Robustness of Cantor Diffractals

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Abstract: Diffractals are electromagnetic waves diffracted by a fractal aperture. In an earlier paper, we reported an important property of Cantor diffractals, that of *redundancy* [R. Verma et. al., Opt. Express 20, 8250 (2012)]. In this paper, we report another important property, that of *robustness*. The question we address is: How much disorder in the Cantor grating can be accommodated by diffractals to continue to yield faithfully its fractal dimension and generator? This answer is of consequence in a number of physical problems involving fractal architecture.

OCIS codes: (050.0050) Diffraction and gratings; (050.1220) Apertures; (290.5880) Scattering, rough surfaces.

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#184594 - $15.00 USD Received 31 Jan 2013; revised 12 Mar 2013; accepted 14 Mar 2013; published 26 Mar 2013

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1. Introduction

Fractal geometries provide a description for many forms in Nature [2–4]. They exhibit self-similar and scale-invariant properties at all levels of magnification and are characterized by a non-integer fractal dimension. Of late, fractal architecture in optical elements is drawing a lot of interest due to promising technological applications associated with them. For example, fractal irises or zone plates are found to exhibit improved imaging capabilities, namely, presence of multiple foci along the optical axis, an increase in the depth of field and a decrease in the chromatic aberration [5–7]. They therefore offer the possibility of polychromatic imaging using broadband sources. Similar attractions have also been observed in the context of fractal photon sieves, and fractal axicons [8, 9]. In another context, diffractograms of fractal superlattices are also capturing the attention of solid state physicists [10–12]. Their study is predicted to yield novel theories of diffraction from aperiodic structures. Fractal organization has also been observed in the modes of some unstable lasers [13, 14].

Diffractals are electro-magnetic waves that have encountered a fractal aperture [15]. They propagate in space and time to yield its diffraction profile. The short wavelength regime of the electromagnetic waves allows an exploration of the ever finer levels of the fractal architecture and conveys important information regarding its properties such as roughness, texture, lacunarity and fractal dimension. Since geometric optics is not applicable on these length scales, scattering methods hold a special appeal for the study of fractal structures [16–20].

The Cantor set is a fractal characterized by a fractal dimension $d_f \approx 0.63$. Being the simplest to construct, it is not surprising that most manmade fractal creations are Cantor fractals [5–12, 21]. Therefore, understanding Cantor diffractals and their properties is significant not only in the context of fractal mathematics but also for technologies based on them. In a recent paper, we reported an important property of Cantor diffractals, that of *redundancy* [1]. We found that the Fraunhofer diffraction pattern comprises of self-similar bands, each containing complete information about the fractal aperture and its properties. This redundancy allowed the reconstruction of the Cantor grating by an inverse Fourier transformation of an *arbitrary* band although it excluded the zero spatial frequency component and contained an insignificant energy content in it. Furthermore, the fractal generator could be obtained from yet smaller fragments of the arbitrary band.

In this paper, we report another important property of Cantor diffractals, that of *robustness*. Literally, robustness refers to a sturdiness in construction even under adverse conditions. The question we address is: How much malformation or disorder in the Cantor grating can be accommodated by the diffractals to continue to yield faithfully its fractal dimension and fractal generator? The answer is of consequence in a number of physical problems involving fractal organization. For example, it can provide a bound for the inherent manufacturing defects that can be tolerated in cantor irises, axicons and sieves, to keep their performance capabilities intact.

We introduce disorder in the Cantor grating by blocking a fraction of randomly chosen *transparencies*, or by opening a fraction of the randomly chosen *opacities*, or by doing both simultaneously. The corresponding diffraction pattern also exhibits redundancy. We can therefore obtain the fractal dimension and the generator from an arbitrary band (primary or any of the secondary bands) and compare these with evaluations of the un-disordered Cantor aperture to quantify robustness to disorder. Our study indicates that unambiguous recovery is possible for as much as 50% disorder if the order of the grating is large.

The paper is organized as follows. In Section 2, we present the mathematical formulation of the disordered Cantor grating and obtain the corresponding disordered diffractals. Section 3 presents our study on robustness of the fractal dimension and fractal generator to disorder in the Cantor grating. An experimental verification of these results using the "$4f$ configuration" is also provided. Finally, we discuss the implications of the twin properties of redundancy and robustness and their applications in Section 4.

