Edge-bipancyclicity of conditional faulty hypercubes

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Abstract

Xu et al. showed that for any set of faulty edges \(F\) of an \(n\)-dimensional hypercube \(Q_n\) with \(|F| \leq n - 1\), each edge of \(Q_n - F\) lies on a cycle of every even length from 6 to \(2^n\), \(n \geq 4\), provided not all edges in \(F\) are incident with the same vertex. In this paper, we show that, under similar condition, the number of faulty edges can be much greater and the same result still holds. More precisely, we show that, for up to \(|F| = 2n - 5\) faulty edges, each edge of the faulty hypercube \(Q_n - F\) lies on a cycle of every even length from 6 to \(2^n\) with each vertex having at least two healthy edges adjacent to it, for \(n \geq 3\). Moreover, this result is optimal in the sense that there is a set \(F\) of \(2n - 4\) conditional faulty edges in \(Q_n\) such that \(Q_n - F\) contains no hamiltonian cycle.

Keywords: Cycles; Pancyclic; Conditional fault; Hypercube; Fault-tolerant; Interconnection networks

1. Introduction

The ring embedding problem, which deals with all the possible lengths of the cycles in a given graph, is investigated in a lot of interconnection networks [2–4]. If a graph contains cycles of all lengths, it is called pancyclic [7]. Bipancyclicity is essentially a restriction of the concept of pancyclicity to cycles of even lengths. A bipartite graph is vertex-bipancyclic [6] if every vertex lies on a cycle of every even length from 4 to \(|V(G)|\). Similarly, a bipartite graph is edge-bipancyclic if every edge lies on a cycle of every even length from 4 to \(|V(G)|\). A bipartite graph is \(k\)-edge-fault-tolerant edge-bipancyclic if \(G - F\) remains edge-bipancyclic for any set of faulty edges \(F \subseteq E(G)\) with \(|F| \leq k\). A path \(P\) is a sequence of adjacent vertices, written as \(\langle v_0, v_1, \ldots, v_m \rangle\). The length of a path \(P\), denoted by \(l(P)\), is the number of edges in \(P\). A hamiltonian cycle is a cycle which includes every vertex of \(G\). In addition, we call \(e\) a healthy edge when \(e\) is fault-free in a graph.

Chan and Lee [1] considered an injured \(n\)-dimensional hypercube where each vertex is incident with at least two healthy edges, and proved that it still contains a hamiltonian cycle even it has \((2n - 5)\) edge faults. Tsai [8] proved that such injured hypercube \(Q_n\) contains a cycle of every even length from 4 to \(2^n\), even if it has up to \((2n - 5)\) edge faults. Recently, Xu et al. [9] showed that for any set of faulty edges \(F\) of \(Q_n\) with \(|F| \leq n - 1\), each edge of \(Q_n - F\) lies on a cycle of every even length from 6 to \(2^n\), \(n \geq 4\), provided not...
all faulty edges are incident with the same vertex. We observe that not all faulty edges are incident with the same vertex is equivalent to stating that each vertex has at least two healthy edges adjacent to it, if $|F| \leq n - 1$.

In this paper, we consider a set of faulty edges satisfying the condition that each vertex of $Q_n - F$ is incident with at least two healthy edges. Such a set of faulty edges $F$ is called a set of conditional faulty edges and $Q_n - F$ is called a conditional faulty hypercube. We find that under this condition, the number of faulty edges can be much greater and the same result still holds. We show that, for up to $|F| = 2n - 5$ conditional faulty edges, each edge of a faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to $2^n$, for $n \geq 3$. We observe that, if $|F| < 2n - 5$, we may arbitrarily delete some more edges to make a faulty edge set $F' \supseteq F$ and $|F'| = 2n - 5$. If our result holds for $F'$, it holds for $F$. From now on, we shall assume $|F| = 2n - 5$.

The above result is optimal in the sense that the result cannot be guaranteed, if there are $2n - 4$ conditional faulty edges. For example, take a cycle of length four in $Q_n$, let $(u_1, u_2, u_3, u_4)$ be the consecutive vertices on this cycle. Suppose that all the $(n - 2)$ edges incident with vertex $u_1$ (respectively, vertex $u_3$) are faulty except those two edges on the four cycle are healthy. There are $2(n - 2)$ conditional faulty edges. Then there does not exist a hamiltonian cycle in this faulty $Q_n$, for $n \geq 3$.

