DIFFERENTIAL TRANSFORM METHOD FOR SOLVING FUZZY FRACTIONAL INITIAL VALUE PROBLEMS

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Abstract:
In this paper we present an approximate analytical solution for fuzzy fractional initial value problems (FFIVP's) of the form:

\[ y^{(q)}(x) = f(x, y(x)), \quad p - 1 < q \leq p, \quad p \in \mathbb{R} \]

\[ y^{(j)}(a) = y_{ja}, \quad j = 0, 1, ..., p - 1 \]

where the fuzzyness appeared in the initial conditions, to be fuzzy numbers, using the differential transform method. The solution of our model equations are calculated in the form of convergent series with easily computable components. Some examples are solved as illustrations using symbolic computation. The numerical results show that the followed approach is easy to implement and accurate when applied to fractional fuzzy initial value problems.

Keywords: Fractional calculus, Fuzzy differential equations, differential transform method.

1. Introduction:
Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models, which has been applied to a wide variety of real problems, for instance, the golden mean [Datta, 2003], practical systems [El-Nachie (a), 3005], quantum optics and gravity [El-Nachie (b), 2005], medicine [Abbod, 2001] and engineering problems.

The concept of fuzzy sets which was originally introduced by Zadeh [Zadeh, 1965] led to the definition of the fuzzy number and its implementation in fuzzy control [Change, 1972] and approximate reasoning problems [Zadeh, 1975], [Zadeh, 1983]. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [Mizumoto, 1976], [Mizumoto, 1979], Nahmias [Nahmias, 1978], Dubois and Prade [Dubois, 1978], [Dubois, 1980] and Ralescu [Ralescu, 1979], all of which observed the fuzzy number as a collection of \( \alpha \)-levels, \( 0 \leq \alpha \leq 1 \), [Negoiita, 1975].

In this paper, the approximate solution of fuzzy fractional order differential equation will be discussed, in which fractional differential equation could be considered as an important type of differential equations, where the differintegration that appears in the equation is of non-integer order.

Real life problems with fractional differential equations are of great importance, since fractional differential equations accumulate the whole information of the function in a weighted form. This has many applications in physics, chemistry, engineering, etc. For that reason, we need for a method to solve such equation, effectively, easy to use and applied in
different problems. The differential transform method will be used here to solve fuzzy fractional order differential equations.

The differential transform method was first applied in the engineering domain in [Zhou, 1986]. In general, the differential transform method is applied to the solution of electric circuit problems. The differential transform method is a numerical method based on the Taylor series expansion, which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the differential transform method obtains a polynomial series solution by means of iterative procedure. Recently, the application of differential transform method is successfully extended to obtain analytical approximate solutions to linear and nonlinear ordinary differential equations of fractional order [Arikoglu, 2006], [Erturk, 2008], and integro-differential equations of fractional order, [Artikoglu, 2009], [Nazari, 2010].

2. Fuzzy Sets:
In this section, we shall present some basic definitions of fuzzy sets including the definition of fuzzy numbers and fuzzy functions.

Definition (1), [Zadeh, 1965]:
Let X be any set of elements. A fuzzy set $\tilde{A}$ is characterized by a membership function $\mu_{\tilde{A}} : X \rightarrow [0, 1]$, and may be written as the set of points

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}.$$

Definition (2), [Zimmermann, 1985]:
The crisp set of elements that belong to the set $\tilde{A}$ at least to the degree $\alpha$ is called the weak $\alpha$-level set (or weak $\alpha$-cut), and is defined by:

$$A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$$

while the strong $\alpha$-level set (or strong $\alpha$-cut) is defined by:

$$A'_\alpha = \{x \in X : \mu_{\tilde{A}}(x) > \alpha\}$$

Definition (3), [Zadeh, 1965]:
A fuzzy subset $\tilde{A}$ of a universal space $X$ is convex if and only if the sets $A_\alpha$ are convex, $\forall \alpha \in [0, 1]$.
Or equivalently, we can define convex fuzzy set directly by using its membership function to satisfy:

$$\mu_{\tilde{A}}[\lambda x_1 + (1-\lambda)x_2] \geq \min \{ \mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2) \}$$

for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$.