2. Mathematical formulation

Let $R_0(x) = \mathrm{rect}(\varepsilon_0 = 2a, x = 0)$, where $\mathrm{rect}(w,x)$ is a rectangle function of width w placed symmetrically about point x. Defining $R_n(x) = \mathrm{rect}(\varepsilon_n = \varepsilon_{n-1}/3, x = 0)$, the n-th generation Cantor grating is given by [1]:

$$G_n^0(x) = R_n(x) * \Delta_n^0(x), \quad n \geq 1, \tag{1}$$

where $\Delta_n^0(x) = \sum_{i=1}^{N} \delta(x - x_i)$, $\quad x_i = \pm 2a/3 \pm 2a/3^2 \pm \cdots \pm 2a/3^n$. The superscript 0 here refers to an un-disordered grating and $N = 2^n$.

Suppose a fraction t of the N transparencies in the n-th generation Cantor grating are to be disordered or blocked. To do so, we generate random numbers $\{X(i);\ i = 1, 2, \cdot\cdot, N\}$ which take value 0 with probability t and value 1 with probability $1 - t$ by the following procedure:

String$(N, t, X(N))$

Do $i = 1, N$

Draw a random number $0 \leq r < 1$ using a standard random number generator

If $r < t$, $X(i) = 0$ else $X(i) = 1$

Enddo

In the presence of disorder of strength t, the aperture function in Eq. (1) therefore becomes

$$G_n^t(x) = R_n(x) * \Delta_n^t(x), \quad n \geq 1, \tag{2}$$

$$= R_n(x) * \sum_{i=1}^{N} X(i)\, \delta(x - x_i) \quad x_i = \pm 2a/3 \pm 2a/3^2 \pm \cdots \pm 2a/3^n. \tag{3}$$

Now suppose a fraction p of the $M = 2^{n-1}$ opacities are replaced by a transparency. As in the preceding case, **String**$(M, p, Y(N))$ can be used to generate $\{Y(i);\ i = 1, 2, \cdot\cdot, M\}$ which take values 0 and 1 with probabilities p and $1 - p$ respectively. The corresponding disordered aperture function is then given by

$$G_n^p(x) = R_n(x) * \Delta_{n-1}^p(x), \quad n \geq 1, \tag{4}$$

$$= R_n(x) * \sum_{i=1}^{M} (Y(i) - 1)\, \delta(x - y_i), \quad y_i = \pm 2a/3 \pm 2a/3^2 \pm \cdots \pm 2a/3^{n-1}. \tag{5}$$

It is easy to generalize to the case when transparencies are blocked at random and opacities are opened at random, both operations performed simultaneously. Suppose a fraction q of the total number $L = N + M$ is now disordered. Generating $\{Z(i);\ i = 1, 2, \cdot\cdot, L\}$ using **String**$(L, q, Z(L))$, the aperture function can be written as

$$G_n^q(x) = R_n(x) * \left(\Delta_n^q(x) + \Delta_{n-1}^q(x)\right), \quad n \geq 1, \tag{6}$$

$$= R_n(x) * \left(\sum_{i=1}^{N} Z(i)\, \delta(x - x_i) + (Z(i) - 1)\, \delta(x - y_i)\right), \tag{7}$$

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where x_i and y_i are as defined in Eqs. (3) and (5) respectively.

In Fig. 1(a), we plot the aperture function for the $n = 4$ Cantor grating using Eq. (1). We have chosen $2a = 1$ here and in all our subsequent evaluations for convenience. Its disordered counterpart, obtained for $q = 1/4$ in Eq. (6), is plotted in Fig. 1(b). To obtain reliable numerics, it is essential to average data over several realizations of disorder generated using **String**.

Fig. 1. n = 4 Cantor grating (a) without disorder and (b) with $q = 1/4$ or 25 % disorder.

The diffraction pattern or the amplitude can be obtained by a Fourier transformation of the aperture function. Scattering experiments however, measure the intensity which is the square of the amplitude. For the Cantor grating, it is given by [1]:

$$I_n^0(f) = \left| \int_{-a}^{a} \mathrm{d}x \; \mathrm{e}^{-i2\pi f x} G_n^0(x) \right|^2 = \left[\left(\frac{2}{3}\right)^n 2a \,\mathrm{sinc}\left(\frac{2\pi a f}{3^n}\right) \right]^2 \left\{ \prod_{m=1}^{n} \cos\left(\frac{4\pi a f}{3^m}\right) \right\}^2, \qquad (8)$$