We now give a formal definition of a hypercube. An $n$-dimensional hypercube is denoted by $Q_n$ with the vertex set $V(Q_n)$ and the edge set $E(Q_n)$. Each vertex $u$ of $Q_n$ can be distinctly labeled by a $n$-bit binary strings, $u = u_{n-1}u_{n-2}...u_10$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. Let $u$ and $v$ be two adjacent vertices. If the binary labels of $u$ and $v$ differ in $i$th position, then the edge between them is said to be in $i$th dimension and the edge $(u, v)$ is called an $i$th dimension edge. Let $i$ be a fixed position, we use $Q^0_{n-1}$ to denote the subgraph of $Q_n$ induced by $\{u \in V(Q_n) | u_i = 0\}$ and $Q^1_{n-1}$ to denote the subgraph of $Q_n$ induced by $\{u \in V(Q_n) | u_i = 1\}$. We say that $Q_n$ is decomposed into $Q^0_{n-1}$ and $Q^1_{n-1}$ by dimension $i$, and $Q^0_{n-1}$ and $Q^1_{n-1}$ are $(n - 1)$-dimensional subcube of $Q_n$ induced by the vertices with the $i$th bit position being 0 and 1, respectively. $Q^0_{n-1}$ and $Q^1_{n-1}$ are all isomorphic to $Q_{n-1}$. For each vertex $u \in V(Q^0_{n-1})$, there is exactly one vertex in $Q^1_{n-1}$, denoted by $u^{(1)}$, such that $(u, u^{(1)}) \in E(Q_n)$. Conversely, for each $u \in V(Q^1_{n-1})$, there is one vertex in $Q^0_{n-1}$, denoted by $u^{(0)}$, such that $(u, u^{(0)}) \in E(Q_n)$. Let $D_i$ be the set of all edges with one end in $Q^0_{n-1}$ and the other in $Q^1_{n-1}$. These edges are called crossing edges in the $i$th dimension between $Q^0_{n-1}$ and $Q^1_{n-1}$. We also call $D_i$ the set of all $i$th dimension edges. Consequently, $|D_i| = 2^{n-1}$ for all $0 \leq i \leq n - 1$.

## 2. Some preliminaries

To prove our main theorem, we need some preliminary results.

**Lemma 1.** (See [5].) $Q_n$ is edge-bipancyclic, and is $(n - 2)$-edge-fault-tolerant edge-bipancyclic, for $n \geq 3$.

**Lemma 2.** (See [9].) Each edge of $Q_4 - F$ lies on a cycle of every even length from 6 to $2^n$ for any $F \subset E(Q_4)$ with $|F| = 3$, provided not all the faulty edges in $F$ are incident with the same vertex.

**Lemma 3.** (See [9].) Any two edges in $Q_n$ are included in a hamiltonian cycle, for $n \geq 2$.

The above lemma can be improved; In addition, we have the following lemmas to simplify our proof.

**Lemma 4.** Let $C_0 = \langle u, P_0, v, u \rangle$ be a cycle in $Q^0_{n-1}$ with its even length from $l_0$ to $2^{n-1}$, and $C_1 = \langle u^{(1)}, P_1, v^{(1)}, u^{(1)} \rangle$ be a cycle in $Q^1_{n-1}$ with its even length from $l_0$ to $2^{n-1}$. Then $C = \langle u, P_0, v, v^{(1)}, P_1, u^{(1)}, u \rangle$ is a cycle in $Q_n$ with its even length from $l_0 + l_1$ to $2^n$.

**Proof.** The proof of this lemma is omitted. \qed

**Lemma 5.** Let $Q_n$ be an $n$-dimensional hypercube, $n \geq 2$, and let $e_1$ and $e_2$ be two edges in the same dimension $i$. Then there exists another dimension $j \neq i$ such that decomposing $Q_n$ into $Q^0_{n-1}$ and $Q^1_{n-1}$ by dimension $j$, we have (1) neither $e_1$ nor $e_2$ is a crossing edge, (2) not $e_1$ and $e_2$ are in the same subcube.

**Proof.** Let $e_1 = (a, b)$ and $e_2 = (s, t)$ be two edges in the same dimension $i$. Let $a = a_n...a_i...a_1$ and $s = s_n...s_i...s_1$. Then $b = a_n...a_i...a_1$ and $t = s_n...s_i...s_1$. Since $e_1 \neq e_2$ and $n \geq 2$, there exists another dimension $j \neq i$, such that $a_j \neq s_j$. We decompose $Q_n$ into $Q^0_{n-1}$ and $Q^1_{n-1}$ by dimension $j$. Then, $e_1$ and $e_2$ are not crossing edges and are in the different subcubes. \qed

**Lemma 6.** Consider an $n$-dimensional hypercube $Q_n$, for $n \geq 4$. Let $e_0$, $e_1$ and $e_2$ be any three edges in $Q_n$, there is a cycle $C$ containing $e_1$ and $e_2$ in $Q_n - \{e_0\}$ with the length $l(C) = 2^n, 2^n - 2$ and $2^n - 4$. 