Definition (4), [Zimmermann, 1985]:
A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line $\mathbb{R}$, such that:

1. There exists exactly one $x_0 \in \mathbb{R}$, with $\mu_{\tilde{M}}(x_0) = 1$ ($x_0$ is called the mean value of $\tilde{M}$).
2. $\mu_M(x)$ is piecewise continuous.

Now, the following two remarks illustrates the representation of a fuzzy number and fuzzy functions in terms of its $\alpha$-level sets, because they are more convenient to use in applications.

**Remark (1):**
A fuzzy number $\tilde{M}$ may be uniquely represented in terms of its $\alpha$-level sets, as the following closed intervals of the real line:

$$M_\alpha = [m - \sqrt{1 - \alpha}, m + \sqrt{1 - \alpha}]$$

or

$$M_\alpha = [\alpha m, \frac{1}{\alpha} m]$$

Where $m$ is the mean value of $\tilde{M}$ and $\alpha \in [0, 1]$. This fuzzy number may be written as $M_\alpha = [\underline{M}, \bar{M}]$, where $\underline{M}$ refers to the greatest lower bound of $M_\alpha$ and $\bar{M}$ to the least upper bound of $M_\alpha$.

**Remark (2):**
Similar to the second approach given in remark (1), one can fuzzyfy any crisp or nonfuzzy function $f$, by letting:

$$\tilde{f}(x) = \alpha f(x), \quad \bar{f}(x) = \frac{1}{\alpha} f(x), \quad x \in X, \quad \alpha \in (0, 1]$$

and hence the fuzzy function $\tilde{f}$ in terms of its $\alpha$-levels is given by $f_\alpha = [\tilde{f}, \bar{f}]$.

### 3. Riemann-Liouville and Caputo Fractional Derivatives

There are various types of definitions for the fractional derivatives of order $q > 0$, the most commonly used definitions among various definitions of fractional derivatives of order $q > 0$ are the Riemann-Liouville and Caputo formulas. Ones which use fractional integrals and derivatives of the whole order. The difference between the two definitions in the order of evaluation. Riemann-Liouville fractional integration of order $q$ is defined as:

$$J_t^q f(x) = \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-t)^{q-1} f(t) dt, \quad q > 0, \quad x > 0 \quad \text{............................................................ (1)}$$

The following equations define Riemann-Liouville and Caputo fractional derivatives of order $q$, respectively:

$$D_t^q f(x) = \frac{d^m}{dx^m} \left[ J_t^{m-q} f(x) \right] \quad \text{............................................................ (2)}$$

$$D_{*t}^q f(x) = J_t^{m-q} \left[ \frac{d^m}{dx^m} f(x) \right]$$

where $m - 1 \leq q < m$ and $m \in \mathbb{N}$. From (1) and (2), we have:

The differential transform of the \( k \)\(^{th} \) derivative of the function \( f \), is defined by:

\[
F(x) = \frac{1}{k!} \left( \frac{d^kf(x)}{dx^k} \right)_{x=x_0}
\]

and the differential inverse transform of \( F(x) \), is defined as:

\[
f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k
\]

From (4) and (5), we get:

\[
f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} \left( \frac{d^kf(x)}{dx^k} \right)_{x=x_0}
\]

which implies that the differential transform is derived from Taylor series expansion, but the method does not evaluate derivatives symbolically.

However, the corresponding derivatives are calculated recursively, and are defined by the transformed equation of the original functions. In practice, the function \( f \) is expressed by:

\[
f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k
\]

so the differential transform method is a numerical method based on Taylor series expansion, which constructs a solution in terms of polynomials.

