n x$. It follows by weakly asymptotically regular and completely continuous of $T$, that $Ty = y$. Hence $W_n(x) \subseteq F(T)$, we also have $\lim_{n \to \infty} \| T^n y - y \| = 0$. By Lemma 2.3, we have $\rho_x = 0$. It implies that $T^n x \to y$ and hence $\{T^n x\}$ converges weakly to fixed point of $T$. In fact, if $x$ is in $M$ then form $\| x_n - T^n x_n \| \to 0$ as $n \to \infty$, we have

$$f(y) = \mu_n \| x_n - y \| \leq \mu_n \| x_n - T^n x_n \| + \mu_n \| T^n x_n - T^n x \| + \mu_n \| T^n x - y \| \leq k_n \mu_n \| x_n - x \| = k_n f(x).$$

Thus

$$f(y) \leq f(x) = \inf_{z \in K} f(z) \leq f(y)$$

This show that $y$ belongs to $M$ and hence $M$ satisfies the property $(w)$. By Lemma 2.4 that $T$ has a fixed point $z_0 \in M$. Next, to show that $\{x_n\}$ converges strongly to a fixed point of $T$. We note that, for any $w \in F(T)$,

$$\langle x_n - T^n x_n, J(x_n - w) \rangle = \langle x_n - w, J(x_n - w) \rangle + \langle w - T^n x_n, J(x_n - w) \rangle \geq \| x_n - w \|^2 - \| w - T^n x_n \| \| x_n - w \| \geq -(k_n) \| x_n - w \|^2 \geq -(k_n - 1)d^2,$$

where $d = \text{diam } C$. Since $x_n$ is a fixed point of $S_n$, it follows that

$$x_n - T^n x_n = \frac{k_n - t_n}{t_n} (u - x_n)$$

and from last inequality above, we get

$$\langle x_n - u, J(x_n - w) \rangle \leq s_n d^2,$$

where $s_n = \frac{t_n(k_n - 1)}{(k_n - t_n)} \to 0$ as $n \to \infty$. Putting $w = z_0$, so we have

$$\langle x_n - u, J(x_n - z_0) \rangle \leq s_n d^2.$$

(3.4)

Since $z_0$ is the minimizer of the function $f$ on $C$, by Lemma 2.1 we have

$$\mu_n \langle z - z_0, J(x_n - z_0) \rangle \leq 0 \text{ for all } z \in C.$$

So, putting $z = u$, we have

$$\mu_n \langle u - z_0, J(x_n - z_0) \rangle \leq 0.$$

(3.5)

(3.6)

From (3.5) and (3.6) then, we have

$$\mu_n \langle x_n - z_0, J(x_n - z_0) \rangle = \mu_n \| x_n - z_0 \|^2 \leq 0.$$

(3.7)
Thus, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \lim_{k \to \infty} x_{n_k} = z_0 \). To complete the proof, suppose there is another subsequence \( \{x_{m_k}\} \) of \( \{x_n\} \) which converges strongly to (say) \( z' \). Then \( z' \) is a fixed point of \( T \). It then follows from (3.4) that

\[
\langle z - u, J(z - z') \rangle \leq 0
\]

and

\[
\langle z' - u, J(z' - z) \rangle \leq 0.
\]

Combining these two inequalities yields

\[
\langle z - z', J(z - z') \rangle = \|z - z'\|^2 = 0
\]

and hence \( z = z' \). Therefore \( \{x_n\} \) converges strongly to a fixed point of \( T \). □

Let \( X \) be a Banach space and let \( C \) be a nonempty convex subset of \( X \). Then for \( x \in C \) we define the inward set \( I_C(x) \) as follows:

\[
I_C(x) = \{ y \in X : y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0 \}.
\]

A mapping \( T : C \to X \) is said to be inward if \( Tx \in I_C(x) \) for all \( x \in C \). \( T \) is also said to be weakly inward if for each \( x \in C \), \( Tx \) be long to the closure of \( I_C(x) \).

Using Theorem 3.3, we can prove the following two strong convergence theorems.