Lemma 4. To prove this lemma, we consider the following two cases:

**Case 1:** Both $e_1$ and $e_2$ are in the same dimension, say dimension $i$. By Lemma 5, we can choose a dimension $j$ such that $e_1$ and $e_2$ are in different subcubes. Without loss of generality, we assume that $e_1$ is in $Q_{n-1}^0$ and $e_2$ is in $Q_{n-1}^1$. We then consider two cases:

1. $e_0$ is not a crossing edge. We assume, without loss of generality, that $e_0$ is in $Q_{n-1}^0$. By Lemma 1, in $Q_{n-1}^0 = \{e_0\}$, there exists a cycle $C_0$ of every even length $4 \leq l(C_0) \leq 2^{n-1}$ going through $e_1$. Since $n \geq 4$, we can choose an edge $(u, v)$ on cycle $C_0$ such that $(u, v) \neq e_1$, and $(u^{(1)}, v^{(1)}) \neq e_2$. By Lemma 3, in $Q_{n-1}^0$, there exists a cycle $C_1$ of length $2^{n-1}$ going through $e_2$ and $(u^{(1)}, v^{(1)})$. Thus, the conclusion follows according to Lemma 4.

2. $e_0$ is a crossing edge. By Lemma 1, there exists a cycle $C_0$ of every even length $4 \leq l(C_0) \leq 2^{n-1}$ going through $e_1$ in $Q_{n-1}^0$. We can choose an edge $(u, v)$ on cycle $C_0$ such that $(u, v)$ is not adjacent to $e_0$ and $(u, v) \neq e_1$ and $(u^{(1)}, v^{(1)}) \neq e_2$. Since $n \geq 4$, we can choose an edge $(u, v)$ on cycle $C_0$ such that $(u, v)$ is not adjacent to $e_0$ and $(u, v) \neq e_1$. By Lemma 3, in $Q_{n-1}^0$, there exists a cycle $C_1$ of length $2^{n-1}$ going through $e_2$ and $(u^{(1)}, v^{(1)})$. Thus, the conclusion follows according to Lemma 4.

**Case 2:** $e_1$ and $e_2$ are in different dimensions. Suppose that $e_0$ is in the $i$th dimension. We decompose $Q_n$ into $Q_{n-1}^0$ and $Q_{n-1}^1$ by dimension $i$. Then, $e_0$ is a crossing edge. Next, we consider two further cases:

1. Either $e_1$ or $e_2$ is a crossing edge. Without loss of generality, we assume that $e_1$ is a crossing edge, and $e_2$ is in $Q_{n-1}^0$. Let $e_1 = (u, u^{(1)})$, where $u \in V(Q_{n-1}^0)$ and $u^{(1)} \in V(Q_{n-1}^1)$. Since $n \geq 4$, there is a neighbor of $u$, say $v$, such that $(u, v) \neq e_2$ and $(u, v)$ is not adjacent to $e_0$. By Lemma 3, there exists a Hamiltonian cycle $C_0$ going through $(u, v)$ and $e_2$ in $Q_{n-1}^0$. By Lemma 1, in $Q_{n-1}^1$, there exists a cycle $C_1$ of every even length $4 \leq l(C_1) \leq 2^{n-1}$ going through $(u^{(1)}, v^{(1)})$. Thus, the conclusion follows.

2. Both $e_1$ and $e_2$ are not crossing edges. If $e_1$ and $e_2$ are in different subcubes, this subcase is similar to case 1.2, and the proof is omitted. Otherwise, both $e_1$ and $e_2$ are in the same subcube. We assume, without loss of generality, that $e_1$ and $e_2$ are in $Q_{n-1}^0$. By Lemma 3, there exists a Hamiltonian cycle $C_0$ going through $e_1$ and $e_2$ in $Q_{n-1}^0$, and $l(C_0) = 2^{n-1}$. Since $n \geq 4$, there is a third edge $(u, v)$ other than $e_1$ and $e_2$ on cycle $C_0$, and $(u, v)$ is not adjacent to $e_0$. By Lemma 1, there exists a cycle $C_1$ of every even length $4 \leq l(C_1) \leq 2^{n-1}$ going through $(u^{(1)}, v^{(1)})$ in $Q_{n-1}^1$. By Lemma 4, the conclusion follows. □