5. The Differential Transform Method for Solving Fuzzy Initial Value Problems of Fractional Order:

Let us expand the analytic function \( f \) as the fractional power series:

\[
f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^{k/\alpha}
\]

where \( \alpha \) is the order of the fraction and \( F(k) \) is the fractional differential transform of \( f \). In order to avoid the fractional initial and boundary conditions, we define the fractional derivative in the Caputo sense:

The relation between the Riemann-Liouville and Caputo operators is given by:

\[
D^q_{x_0} f(x) = D^q_{x_0} \left[ f(x) - \sum_{k=0}^{\infty} \frac{1}{k!} (x-x_0)^k f^{(k)}(x_0) \right]
\]

Replacing \( f(x) \) by:
in (3) and using (7) we obtain the fractional derivative in the Caputo sense, as:

\[
D^q \ast_{x_0} f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{x_0}^{x} \frac{f(t) - \sum_{k=0}^{m-1} \frac{1}{k!} (t-x_0)^k f^{(k)}(x_0)}{(t-x)^{1+q-m}} \, dt
\]

Since the initial conditions are implemented by the integer-order derivative, the transformations of the initial conditions for \( k = 0, 1, \ldots, (\alpha q - 1) \), are defined by:

\[
F(x) = \begin{cases} 0, & k \not\in \mathbb{Z}^+ \\ \frac{k}{\alpha} & k \in \mathbb{Z}^+ \end{cases}
\]

where \( q \) is the order of the corresponding fractional equation [Erturtk, 2008].

The following theorems that may be deduced from eqs.(3) and (6) are given below, for proofs and details see [Arikoglu, 2006].

**Theorem (1):**
If \( f(x) = g(x) \pm h(x) \), then \( F(k) = G(k) \pm H(k) \), where \( F, G \) and \( H \) are the differential transforms of \( f, g \) and \( h \), respectively.

**Theorem (2):**
If \( f(x) = g(x) h(x) \), then \( F(k) = \sum_{\ell=0}^{k} G(\ell) H(k-\ell) \), where \( F, G \) and \( H \) are the differential transforms of \( f, g \) and \( h \), respectively.

**Theorem (3):**
If \( f(x) = g_1(x) g_2(x) \ldots g_{n-1}(x) g_n(x) \), then:

\[
F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1) G_2(k_2-k_1) \cdots G_{n-1}(k_{n-1}-k_{n-2}) G_n(k-k_{n-1})
\]

where \( G_1, G_2, \ldots, G_n \) are the differential transforms of \( g_1, g_2, \ldots, g_n \); respectively.

**Theorem (4):**
If \( f(x) = (x-x_0)^p \), then \( F(k) = \delta(k - \alpha p) \), where:

\[
\delta(k) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases}
\]
Theorem (5):

If \( f(x) = D^q_{x_0} [g(x)] \), then \( F(x) = \frac{\Gamma\left(q + 1 + \frac{k}{\alpha}\right)}{\Gamma\left(1 + \frac{k}{\alpha}\right)} G(k + \alpha q) \).

Theorem (6):

For the production of fractional derivatives in the most general form:

\[
f(x) = \frac{d^{q_1}}{dx^{q_1}} [g_1(x)] \frac{d^{q_2}}{dx^{q_2}} [g_2(x)] \ldots \frac{d^{q_n}}{dx^{q_n}} [g_n(x)]
\]

then:

\[
F(k) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \ldots \sum_{k_1=0}^{k_2} \frac{\Gamma\left(q_1 + 1 + \frac{k_1}{\alpha}\right) \Gamma\left(q_2 + 1 + \frac{k_2 - k_1}{\alpha}\right) \ldots \Gamma\left(q_{n-1} + 1 + \frac{k_{n-1} - k_{n-2}}{\alpha}\right) \Gamma\left(q_n + 1 + \frac{k - k_{n-1}}{\alpha}\right)}{\Gamma\left(1 + \frac{k_1}{\alpha}\right) \Gamma\left(1 + \frac{k_2 - k_1}{\alpha}\right) \ldots \Gamma\left(1 + \frac{k_{n-1} - k_{n-2}}{\alpha}\right) \Gamma\left(1 + \frac{k - k_{n-1}}{\alpha}\right)} \alpha_{q_i} \ldots G_{n-1}(k_{n-1} - k_{n-2} + \alpha q_{n-1}) G_n(k - k_{n-1} + \alpha q_n)
\]