where f is the spatial frequency of the scattered wave having dimensions of inverse length and $\mathrm{sinc}(x) = \sin(x)/x$. It is simple algebra to check that $I_n^0(f) = (2/3)^2 I_{n-1}^0(f/3)$, which illustrates self-similarity and scale-invariance characteristic of fractal structures [1]. The intensity profile comprises of well-separated bands with prominent amplitude variation due to the enveloping sinc function. The internal structure, represented by the cosine terms, is a result of interference from diffractals emanating from the transparencies. The term $\{\cdots\}^2$ in Eq. (8) is the structure factor $S(f)$. The fractal dimension d_f is customarily obtained from the integrated structure factor [20,22,23]:

$$\overline{S(f)} \;=\; \frac{1}{\Delta f} \int_{f}^{f+\Delta f} dq \; S(q) \;\propto f^{-d_f}. \qquad (9)$$

The intensity profile of the disordered grating $I_n^s(f)$, where $s = t,\ p$ or q as the case may be, can be obtained by substituting $G_n^0(x)$ by $G_n^s(x)$ in Eq. (8). This disordering does not affect the enveloping sinc function, but modifies $S(f)$, and consequently d_f. We demonstrate now that the inherent self-similarity in the interference pattern compensates for this change in a major way.

3. Robustness of Diffractals

First, we study the robustness of the fractal dimension d_f to disorder in the Cantor aperture. We present results for the most general case when both transparencies and opacities are disordered. All the data that we present has been averaged over 30 disorder realizations generated using procedure **String**. Table 1 presents the disorder averaged fractal dimension $\langle d_f \rangle$ calculated from the first secondary band for different values of disorder q and order n. The angular brackets $\langle \cdots \rangle$ indicate an averaging over disorder realizations. These data clearly reveal that the fractal dimension is correct to the first significant digit of the un-disordered value even if q is as large as 50% for higher n.

Next we investigate the effect of disorder on the reconstruction of the fractal generator from disordered diffractals. From Eq. (8), it can be seen that the internal structure of any band contains information about all the n length scales in the fractal grating. Further, the zeros of

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Table 1. $\langle d_f \rangle$ as a function of disorder q for Cantor gratings of order n.

q(%)	6.25	12.5	25.0	50.0
n = 4	0.62860	0.60197	0.55984	0.53488
n = 6	0.62343	0.61532	0.59834	0.57844
n = 8	0.62621	0.62229	0.61497	0.60024

the m-th cosine term are given by $f_m = \pm(2m_o+1)3^m/4$; $m_o = 0,1,2,...$ As $f_m = 3f_{m-1}$, the zeros of orders $m' < m$ are nested in those of order m. The periods of all cosine terms are therefore commensurate. As a consequence, the energy content enclosed between zeros of $\cos\left(2f/3^m\right), \cos\left(2f/3^{m-1}\right), \cdots, \cos\left(2f/3^2\right)$ is also sufficient to yield reconstructions of orders $m-1, m-2, \cdots, 1$ respectively [1]. Thus the fractal generator, corresponding to $n=1$, can be reconstructed by Fourier transformation of a fragment of an arbitrary band.

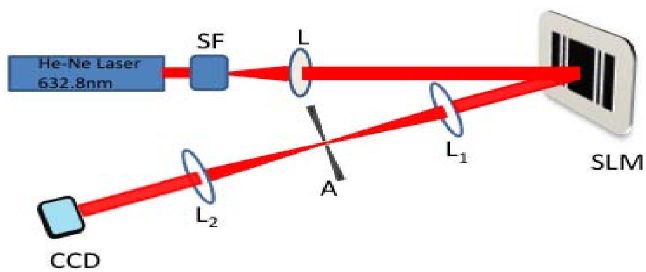

Fig. 2. Schematic representation of the $4f$ arrangement.

Figure 2 shows the $4f$ optical arrangement [1, 24] used for the experimental verification of the above statements. The disordered Cantor grating with $n=4$ and $q=0.25$ is displayed on the Holoeye LCR-2500 reflective type LC Spatial Light Modulator (SLM), with pixel size 19μ and resolution 1024×768, is illuminated with a coherent and collimated laser light of wavelength = 632.8 nm. The lens L_1 (f = 13.5 cm) Fourier transforms the displayed grating to yield the diffraction pattern on the Fourier plane F_1. An aperture A (of adjustable width) allows only a part of the diffraction pattern to be incident on the lens L_2 (f = 13.5 cm) which performs Fourier transformation once again to yield reconstruction of the grating. Figure 3(a) shows the disordered Cantor grating that was displayed on the SLM. The corresponding intensity profile observed on F_1 is depicted in Fig. 3(b). Figures 3(c) - 3(f) are reconstructions via L_2 from nested clips of first secondary band obtained using the description provided in the preceding paragraph. These reconstructions, in addition to yielding the generator (from Fig. 3(f)), also provide an experimental verification of redundancy in disordered diffractals. Higher generation gratings, which can accommodate larger disorder, could not be used due to issues of resolution. We have however been able to verify robustness theoretically for order 6 gratings with 50% disorder using Matlab simulations. But here too, resolution comes in the way when $n > 6$.