Let $F$ be a set of faulty edges of $Q_n$. Suppose that we decompose $Q_n$ into $Q_{n-1}^0$ and $Q_{n-1}^1$ by dimension $j$, and let $F_L = F \cap E(Q_{n-1}^0)$, $F_R = F \cap E(Q_{n-1}^1)$. Suppose that $F$ is a set of conditional faulty edges of $Q_n$. If we arbitrarily decompose $Q_n$ into $Q_{n-1}^0$ and $Q_{n-1}^1$ by a dimension, $F_L$ and $F_R$ may not be conditional faulty edges in $Q_{n-1}^0$ and $Q_{n-1}^1$, respectively. However, we will show that it is always possible to find some suitable dimension such that decomposing by this dimension, both $F_L$ and $F_R$ are conditional faulty sets in $Q_{n-1}^0$ and $Q_{n-1}^1$, respectively.

**Lemma 7.** Consider an $n$-dimensional hypercube $Q_n$, for $n \geq 4$. Let $F$ be a set of conditional faulty edges with $|F| = 2n - 5$. There are at most two vertices in $Q_n$ incident with $(n - 2)$ faulty edges.

**Proof.** If there are three vertices in $Q_n$ incident with $(n - 2)$ faulty edges, the number of faulty edge $F$ is at least $3n - 8$. However, $(3n - 8) > (2n - 5)$ for all $n \geq 4$ which is a contradiction. □

**Lemma 8.** Consider an $n$-dimensional hypercube $Q_n$, $n \geq 4$. Let $F$ be a set of conditional faulty edges with $|F| = 2n - 5$. If there are two vertices $x$ and $y$ both incident with $n - 2$ faulty edges, then $x$ and $y$ are adjacent in $Q_n$ and the edge $(x, y)$ is a faulty edge. Suppose that $(x, y)$ is in dimension $j$. Then decomposing $Q_n$ into $Q_{n-1}^0$ and $Q_{n-1}^1$ by dimension $j$, both $F_L$ and $F_R$ are sets of conditional faulty edges in $Q_{n-1}^0$ and $Q_{n-1}^1$, respectively. Moreover, $|F_L| \leq 2n - 6$ and $|F_R| \leq 2n - 6$.

**Proof.** If there are two vertices $x$ and $y$ in $Q_n$ incident with $(n - 2)$ faulty edges, then these two vertices are connected by a faulty edge. Otherwise, $|F| = 2(n - 2) = 2n - 4 > 2n - 5$ which is a contradiction. Suppose the edge $(x, y)$ is in dimension $j$, we decompose $Q_n$ into two subcubes. It is clearly that each vertex in $Q_{n-1}^0$ and $Q_{n-1}^1$ is still incident with at least two healthy edges, and both $F_L$ and $F_R$ are conditional faulty edges in $Q_{n-1}^0$ and $Q_{n-1}^1$, respectively. Then, $|F_L| \leq |F_R| = n - 3 \leq 2n - 6$, for $n \geq 4$. □

**Lemma 9.** Consider an $n$-dimensional hypercube $Q_n$, for $n \geq 4$. Let $F$ be a set of conditional faulty edges with $|F| = 2n - 5$. Suppose that there exists exactly one vertex $x$ having $(n - 2)$ faulty edges incident with it. Since $n - 2 \geq 2$, let $e_1$ and $e_2$ be two faulty edges incident with $x$, and let $e_1$ and $e_2$ be $j$th and $k$th dimension edges, respectively. Then decomposing $Q_n$ into $Q_{n-1}^0$ and $Q_{n-1}^1$ by either one of these two dimensions $j$ and
Lemma 10. Let $Q_n$ be an $n$-dimensional hypercube, $F$ be a set of faulty edges with $|F| \geq 2$, and $e$ be a healthy edge, $n \geq 2$. Then there exists a dimension $j$, decomposing $Q_n$ into $Q^0_{n-1}$ and $Q^1_{n-1}$ by this dimension, such that $e$ is not a crossing edge and not all the faulty edges are in the same subcube.