**Theorem 3.4** Suppose that \( X \) is a real uniformly convex Banach space with uniformly Gâteaux differentiable norm, \( C \) is a nonempty bounded closed convex subset of \( X \) and \( T : C \to X \) is an asymptotically nonexpansive non-self mapping satisfying the weak inwardness condition. Let \( u \in C \). Then, a mapping \( G_n \) on \( C \) given by

\[
G_n(x) = \left( 1 - \frac{t_n}{k_n} \right) u + \frac{t_n}{k_n}(PT)^nx \quad \text{for all } x \in C
\]

(3.8)

has a unique fixed point \( x_n \) in \( C \), where \( P \) is a sunny nonexpansive retraction of \( X \) onto \( C \). Further, if \( T \) is weakly asymptotically regular and completely continuous, then \( \{x_n\} \) converges strongly to a fixed point of \( T \).

**Proof.** It follows by the Banach contraction principle that there exists a unique fixed point \( x_n \) of \( G_n \) in \( C \) such that

\[
x_n = \left( 1 - \frac{t_n}{k_n} \right) u + \frac{t_n}{k_n}(PT)^nx_n.
\]

By Theorem 3.3, we obtain that \( \{x_n\} \) converges strongly to a fixed point \( z \) of \( PT \). Next, let us show that \( z \in F(T) \). Since \( z = PTz \) and \( P \) is a sunny nonexpansive retraction of \( X \) onto \( C \), from Lemma 2.2, we have

\[
\langle Tz - z, J(z - v) \rangle \geq 0
\]
for all $v \in C$. On the other hand, $Tz$ belong to the closure of $I_C(z)$ by the weak inwardness condition. Hence for each integer $n \geq 1$, there exists $z_n \in C$ and $a_n \geq 0$ such that the sequence

$$y_n := z + a_n(z_n - z) \rightarrow Tz.$$ 

Since

$$0 \leq a_n\langle Tz - z, J(z - z_n) \rangle = \langle Tz - z, J(a_n(z - z_n)) \rangle = \langle Tz - z, J(z - y_n) \rangle,$$

we have

$$0 \leq \langle Tz - z, J(z - Tz) \rangle = -\|Tz - z\|^2$$

and hence $Tz = z$. □

**Theorem 3.5** Suppose that $X$ is a real uniformly convex Banach space with uniformly Gâteaux differentiable norm, $C$ is a nonempty bounded closed convex subset of $X$ and $T : C \rightarrow X$ is an asymptotically nonexpansive non-self mapping satisfying the weak inwardness condition. Let $u \in C$. Then, a mapping $U_n$ on $C$ given by

$$U_n(y) = P \left( \left( 1 - \frac{t_n}{k_n} \right) u + \frac{t_n}{k_n} T(PT)^{n-1} y \right) \quad \text{for all } y \in C \quad (3.9)$$

has a unique fixed point $y_n$ in $C$, where $P$ is a sunny nonexpansive retraction of $X$ onto $C$. Further, if $T$ is weakly asymptotically regular and completely continuous then $\{y_n\}$ converges strongly to a fixed point of $T$.

**Proof.** It follows by the Banach contraction principle that there exists a unique fixed point $y_n$ of $U_n$ in $C$ such that

$$y_n = P \left( \left( 1 - \frac{t_n}{k_n} \right) u + \frac{t_n}{k_n} T(PT)^{n-1} y_n \right).$$

As in proof of Theorem 3.3, if $f(z) = \mu_n \|y_n - z\|$ for every $z \in C$ and $M = \{u \in C : f(u) = \inf_{z \in C} f(z)\}$, we have a fixed point $y$ of $PT$ in $M$. Hence by Lemma 2.2,

$$\langle Ty - v, J(y - v) \rangle \geq 0 \quad \text{for all } v \in C.$$

Note that $Ty$ belong to the closure of $I_C(y)$ by the weak inwardness condition. Hence, for each integer $n \geq 1$, there exist $z_n \in C$ and $a_n \geq 0$ such that the sequence

$$x_n := y + a_n(z_n - y) \rightarrow Ty.$$

As in the proof of Theorem 3.4, we have $Ty = y$. For any $w \in F(T)$, we have

$$\frac{t_n}{k_n} (w - u) + u = \frac{t_n}{k_n} w + (1 - \frac{t_n}{k_n})u = P(\frac{t_n}{k_n} w + (1 - \frac{t_n}{k_n})u), \quad \text{for } n = 1, 2, \ldots.
Combining (3.10) and (3.11), we get