where \( \alpha_{q_i} \in \mathbb{R}^+ \), for \( i = 1, 2, \ldots, n \); and \( G_1, G_2, \ldots, G_n \) are the differential transforms of \( D^{q_1} g_1, D^{q_2} g_2, \ldots, D^{q_n} g_n \), respectively.

Now, as an application of the differential transform method for solving fuzzy initial value problems of fractional order, consider the FFIVP's:

\[
y^{(q)}(x) = f(x, y(x)), \quad p - 1 < q \leq p, \quad p \in \mathbb{R}
\]

\[
y^{(j)}(a) = \tilde{y}_{ja}, \quad j = 0, 1, \ldots, p - 1
\]

where \( q \) is the order of the differential equation and \( \tilde{y}_{ja} \) are fuzzy numbers, for all \( j \). Therefore, the solution of eq.(9) will be a fuzzy function of the form \( y = [\underline{y}, \overline{y}] \), where \( \underline{y} \) and \( \overline{y} \) refers to the lower and upper solutions of \( \tilde{y} \), respectively.

Now, to find the lower solution \( \underline{y} \), we must solve using the differential transform method, the nonfuzzy fractional initial value problem:

\[
\underline{y}^{(q)}(x) = f(x, \underline{y}(x)), \quad p - 1 < q \leq p
\]

\[
\underline{y}^{(j)}(a) = y_{ja} - \sqrt{1 - \alpha}, \quad j = 0, 1, \ldots, p - 1
\]

where \( y_{ja} \) is the mean value of the fuzzy number \( \tilde{y}_{ja} \), for all \( j \).
Similarly, to find the upper solution $\bar{y}$, we must solve using the differential transform method, the nonfuzzy fractional initial value problem:

$$\bar{y}^{(q)}(x) = f(x, \bar{y}(x)), \quad p - 1 < q \leq p$$

$$\bar{y}^{(j)}(a) = y_{j_a} + \sqrt{1-\alpha}, \quad j = 0, 1, \ldots, p - 1$$

6. Illustrative Examples:

In this section, we shall present two FFIVP's, linear and nonlinear, and solved using the above method of solution.

**Example (1):**
Consider the following linear FFIVP:

$$D^q y(x) = -y(x), \quad 1 < q \leq 2, \quad x \geq 0$$

with initial conditions:

$$y(0) = 1, \quad y'(0) = 0$$

The solution will be of the form $[\underline{y}, \bar{y}]$.

To find $\underline{y}$, we must solve the problem:

$$D^q \underline{y}(x) = -\underline{y}(x), \quad 1 < q \leq 2, \quad x \geq 0$$

with the initial conditions:

$$\underline{y}(0) = 1 - \sqrt{1-\alpha}, \quad \underline{y}'(0) = -\sqrt{1-\alpha}$$

and application of the differential transform method, gives:

$$Y(k + \beta q) = \frac{\Gamma\left(1+\frac{k}{\beta}\right)}{\Gamma\left(q+1+\frac{k}{\beta}\right)}[-Y(k)]$$

with the initial conditions, for $k = 0, 1, \ldots, q \beta - 1$

$$Y(k) = \begin{cases} 
\frac{1}{(k/\beta)!} \left( \frac{d^{k/\beta} \underline{y}(x)}{dx^{k/\beta}} \right)_{x=0}, & \text{if} \quad \frac{k}{\beta} \in \mathbb{R}^+ \\
0, & \text{if} \quad \frac{k}{\beta} \notin \mathbb{R}^+
\end{cases}$$

and to find $\bar{y}$, the following problem must be solved:

$$D^q \bar{y}(x) = -\bar{y}(x), \quad 1 < q \leq 2, \quad x \geq 0$$

with initial conditions:

$$\bar{y}(0) = 1 + \sqrt{1-\alpha}, \quad \bar{y}'(0) = \sqrt{1-\alpha}$$

Similarly, by the differential transform method, we get:
\[ \bar{Y}(k + \beta q) = \frac{\Gamma\left(1 + \frac{k}{\beta}\right)}{\Gamma\left(q + 1 + \frac{k}{\beta}\right)} \{ -\bar{Y}(k) \} \]

with the initial conditions, for \( k = 0, 1, \ldots, \beta q - 1 \)

\[
\bar{Y}(k) = \begin{cases} 
\frac{1}{(k/\beta)!} \left( \frac{d^{\frac{k}{\beta}} Y(x)}{dx^{\frac{k}{\beta}}} \right)_{x=0} & \text{if } k \in \mathbb{N}^+ \\
0, & \text{if } k \notin \mathbb{N}^+ 
\end{cases}
\]

As a special case, if \( q = 3/2 \) and \( \beta = 2 \), then the solution of the above problem will be:

\[
\bar{Y}(k + 3) = \frac{\Gamma\left(1 + \frac{k}{2}\right)}{\Gamma\left(\frac{5}{2} + \frac{k}{2}\right)} \{ -\bar{Y}(k) \} \] ...................................................... (10)

with the initial conditions:

\[
\bar{Y}(0) = 1 - \sqrt{1 - \alpha}, \quad \bar{Y}(1) = 0, \quad \bar{Y}(2) = -\sqrt{1 - \alpha} \] ...................................................... (11)

By using (10) and (11), we get:

\[
\begin{align*}
\bar{Y}(3) &= \frac{-(1 - \sqrt{1 - \alpha})}{\Gamma(5/2)}, \quad \bar{Y}(4) = 0, \quad \bar{Y}(5) = \frac{\sqrt{1 - \alpha}}{\Gamma(7/2)}, \quad \bar{Y}(6) = \frac{1 - \sqrt{1 - \alpha}}{\Gamma(4)}, \\
\bar{Y}(7) &= 0, \quad \bar{Y}(8) = \frac{-\sqrt{1 - \alpha}}{\Gamma(5)}, \quad \bar{Y}(9) = \frac{-(1 - \sqrt{1 - \alpha})}{\Gamma(11/2)}, \quad \bar{Y}(10) = 0
\end{align*}
\]

By using (5) up to 10-terms, we have:

\[
y(x) = 1 - \sqrt{1 - \alpha} - x\sqrt{1 - \alpha} - \frac{1 - \sqrt{1 - \alpha}}{\Gamma(5/2)} x^{3/2} + \frac{\sqrt{1 - \alpha}}{\Gamma(7/2)} x^{5/2} + \frac{1 - \sqrt{1 - \alpha}}{\Gamma(4)} x^3 - \frac{\sqrt{1 - \alpha}}{\Gamma(5)} x^4 - \frac{1 - \sqrt{1 - \alpha}}{\Gamma(11/2)} x^{9/2}
\]

and in a similar manner:

\[
\bar{Y}(k + 3) = \frac{\Gamma\left(1 + \frac{k}{2}\right)}{\Gamma\left(\frac{5}{2} + \frac{k}{2}\right)} \{ -\bar{Y}(k) \} \] ...................................................... (12)

with the initial conditions:

\[
\bar{Y}(0) = 1 + \sqrt{1 - \alpha}, \quad \bar{Y}(1) = 0, \quad \bar{Y}(2) = \sqrt{1 - \alpha} \] ...................................................... (13)