Proof. Suppose that $e = (u, v)$ is in dimension $i$. If there is a faulty edge $f$ not in dimension $i$, say in dimension $j$, we decompose $Q_n$ into $Q^0_{n-1}$ and $Q^1_{n-1}$ by dimension $j$. Then $f$ is a crossing edge but $e$ is not, and all the faulty edges are not in the same subcube. Otherwise, all the faulty edges are in the same dimension $i$ as $e$ is in. We now choose any two faulty edges $f_1$ and $f_2$ in $F$. By Lemma 5, $Q_n$ can be decomposed into $Q^0_{n-1}$ and $Q^1_{n-1}$ by some dimension $j \neq i$ such that edges $f_1$ and $f_2$ are not in the same subcube, and $e$ is not a crossing edge. □

3. Main theorem

We now prove our main result.

Theorem 1. Let $Q_n$ be an $n$-dimensional hypercube, and $F$ be a set of conditional faulty edges with $|F| \leq 2n - 6$. Then each edge of the conditional faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to $2^n$, for $n \geq 3$.

Proof. We prove this theorem by induction on $n$. For $n = 3$, since $2n - 6 = n - 2$, by Lemma 1, the result is true. For $n = 4$, $2n - 6 = n - 1$, by Lemma 2, the result holds. Assume the theorem holds for $n - 1$, for some $n \geq 5$, we shall show that it is true for $n$.

As we mentioned before, we may assume $|F| = 2n - 6$. Let $e = (u, v)$ be an edge in $Q_n - F$. We shall find a cycle of every even length from 6 to $2^n$ passing through $e$ in $Q_n - F$. Assume that $e$ is an $i$th dimension edge, $e \in D_i$, for some $i \in \{1, 2, \ldots, n\}$. The proof is divided into three major cases:

Case 1: There are two vertices $x$ and $y$ in $Q_n$ incident with $(n - 2)$ faulty edges. By Lemma 8, $(x, y)$ is an edge in $Q_n$ and is a faulty edge. We denote this edge by $e_f$. Suppose that $e_f$ is a $j$th dimension edge. We decompose $Q_n$ into $Q^0_{n-1}$ and $Q^1_{n-1}$ by dimension $j$. We then consider two further cases:

1.1: $e_f = (x, y)$ and $e = (u, v)$ are in the same dimension. Thus, $j = i$ and $e_f \in D_i$ (Fig. 1(a)). In this case, $e$ is an edge crossing $Q^0_{n-1}$ and $Q^1_{n-1}$. Without loss of generality, assume that $u \in V(Q^0_{n-1})$ and $v \in V(Q^1_{n-1})$. Since $n \geq 5$, $u$ has a neighboring vertex $w \in V(Q^0_{n-1})$, by the definition of hypercube, $w^{(1)}$ is a neighbor of $v$ such that the edge $(w, w^{(1)})$ is a healthy edge and $(w, w^{(1)})$ is a crossing edge between

Fig. 1. Illustration for theorem.
Let $Q^0_{n-1}$ and $Q^1_{n-1}$. By Lemma 1, there exists a cycle $C_0$ in $Q^0_{n-1} - F_L$ passing through $(u, w)$ of every even length $4 \leq l(C_0) \leq 2^{n-1}$ and a cycle $C_1$ in $Q^0_{n-1} - F_R$ going through $(v, w^{(1)})$ of every even length $4 \leq l(C_1) \leq 2^{n-1}$. We write $C_0$ as $(u, P_0, w, u)$, and $C_1$ as $(v, P_1, w^{(1)}, v)$. Thus, $(u, P_0, w, w^{(1)}, v, u)$ is a cycle of length 6 with $l(P_0) = 3$. By Lemma 4, $(u, P_0, w, w^{(1)}, P_1, v, u)$ can form a cycle of every even length from 8 to $2^n$ through $e$ in $Q_n - F$.

**1.2:** $e_f$ and $e$ are in different dimensions. Thus, $j \neq i$ and $e_f \notin D_j$ (Fig. 1(b)). In this case, $e$ is in $Q^0_{n-1}$ or $Q^1_{n-1}$. Without loss of generality, we may assume that $e \in E(Q^0_{n-1})$. By Lemma 1, there exists a cycle $C$ in $Q^0_{n-1} - F_L$ going through the edge $e$ of every even length $l$, $6 \leq l \leq 2^{n-1}$. Let $C_0$ be a cycle of length $2^{n-1} - 2$ or $2^{n-1}$ passing through $e$ in $Q^0_{n-1} - F_L$. Since $n \geq 5$, there exists an edge $(s, t)$ on $C_0$ such that neither $s$ nor $t$ is adjacent to $e_f$ and $(s, t) \neq e$. By definition, $(s^{(1)}, t^{(1)})$ is an edge in $Q^0_{n-1}$ and $(s^{(1)}, t^{(1)})$ is a cycle in $Q^0_{n-1}$ and $(s^{(1)}, t^{(1)})$ is an edge of every even length $4 \leq l(C_1) \leq 2^{n-1}$. Thus, the conclusion follows according to Lemma 4.

**Case 2:** There is exactly one vertex in $Q_n$ incident with $(n-2)$ faulty edges. Let $x$ be the vertex having $(n-2)$ faulty edges incident with it. Let $f_1$ and $f_2$ be two faulty edges incident with $x$, so $f_1$ and $f_2$ are in different dimensions $j$ and $k$. By Lemma 9, decomposing $Q_n$ into $Q^0_{n-1}$ and $Q^1_{n-1}$ by either $j$th or $k$th dimension, both $F_L = F \cap E(Q^1_{n-1})$ and $F_R = F \cap E(Q^0_{n-1})$ are sets of conditional faulty edges in $Q^0_{n-1}$ and $Q^1_{n-1}$, respectively. Between dimension $j$ and $k$, we choose one to decompose $Q_n$ into $Q^0_{n-1}$ and $Q^1_{n-1}$ say dimension $j$, such that the required edge $e$ is not a crossing edge. Therefore, there is an edge $e_f$ crossing $Q^0_{n-1}$ and $Q^1_{n-1}$, we denote this edge by $e_f$, and $e_f \notin F \cap D_j$ is incident with $x$. Without loss of generality, we may assume that $x \in V(Q^0_{n-1})$.

**2.1:** Suppose $|F_L| \leq 2n - 7$ and $|F_R| \leq 2n - 7$ (Fig. 1(c)). Without loss of generality, we further assume that $e \in E(Q^0_{n-1})$. By induction hypothesis, there exists a cycle $C$ in $Q^0_{n-1} - F_L$ of every even length $6 \leq l(C) \leq 2^{n-1}$ passing through $e$. Let $C_0$ be a cycle of length $2^{n-1} - 4 \leq l(C_0) \leq 2^{n-1}$ through $e$ in $Q^0_{n-1} - F_L$. Since $|C_0 - e| \geq 2^{n-1} - 4 - 1 > 2(2n - 5) \geq 2|F \cap D_j|$, for all $n \geq 5$. There exists an edge $(s, t)$ on $C_0$ such that $(s, t)$ is not $e$, and both $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. By induction hypothesis, there exists a cycle $C_1$ in $Q^1_{n-1} - F_R$ of every even length $6 \leq l(C_1) \leq 2^{n-1}$ passing through $(s^{(1)}, t^{(1)})$. By Lemma 4, the conclusion follows.

**2.2:** $|F_L| = 2n - 6$. In this case, $|F \cap D_j| = 1$ and $|F \cap E(Q^1_{n-1})| = |F_R| = 0$.

**2.2.1:** $e$ is in subcube $Q^0_{n-1}$. To find a cycle of length 6 passing through $e = (u, v)$, we discuss the case that whether $e$ is incident with $x$ or not. If $e$ is incident with $x$, without loss of generality, we assume that $u = x$ (Fig. 1(d)). Thus, $(u, v^{(1)})$ is a healthy edge. Since $F_L$ is a set of conditional faulty edges in $Q^0_{n-1}$, vertex $u = x$ has two healthy edges incident with it. Let $w$ be a neighbor of $u$ in $Q^0_{n-1}$ such that $(w, u)$ and $(w, u^{(1)})$ are healthy edges and $w \neq v$. Thus, $(u, v, v^{(1)}, u^{(1)}, w, u)$ is a cycle of length 6 in $Q_n - F$. Otherwise, $e$ is not incident with $x$, then $(u, v^{(1)})$ and $(w, v^{(1)})$ are healthy edges (Fig. 1(e)). By Lemma 1, there exists a cycle $C_1 = (u^{(1)}, P_1, v^{(1)}, u^{(1)})$ of length four in $Q^0_{n-1}$ through the edge $(u^{(1)}, v^{(1)})$. Thus, $(u, v^{(1)}, P_1, v^{(1)}, v, u)$ is a cycle of length 6 in $Q_n - F$, where $l(P_1) = 3$.

Let $e_1$ be a faulty edge in $Q^0_{n-1}$ that is not adjacent to $e_f$. Though $e_1$ is a faulty edge, we treat it as a healthy edge temporarily, then the total number of faulty edge in $Q^0_{n-1}$ is $2n - 7$. By induction hypothesis, there exists a cycle $C_0$ of every even length $6 \leq l(C_0) \leq 2^{n-1}$ going through $e$ in $Q^0_{n-1} - (F_L - \{e_1\})$. If $C_0$ passes $e_1$, we choose $e_1$, or else, we choose any one edge other then $e$ on $C_0$ which is not adjacent to $e_f$. Let the chosen edge be denoted by $(s, t)$. We write cycle $C_0$ as $(s, P_0, t, s)$. Since $|F \cap D_j| = 1$ and $|F_R| = 0$, $(s, s^{(1)})$, $(t, t^{(1)})$ and $(s^{(1)}, t^{(1)})$ are all healthy edges. Thus, $(s, P_0, t, s^{(1)}, s^{(1)}, s)$ is a cycle of length $8$ in $Q_n - F$ if $l(P_0) = 5$. Suppose that $10 \leq l \leq 2^n$ and $l$ is even. By Lemma 1, in $Q^0_{n-1}$, there exists a cycle $C_3$ of length $4 \leq l(C_3) \leq 2^{n-1}$ passing through $(s^{(1)}, t^{(1)})$. We write $C_3$ as $(s^{(1)}, P_3, t^{(1)}, s^{(1)})$. By Lemma 4, $(s, P_0, t, s^{(1)}, P_3, s^{(1)}, s, t)$ is a cycle of length $l$ through $e$ in $Q_n - F$.

**2.2.2:** $e$ is in subcube $Q^1_{n-1}$ (Fig. 1(f)). By Lemma 1, there exists a cycle $C$ of every even length $4 \leq l \leq 2^{n-1}$ passing through $e$ in $Q^1_{n-1}$. Suppose that $2^{n-1} + 2 \leq l \leq 2^n$ and $l$ is even. Since $F_L$ is a set of conditional faulty edges, there are at most $(n-3)$ faulty edges adjacent to $e_f$ in $Q^0_{n-1}$. For $n \geq 5$, $n - 3 \geq 2$, we can choose a faulty edge $e_2 = (s, t)$ in $Q^0_{n-1}$ such that $e_2$ is not adjacent to $e_f$ and $(s^{(1)}, t^{(1)})$ is not $e$. Treating the edge $e_2$ as a healthy edge, by induction hypothesis, there exists a cycle $C_0$ of length $6 \leq l(C_0) \leq 2^{n-1}$ going through $e_2$ in $Q^0_{n-1} - (F_L - \{e_2\})$. We observe that $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. By Lemma 6, there exists a
cycle \( C_1 \) of every length \( 2^{n-1} - 4, 2^{n-1} - 2, \) or \( 2^{n-1} \) through \((s^{(1)}, t^{(1)})\) and \( e \) in \( Q^{1}_{n-1} \). By Lemma 4, the conclusion follows.

**Case 3:** Every vertex in \( Q_n \) is incident with at most \((n - 3)\) faulty edges. In this case, suppose that \( e = (u, v) \) is in dimension \( i \). By Lemma 10, \( Q_n \) can be decomposed into \( Q^{0}_{n-1} \) and \( Q^{1}_{n-1} \) by a dimension \( j \) different from \( i \) such that \( e \) is not a crossing edge and not all the faulty edges are in the same subcube. Then \(|F_L| \leq 2n - 6\) and \(|F_R| \leq 2n - 6\). Next, we consider two further cases:

**3.1:** At least one faulty edge is a \( j \)th dimension edge. Thus, \(|F \cap D_j| \neq 0\).

We then consider two cases: (a) \(|F_L| \leq 2n - 7\) and \(|F_R| \leq 2n - 7\), and (b) \(|F_L| = 2n - 6\) or \(|F_R| = 2n - 6\). The proof of this subcase is exactly the same as that of case 2.

**3.2:** None of the faulty edges is a \( j \)th dimension edge. Thus, \(|F \cap D_j| = 0\).

**3.2.1:** \(|F_L| \leq 2n - 7\) and \(|F_R| \leq 2n - 7\). Without loss of generality, we may assume that \( e \in E(Q^{0}_{n-1}) \). By induction hypothesis, there exists a cycle \( C \) of every even length \( 6 \leq l(C) \leq 2^{n-1} \) in \( Q^{0}_{n-1} \) passing through \( e \). Let \( C_0 \) be a cycle of every even length \( 2^{n-1} - 4 \leq l(C_0) \leq 2^{n-1} \) going through \( e \) in \( Q^{0}_{n-1} \). There exists an edge \((s, t)\) other than \( e \) in \( C_0 \). Since \(|F \cap D_j| = 0\), \((s, t^{(1)})\) and \((t, t^{(1)})\) are healthy edges. We write \( C_0 \) as \((s, P_0, t, s)\). By induction hypothesis, there exists a cycle \( C_1 \) of every even length \( 6 \leq l(C_1) \leq 2^{n-1} \) in \( Q^{1}_{n-1} \). Thus, the conclusion follows according to Lemma 4.