Putting \(z\)

From Lemma 2.1, it follows that

where \(b\)

So, we have

Hence

\[
\|\left(y_n - u\right) - \frac{t_n}{k_n} (w - u)\|^2 = \|P\left(\frac{t_n}{k_n} (T(PT)^{n-1} y_n + (1 - \frac{t_n}{k_n}) u) - u - \frac{t_n}{k_n} (w - u)\right)\|^2
\]

\[
= \|P\left(\frac{t_n}{k_n} (T(PT)^{n-1} y_n - u) + u\right) - u - \frac{t_n}{k_n} (w - u)\|^2
\]

\[
= \|\frac{t_n}{k_n} (T(PT)^{n-1} y_n - u) + u\|^2 - \frac{t_n}{k_n} (w - u)\|^2
\]

\[
\leq \left(\frac{t_n}{k_n}\right)^2 \|T(PT)^{n-1} y_n - T(PT)^{n-1} w\|^2
\]

\[
= \frac{t_n^2}{k_n} \|\left(y_n - u\right) - (w - u)\|^2.
\]

So, we have

\[
0 \geq \|\left(y_n - u\right) - \frac{t_n}{k_n} (w - u)\|^2 - \|t_n (y_n - u) - t_n (w - u)\|^2
\]

\[
= 2 \langle (y_n - u) - \frac{t_n}{k_n} (w - u) - t_n (y_n + u) + t_n (w - u), J(t_n (y_n - w))\rangle
\]

\[
= 2 \langle (1 - t_n) (y_n - u) - \frac{t_n}{k_n} (1 - k_n) (w - u), J(t_n (y_n - w))\rangle
\]

\[
= 2 (1 - t_n) t_n \langle y_n - u - \frac{t_n}{k_n} (1 - k_n) (w - u), J(y_n - w)\rangle
\]

\[
= 2 (1 - t_n) t_n \langle y_n - u - b_n (w - u), J(y_n - w)\rangle,
\]

where \(b_n = \frac{t_n (1 - k_n)}{k_n (1 - t_n)} \rightarrow 0\) as \(n \rightarrow \infty\), and hence

\[
\langle y_n - u, J(y_n - w)\rangle \leq b_n \langle (w - u), J(y_n - w)\rangle. \tag{3.10}
\]

Thus putting \(w = y\),

\[
\langle y_n - u, J(y_n - y)\rangle \leq b_n \langle (y - u), J(y_n - y)\rangle. \tag{3.11}
\]

From Lemma 2.1, it follows that

\[
\mu_n \langle z - y, J(y_n - y)\rangle \leq 0\) for all \(z \in C\).
\]

Putting \(z = u\), we have

\[
\mu_n \langle u - y, J(y_n - y)\rangle \leq 0. \tag{3.12}
\]

Combining (3.10) and (3.11), we get

\[
\mu_n \langle y_n - y, J(y_n - y)\rangle \leq \mu_n b_n \langle (y - u), J(y_n - y)\rangle = 0.
\]
It follows that
\[ \mu_n \| y_n - y \|^2 \leq 0. \tag{3.13} \]
Therefore, there exists a subsequence \((y_{n_k})\) of \((y_n)\) such that \(\lim_{k \to \infty} y_{n_k} = y\). To complete the proof, suppose there is another subsequence \((y_{m_k})\) of \((y_n)\) which converges strongly to (say) \(y'\). Then \(y'\) is a fixed point of \(T\) and \(y' = y\). Therefore \((y_n)\) converges strongly to a fixed point of \(T\). □

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**References**


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