By using (12) and (13), we get:
By using (5) up to 10-terms, we have:

\[
\bar{Y}(3) = \frac{-(1+\sqrt{1-\alpha})}{\Gamma(5/2)}, \quad \bar{Y}(4) = 0, \quad \bar{Y}(5) = \frac{\sqrt{1-\alpha}}{\Gamma(7/2)}, \quad \bar{Y}(6) = \frac{1+\sqrt{1-\alpha}}{\Gamma(4)},
\]

\[
\bar{Y}(7) = 0, \quad \bar{Y}(8) = \frac{-\sqrt{1-\alpha}}{\Gamma(5)}, \quad \bar{Y}(9) = \frac{-(1+\sqrt{1-\alpha})}{\Gamma(11/2)}, \quad \bar{Y}(10) = 0
\]

The following figure illustrate the fuzzy solution of example (1) for \(q = 3/2\), \(\beta = 2\), and for different values of \(\alpha\):

![Graph illustrating the fuzzy solution of example (1) for different values of \(\alpha\).](image)

**Example (2):**

Consider the nonlinear FFIVP:

\[D^qy = y^2 + 1, \quad p - 1 < q \leq p, \quad p \in \mathbb{R}, \quad x \geq 0\]

with initial conditions:

\[y^{(j)}(0) = 0, \quad j = 0, 1, \ldots, p - 1\]

Hence, the solution will take the form \([y, \bar{y}]\)

To find \(\bar{y}\), we must use the differential transform method to solve:
$$D^q y = y^2 + 1, \quad p - 1 < q \leq p, \ p \in \mathbb{N}, \ x \geq 0$$

with initial conditions:

$$y^{(j)}(0) = -\sqrt{1-x}, \ j = 0, 1, \ldots, p - 1$$

Then, we have:

$$Y(k + \beta q) = \frac{\Gamma\left(1 + \frac{k}{\beta}\right)}{\Gamma\left(q + 1 + \frac{k}{\beta}\right)} \left\{ \sum_{k_1=0}^{k} Y(k_1) Y(k - k_1) \right\} + \delta(k)$$

with the initial conditions, for $$k = 0, 1, \ldots, \beta q - 1$$

$$Y(0) = \begin{cases} \frac{1}{(k/\beta)!} \left( \frac{d^{k/\beta} y(x)}{dx^{k/\beta}} \right)_{x=0}, & \text{if } \frac{k}{\beta} \in \mathbb{N}^+ \\ 0, & \text{if } \frac{k}{\beta} \notin \mathbb{N}^+ \end{cases}$$

and in a similar manner to find $$\bar{y}$$, the differential transform method will be used to solve:

$$D^q \bar{y} = \bar{y}^2 + 1, \quad p - 1 < q \leq p, \ p \in \mathbb{N}, \ x \geq 0$$

with initial conditions:

$$\bar{y}^{(j)}(0) = \sqrt{1-x}, \ j = 0, 1, \ldots, p - 1$$

Hence:

$$\bar{Y}(k + \beta q) = \frac{\Gamma\left(1 + \frac{k}{\beta}\right)}{\Gamma\left(q + 1 + \frac{k}{\beta}\right)} \left\{ \sum_{k_1=0}^{k} \bar{Y}(k_1) \bar{Y}(k - k_1) \right\} + \delta(k)$$

With the initial conditions, for $$k = 0, 1, \ldots, \beta q - 1$$

$$\bar{Y}(0) = \begin{cases} \frac{1}{(k/\beta)!} \left( \frac{d^{k/\beta} \bar{y}(x)}{dx^{k/\beta}} \right)_{x=0}, & \text{if } \frac{k}{\beta} \in \mathbb{N}^+ \\ 0, & \text{if } \frac{k}{\beta} \notin \mathbb{N}^+ \end{cases}$$

Now, for $$q = 2.5, \ \beta = 2$$, then the solution of the above problem will be:

$$\bar{Y}(k + 5) = \frac{\Gamma\left(1 + \frac{k}{2}\right)}{\Gamma\left(\frac{7}{2} + \frac{k}{2}\right)} \left\{ \sum_{k_1=0}^{k} \bar{Y}(k_1) \bar{Y}(k - k_1) \right\} + \delta(k)$$