**3.2.2:** Suppose \(|F_L| = 2n - 6\) or \(|F_R| = 2n - 6\), say the former case. In this case, \(|F_R| = 1\). We then consider two cases: (a) \( e \) is in subcube \( Q^{0}_{n-1} \), and (b) \( e \) is in subcube \( Q^{1}_{n-1} \).

(a) \( e = (u, v) \) is in subcube \( Q^{0}_{n-1} \). Since \(|F \cap D_j| = 0\), both \((u, u^{(1)})\) and \((v, v^{(1)})\) are healthy edges. Let \( l \) be an even number with \( 6 \leq l \leq 2^{n-1} \). By Lemma 1, there exists a cycle \( C_1 \) of every even length from \( 4 \) to \( 2^{n-1} \) passing through \((u^{(1)}, v^{(1)})\) in \( Q^{1}_{n-1} \). We write \( C_1 \) as \((u^{(1)}, P_1, v^{(1)}, u^{(1)})\). No matter \((u^{(1)}, v^{(1)})\) is healthy or not, \((u, u^{(1)})\) forms a cycle of length \( l \) through \( e \) in \( Q^0_n \). Suppose that \( 2^{n-1} - 2 \leq l \leq 2^n \). Let \( e_1 \) be a faulty edge in \( Q^{0}_{n-1} \). We may treat \( e_1 \) as a healthy edges temporarily. By induction hypothesis, there exists a cycle \( C_0 \) of length \( 6 \leq l(C_0) \leq 2^{n-1} \) going through \( e \) in \( Q^{0}_{n-1} \). If \( C_0 \) passes the edge \( e_1 \), we choose \( e_1 \) to be deleted. Otherwise, we choose another edge other than \( e \) on cycle \( C_0 \). Let the chosen edge be denoted by \((s, t)\). We write the cycle \( C_0 \) as \((s, P_0, t, s)\). Treating \((s^{(1)}, t^{(1)})\) as a healthy edge, by Lemma 1, there exists a cycle \( C_3 \) of every even length from \( 4 \) to \( 2^{n-1} \) passing through \((s^{(1)}, t^{(1)})\) in \( Q^{1}_{n-1} - \{F_L - (s^{(1)}, t^{(1)})\} \). By Lemma 4, the conclusion follows.

(b) \( e \) is in subcube \( Q^{1}_{n-1} \). Let \( e_1 \) be the only faulty edge in \( Q^{1}_{n-1} \). By Lemma 1, there exists a cycle \( C \) of every even length from \( 6 \) to \( 2^{n-1} \) through \( e \) in \( Q^{1}_{n-1} \). Suppose that \( 2^{n-1} + 2 \leq l \leq 2^n \), and \( l \) is even. Let \( e_0 = (s, t) \) be a faulty edge in \( Q^{0}_{n-1} \) such that \((s^{(1)}, t^{(1)}) \neq e \) and \((s^{(1)}, t^{(1)}) \neq e_1 \). By induction hypothesis, there exists a cycle \( C_0 \) of length \( 6 \leq l(C_0) \leq 2^{n-1} \) in \( Q^{0}_{n-1} \). Going through \( e_0 \). If \((s^{(1)}, t^{(1)}) = e_1 \), treat \( e_1 \) as a healthy edge temporarily, by Lemma 6, there exists a cycle \( C_1 \) of length \( 2^{n-1} - 4 \), \( 2^{n-1} - 2 \), respectively, going through both \((s^{(1)}, t^{(1)})\) and \( e \) in \( Q^{1}_{n-1} \). By Lemma 4, the conclusion follows. Otherwise, if \((s^{(1)}, t^{(1)}) \neq e_1 \), by Lemma 6, there exists a cycle \( C_3 \) of length \( 2^{n-1} - 2 \), or \( 2^{n-1} - 4 \), respectively, going through both \( e \) and \((s^{(1)}, t^{(1)})\) in \( Q^{1}_{n-1} \). Thus, the conclusion follows according to Lemma 4.

This completes the proof. \( \square \)

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**References**