4. Discussion

Fractals have fascinated scientists ever since their discovery by Mandelbrot. The presence of such architectures in technology nowadays only furthers the interest and intrigue in them. The self-similarity in fractals persists at all levels of magnification. Scattering experiments are hence an important tool to probe their properties, especially because geometrical optics is not applicable on the ever finer length scales of the fractal structure. Their exploration is allowed by the

#184594 - $15.00 USD Received 31 Jan 2013; revised 12 Mar 2013; accepted 14 Mar 2013; published 26 Mar 2013

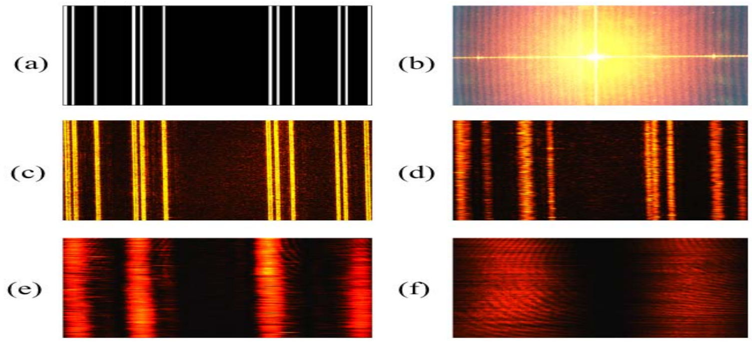

Fig. 3. (a) $n = 4$ Cantor grating with 25 % disorder; (b) Corresponding diffraction pattern; (c) - (f): Reconstructions from nested clips of first secondary band (refer text for details).

short-wavelength regime of the incident radiation. We have discussed two important properties of Cantor diffractals, which are electromagnetic waves that have encountered a (fractal) Cantor grating. Firstly, we have found that the fractal dimension evaluated from disordered diffractals is robust to the presence of random disorder in the Cantor grating. Our observations indicate that this is possible even if almost half of the transparencies and opacities in the grating are the wrong way. The second important property, verified experimentally in this paper, is that of redundancy. The latter allows for recovery of the the aperture function from clips and fragments of the corresponding diffraction field. We applied this property to obtain the fractal generator of the un-disordered Cantor grating from disordered diffractals. So together, robustness and redundancy can be used, to unambiguously identify the *parent* fractal! Said differently, Cantor diffractals can accommodate imperfections of the Cantor grating without significantly altering the diffraction field even if the disorder is as large as 50%.

Robustness and redundancy, arising due to the presence of self-similar length scales in the Cantor grating, impart special characteristics to fractal diffraction optics. There are important technological applications where these properties can be consequential. In this context, we mention two prototypical examples from distinct physical settings. The first one is in the context of optical elements such as cantor irises, axicons and sieves [5–9]. They can provide a bound for the inherent manufacturing defects that can be tolerated to keep their imaging capabilities intact. The second one is in the context of interfaces created in deposition problems [3, 4]. These are stochastic fractals, and their fractal dimension depends on the relative strengths of the underlying microscopic processes such as deposition, fragmentation and diffusion. The formation of defects and cracks are inherent in these interfaces and they grow with (deposition) time due to shadowing of incoming particles. The possibility of identification of the parent fractal despite imperfections in the structure, could provide information about the mechanisms contributing towards interfacial growth. It also interesting to check if robustness and redundancy are universal to aperiodic fractal structures such as Fibonacci and Thue-Morse gratings. We will address these problems in forthcoming papers.

Finally, we emphasize that the twin properties of robustness and redundancy observed in Cantor diffractals are generic to diffractals, irrespective of whether they emanate from deterministic or stochastic fractal apertures. We therefore believe that our work is of significance for a variety of problems with fractal architecture and their technological applications.

Acknowledgments

RV and MKS would like to acknowledge financial support from CSIR, India.

#184594 - $15.00 USD Received 31 Jan 2013; revised 12 Mar 2013; accepted 14 Mar 2013; published 26 Mar 2013