................................................. (14)
\[ \bar{Y}(0) = \sqrt{1-\alpha} \]

\[ \bar{Y}(1) = 0, \quad \bar{Y}(2) = \sqrt{1-\alpha}, \quad \bar{Y}(3) = 0, \quad \bar{Y}(4) = \frac{\sqrt{1-\alpha}}{2} \] .......................... (15)

Using (14) and (15), we have:

\[ \bar{Y}(5) = \frac{2 - \alpha}{\Gamma(7/2)}, \quad \bar{Y}(6) = 0, \quad \bar{Y}(7) = \frac{\Gamma(2)(2 - 2\alpha)}{\Gamma(9/2)}, \quad \bar{Y}(8) = 0, \quad \bar{Y}(9) = \frac{\Gamma(3)(2 - 2\alpha)}{\Gamma(11/2)}, \]

\[ \bar{Y}(10) = \frac{2(2 - \alpha)\sqrt{1-\alpha}}{\Gamma(6)} \]

Then by using eq.(5), up to 10-times, we have:

\[ Y(x) = \sqrt{1-\alpha} + \sqrt{1-\alpha} x + \frac{\sqrt{1-\alpha}}{2} x^2 + \frac{2 - \alpha}{\Gamma(7/2)} x^{5/2} + \frac{\Gamma(2)(2 - 2\alpha)}{\Gamma(9/2)} x^{7/2} + \]

\[ \frac{\Gamma(3)(2 - 2\alpha)}{\Gamma(11/2)} x^{9/2} + \frac{2(2 - \alpha)\sqrt{1-\alpha}}{\Gamma(6)} x^5 \]

Similarly:

\[ Y(k + 5) = \frac{\Gamma\left(\frac{1+k}{2}\right)}{\Gamma\left(\frac{7+k}{2}\right)} \left[ \sum_{k_1=0}^{k} \frac{Y(k_1)Y(k-k_1)}{(k_1)!} \right] + \delta(k) \] .......................... (16)

\[ Y(0) = -\sqrt{1-\alpha}, \quad Y(1) = 0, \quad Y(2) = -\sqrt{1-\alpha}, \quad Y(3) = 0, \quad Y(4) = \frac{-\sqrt{1-\alpha}}{2} \] .......................... (17)

Using (16) and (17), we have:

\[ Y(5) = \frac{2 - \alpha}{\Gamma(7/2)}, \quad Y(6) = 0, \quad Y(7) = \frac{\Gamma(2)(2 - 2\alpha)}{\Gamma(9/2)}, \quad Y(8) = 0, \quad Y(9) = \frac{\Gamma(3)(2 - 2\alpha)}{\Gamma(11/2)}, \]

\[ Y(10) = \frac{-2(2 - \alpha)\sqrt{1-\alpha}}{\Gamma(6)} \]

By using eq.(5) up to 10-terms, we have:

\[ Y(x) = -\sqrt{1-\alpha} - \sqrt{1-\alpha} x - \frac{\sqrt{1-\alpha}}{2} x^2 + \frac{2 - \alpha}{\Gamma(7/2)} x^{5/2} + \frac{\Gamma(2)(2 - 2\alpha)}{\Gamma(9/2)} x^{7/2} + \]

\[ \frac{\Gamma(3)(2 - 2\alpha)}{\Gamma(11/2)} x^{9/2} + \frac{2(2 - \alpha)\sqrt{1-\alpha}}{\Gamma(6)} x^5 \]

The following figure represents the solution for \( q = 5/2, \beta = 2 \) and for different values of \( \alpha \)
7. Conclusions:
1. The crisp solution, i.e., the solution of nonfuzzy fractional differential equations, may be considered as a special case of the solution of the fuzzy fractional differential equations with $\alpha = 1$.
2. The validity of the results may be achieved from the equality of the upper and lower solutions at $\alpha = 1$.
3. The differential transform method proved its validity and accurate results in solving fuzzy fractional differential equations.

8. References: