A hybrid formulation and design of model predictive control for systems under actuator saturation and backlash

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Abstract

In this paper, we develop a hybrid design framework of model predictive controller (MPC) for multivariable systems that simultaneously and explicitly addresses the actuator saturation and backlash. The discrete characteristics of the actuator backlash allows us to mathematically express it as a set of mixed-integer linear inequalities constraint in the inputs. As a result, the constrained MPC design is formulated as solving a mixed-integer quadratic programming (MIQP) problem. Furthermore, the proposed MIQP-based design is applied only in the proximity of steady state operating points after locating the active backlash and providing the estimate of the backlash size. Simulation studies are presented to demonstrate how the hybrid MPC performs when applied to an industrial case study of fluid catalytic cracking unit. It is shown that in the presence of actuator saturation and backlash the closed-loop performance can be improved substantially when applying the hybrid method as compared to the traditional design approaches.

Keywords: Actuator saturation and backlash; Multivariable systems; Hybrid MPC; MIQP

1. Introduction

Model predictive control (MPC) has been known to be one of the leading advanced control algorithms in industrial practice [18,17]. One of its advantages includes the ability to explicitly handle constraint in inputs, states and/or outputs. An explicit internal model is used for the online optimal control calculation at each sampling time while considering the process behavior over a future, finite time horizon. In industrial applications, MPC has been employed to improve product quality and yields by operating a process as close as possible to its constraint [22,4,11,20,25]. Over the past decades, the popularity of MPC continued to increase in many different application areas [11,4]. Despite such an increased popularity in process industries, a number of issues in the MPC applications, e.g. its sensitiveness to model uncertainty, remains. Most recent research has extensively explored the robustness of MPC to uncertainty in the process parameter and/or dynamics, e.g. Blasco et al. [4], Kothare et al. [15], Wu [25], and Chisci et al. [5].

Actuators play an important role in process control systems. Through an actuator, control is brought to bear on a process. They must operate precisely and function reliably. In many cases, actuators manipulate energy flows, mass flows or forces as a response to low energy input signals like electrical voltages or currents, pneumatic and hydraulic pressures or flows [14]. Because of their continuous motion and power amplification, actuators usually undergo wear and aging. The likelihood of actuator systems being malfunction is more significant when these components are installed in harsh environments (e.g. with high temperature, chemical solvents, aggressive media etc.) [1]. Their properties change at least gradually with time, and their performance may diminish. Actuator dynamics are therefore nonlinear, slow with direction-dependent
dynamics, and limited in their action. The actuator, however, is often assumed in the vast majority of control design as linear, instantaneous in operation, and has an unbounded output [8,7]. The actuator dynamics (e.g. backlash, stiction, dead-band, etc.) has been neglected in any controller design and synthesis. Instead, they are treated as an implementation issue rather than design.

As a commonly used actuator in process industries, control valves in most cases not only contain static non-linearity (e.g. saturation), but also dynamic nonlinearity including backlash, stiction, and hysteresis [9]. Dead-band due to backlash and friction/stiction is a major contributor to the valve problems. Mechanical assemblies inside the valve can be a primary source of dead-band [2]. A control valve with excessive dead-band may not even respond to a small change in control actions. As a result, they would produce a sustained oscillation in process variables, decrease the life of control valves, and in most cases, lead to inferior quality end products causing reduced profitability.

The control valves are often more attributed for control performance degradation [2,14,19,7]. As commonly the weakest link of the control loop, the control valve accounts for about 32% of “poor” or “fail” control loops [16]. A recent survey in Yang and Clarke [26] indicated that 30% of all control loops in Canadian paper mills were oscillating because of valve problems. In refinery industries, hidden cycling due to the valve problems was also detected in most loops of a unit, causing a large variation in the final products [19]. In more than 4000 audited loops in all industries as reported by Beckman and Jury [2], it is found that performance for more than 50% of the loops could be significantly improved if some work was done on the valve, actuator and/or positioner-I/P (current-to-pneumatic transducer). Hence, it appears that the potential benefit of such advanced control algorithm as MPC could be made inconsequential because of poor actuators.

In this paper, we study the performance of MPC in the presence of actuator saturation and backlash. It will be shown that the performance of MPC in the presence of actuator backlash can be considerably deteriorated and often lead to sustained oscillations. When the backlash is active, the actuator is not effective in following the command signals dictated by the controller. As a result, a limit cycle is produced in the proximity of the steady state operating points. Even if the controlled system behaves linearly, such a limit cycle remains. One way of reducing the backlash effect would be to explicitly take it into account in the MPC design so that an improved performance could be gained.

Our work is focused on developing a systematic method of designing MPC that considers both actuator saturation and backlash explicitly. The main results of this work are as follows: (1) A hybrid formulation of the input constraint within the MPC design is established as a set of mixed-integer linear inequality constraint. In the hybrid formulation, we include an approximate of the backlash inverse model, where a set of propositional logics is introduced. The logics are then translated into a set of mixed-integer linear inequalities by employing some integer variables. (2) We propose a mixed-integer quadratic programming (MIQP) method to solve the hybrid formulation of MPC design problem. As a result, an MIQP-based design of MPC is established for reducing the backlash effect. In implementing the proposed approach, we need an estimate of the backlash size, and to locate the active backlash in the closed loop. For this purpose, some available techniques of the actuator backlash detection [28,6] can be used.

This paper is organized as follows. In Section 2, we briefly review the standard MPC design, and highlight its possible limitations. Then, we address the actuator backlash dynamics and modeling, together with the backlash compensation techniques. After outlining the basic idea of our approach, we establish the MIQP-based design of MPC in Section 3. Simulation results for the applications of the method to an industrial case study of FCCU process are discussed in Section 4. Finally, we summarize some conclusions drawn from this work.

2. Preliminaries

2.1. Principle and limitations of MPC

MPC is a form of control algorithms in which the current control action is determined by solving online a finite horizon, open-loop optimal control problem by utilizing the current plant measurements as the initial states and an explicit model of the process to predict future outputs of the system. The online computation of predictive control laws is substantially different from the conventional control design, which uses precomputed (or explicit) control laws. This feature gives MPC some degrees of flexibility to handle constraint explicitly.

Fig. 1 shows how MPC is implemented in a feedback loop and Fig. 2 illustrates the basic principle of MPC algorithm. At each control interval, the MPC algorithm determines the solution of an optimization problem by computing a sequence of optimal future manipulated variables adjustments over a finite number of future time instant. It is assumed that there is no control action after time \( k + M – 1 \) and only the first input in the optimal sequence is implemented. At the next sampling time, the optimization problem is resolved with new measurements from the plant. Thus, both the control horizon \( M \) and the prediction horizon \( P \) move or recede ahead by one step as time moves ahead by one step. This is the reason why MPC is also sometimes referred to as receding horizon control (RHC) or moving horizon control (MHC) [22].

Although many processes are inherently nonlinear, most of MPC applications are based on linear dynamic models such as impulse, step, or state-space models, which are usually valid for particular operating points. This may be contributed by the fact that linear empirical models can be identified more easily from test data using system identifi-
cation technique. Also, most of MPC applications to date have been largely in maintaining the process at a desired steady state (e.g. regulatory control problem), rather than move rapidly from one operating point to another (e.g. servo-control problem). For such regulatory control applications, a carefully identified linear model is sufficiently accurate in the neighborhood of a single operating point.

A potential limitation of MPC design methods is its inefficacy of explicitly handling plant model uncertainty. The algorithm minimizes a nominal objective function, where a linear time-invariant model is used to predict the future behavior of the plant. The feedback implementation of MPC is inherently to compensate for model uncertainty/ inaccuracy due to unmeasured disturbances that cause the plant outputs to behave differently from the model prediction. As studied by Tsai et al. [23], the feedback mechanism of MPC does tolerate some model mismatch, but only to a certain extent. Hence, such an “optimal” performance of linear MPC may deteriorate when it is implemented on a physical system which is not reasonably well described by the linear model.

Plant/model mismatch usually arises when a model cannot exactly reproduce the process behavior. This mismatch may be contributed by several factors: (i) uncertainties due to nonlinearities, (ii) uncertainties due to dynamics, and (iii) uncertainties in terms of parametric; which are either unknown, un-modelled or neglected. By referring to Fig. 1, three possible sources of uncertainty may rise from the model of the plant itself – i.e. the model derived to represent the plant is not representative of the actual process; the actuator – the actuator dynamics is neglected and un-modelled in the control design phase; and/or from the sensor dynamics.

Nonlinearity as model uncertainty is present in a variety of forms. Prominently, they are originated from two sources – either the process itself being nonlinear, or more commonly, from the actuator dynamics, which include backlash, stiction, and deadzone. Goodwin [12] stated that a nonsmooth nonlinearity is regularly met in practice. Unless compensated, the effects of nonsmooth nonlinearity can be significant on the control system performance.
2.2. Linear MPC design framework

Assume that the system is described by the following linear time-invariant model:

\[ x(k + 1) = Ax(k) + Bu(k) \]
\[ y(k) = Cx(k) \]  

with \( x(k) \) is the state vector \( \in \mathbb{R}^n \), \( u(k) \) is the input vector \( \in \mathbb{R}^m \), and \( y(k) \) is the controlled output vector \( \in \mathbb{R}^m \). The matrices \( (A, B, C) \) are associated with the state, input and controlled output matrices, respectively.

Assume that \((A,B,C)\) is stabilizable and detectable, and the system is open-loop stable. We define the cost function

\[ J_0 = \sum_{i=0}^{P-1} \|\hat{y}(k+i|k) - r(k+i|k)\|_S^2 + \sum_{i=0}^{M-1} \|\Delta \hat{u}(k+i|k)\|_R^2 \]  

where \( P \) is the prediction horizon and \( M \) is the control horizon. This formulation coincides with that used in majority of the predictive control literature. The cost function penalizes deviations of the predicted controller outputs \( \hat{y}(k+i|k) \) from the reference trajectory vector \( r(k+i|k) \). We also assume that the cost function does not penalize particular values of the input vector \( u(k) \), but only changes of the input vector, \( \Delta u(k) = u(k) - u(k - 1) \). By using the input changes as the free variables in the optimization an integral action is introduced in the MPC controller, i.e. the actual inputs are obtained by integrating the changes in the input.

We select \( M \leq P \) and set \( \Delta \hat{u}(k+i|k) = 0 \) for \( i \geq M \). \( S \) and \( R \) are the weighting matrices on the controlled outputs and manipulated inputs respectively. The weighting matrices \( R \) are sometimes called move suppression factors, since increasing them penalizes changes in the input vector more heavily.

We rewrite (3) as

\[ J_0 = \|Y(k) - R(k)\|^2_S + \|\Delta u(k)\|^2_R \]  

where

\[ Y(k) = \begin{bmatrix} \hat{y}(k+1|k) \\ \vdots \\ \hat{y}(k+P|k) \end{bmatrix}, \quad R(k) = \begin{bmatrix} \hat{r}(k+1|k) \\ \vdots \\ \hat{r}(k+P|k) \end{bmatrix}, \]  

\[ \Delta u(k) = \begin{bmatrix} \Delta \hat{u}(k|k) \\ \vdots \\ \Delta \hat{u}(k+M-1|k) \end{bmatrix} \]

The predicted output vector is given by

\[ Y(k) = \Psi x(k) + \Theta \Delta u(k) \]  

where

\[ \Psi = \begin{bmatrix} A \\ \vdots \\ A^M \\ A^{M+1} \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} A^P \\ \vdots \\ \sum_{i=0}^{P-1} A^i B \\ A \end{bmatrix}, \]  

\[ \Theta = \begin{bmatrix} B & \cdots & 0 \\ AB + B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \sum_{i=0}^{M-1} A^i B & \cdots & AB + B \\ \vdots & \ddots & \vdots \\ \sum_{i=0}^{P-1} A^i B & \cdots & \sum_{i=0}^{P-M} A^i B \end{bmatrix} \]

Define

\[ e(k) = R(k) - \Psi x(k) - \Theta \Delta u(k - 1) \]  

as the measurement corrected vector of future output deviations from the reference trajectory (i.e. tracking error), assuming that all future control moves, \( \Delta u(k) \), are zero. Hence, we can rewrite (4) as follows:

\[ J_0 = \|\Theta \Delta u(k) - e(k)\|^2_S + \|\Delta u(k)\|^2_R \]  

\[ = [\Delta u(k)^T \Theta^T - e(k)^T S \Theta \Delta u(k) - e(k)]^T R \Delta u(k) + \Delta u(k)^T \Theta^T S e(k) \]  

\[ + \Delta u(k)^T [\Theta^T S \Theta + R] \Delta u(k) \]  

which has the form

\[ J = co\text{\textsuperscript{const}} - \Delta u(k)^T G + \Delta u(k)^T H \Delta u(k) \]  

where the gradient vector

\[ G = 2 \Theta^T S e(k) \]  

and the Hessian

\[ H = \Theta^T S \Theta + R \]

Note that term \( co\text{\textsuperscript{const}} = e(k)^T Q e(k) \) is independent of \( \Delta u(k) \) and can be removed from the objective function.

Magnitude and rate constraints on both the input and the output of the plant are expressed as follows:

\[ u_{\text{min}} \leq u(k) \leq u_{\text{max}}, \quad k = 0, 1, \ldots, M - 1 \]  

\[ y_{\text{min}} \leq y(k) \leq y_{\text{max}}, \quad k = 1, 2, \ldots, P - 1 \]  

\[ \Delta u_{\text{min}} \leq \Delta u(k) - \Delta u(k - 1) \leq \Delta u_{\text{max}}, \quad k = 0, 1, \ldots, M - 1 \]
These constraints can be transformed into linear constraints on \( \Delta u \) of the form

\[
L \Delta u(k) \leq K
\]  

(15)

The resulting optimization:

\[
\Delta u^*(k) = \arg \min_{\Delta u(k)} \Delta u^T(k)H \Delta u(k) - \Delta u(k)^T G
\]

(16)

This optimization is a convex quadratic programming (QP) problem obtained from the quadratic cost and linear constraints. Standard numerical procedures are available to efficiently solve this problem.

### 2.3. Backlash models and compensation

Since 1940s, in the early days of classical control theory, the backlash nonlinearity has been recognized as one of the factors severely limiting the performance of feedback systems. Backlash may cause delays, oscillations, and inaccuracy. Backlash is often referred to as slackness, or looseness, of a mechanical connection. This slackness results in a discontinuity of motion when the valves change direction. Consequently, the valve movement will be different from the signal generated from the controller. Theoretically, backlash in a valve is zero; however, virtually in all valves it is near 1% of the input signal span. On most of processes, backlash of 2% or 3% is tolerable only if the controller is not tuned aggressively.

Backlash effect is not noticeable when the change of controller output is large. However, as the movement of the controller output reduces in its magnitude, the impact of backlash is enhanced. It would introduce dead-time to the system, especially when the valve reverses direction due to the loose mechanical connection. The impact of backlash, and hence the dead-time, is further magnified for a small change in the controller output. It is being of major concern especially in regulatory control since the majority of the control signal received is small. Hence, it degrades the controller performance.

Backlash is relatively repeatable, and can be compensated. For controlled systems with a backlash at the plant input, various approaches have been pursued for compensating its effect; for example, Neural networks [21], dithering, and fuzzy logic [24]. The main idea of these approaches is to place an inverse of actuator nonlinearity in series with the feedback controller to produce a pure input–output gain (see Fig. 3). This means that the backlash gap has to be traversed instantaneously. Deriving a backlash model is therefore the key step of realizing its compensation technique.

Fig. 4 describes an input–output map of the actuator backlash. We define two quantities to represent two situations: (i) if \( u_I(t) \) is inside the backlash and on the positive side of the backlash, the distance from the positive boundary is defined as \( d_p \) (see Fig. 5(a)), and (ii) if \( u_I(t) \) is inside the backlash and on negative side, the distance from its negative boundary is defined as \( d_n \) (see Fig. 5(b)). Defining \( r = \frac{d_p}{m} \), we have:

Fig. 3. A nonlinear inverse strategy for backlash compensation adapted from Chow and Clarke [7].

Fig. 4. An input–output map of backlash.
The second alternative is to adopt the idea of nonlinear inverse strategy in compensating backlash. This can be implemented as either to insert a nonlinear inverse compensation of the backlash in the lower level controller (e.g., in series with PID controller), or to place a nonlinear inverse in series with the MPC itself, by following the generic structure of nonlinear inverse strategy as depicted in Fig. 3. The drawback of this approach, however, is its inability to account for actuator saturation. The saturation would reduce the effectiveness of nonlinear inverse strategy in which it can severely hinder the full compensation of the backlash nonlinearity. This means that the backlash may still have a significant impact on deteriorating the controller performance using this inverse compensation if the actuator saturation is active. Clarke [8] pointed out this concern, by mentioning that the invertibility nature of backlash is not achievable in the presence of saturation, where the required inverted signal exceed the maximum saturation limit allowable for the actuator.

3. Proposed method

In this work, we propose a more tactical way of reducing the backlash effect, that is by integrating the backlash model within the MPC design framework. As a result, in the MPC design the backlash nonlinearity together with actuator saturation are addressed simultaneously. By adopting such a tactical approach, we adopt the nonlinear inverse strategy within the MPC design context while explicitly addressing input constraints. This approach is reasonable because the backlash nonlinearity, though non-smooth by nature, has discrete characteristics so that it can be concisely and mathematically expressed by a series of if–then–else rules [7]. The hybrid design framework developed by [3] allows such if–then–else rules to be translated efficiently into a set of linear logical constraint. Ultimately, such formulation results in a system consisting of continuous and discrete components, which is referred to as mixed logical dynamical (MLD) systems.

\[
d_p(t) = \left[ \frac{u_p(t)}{m_b} + d \right] - u_1(t)
\]

\[
d_a(t) = \left[ \frac{u_p(t)}{m_b} + (-d) \right] - u_1(t)
\]

Hence, we obtain the following backlash model:

\[
\begin{align*}
    u_p(t) & = \begin{cases} 
        m_b[u_1(t) - d] & : \Delta u_1(t) > d_p(t - 1) \\
        m_b[u_1(t) - (-d)] & : \Delta u_1(t) < d_a(t - 1) \\
        u_p(t - 1) & : \text{otherwise}
    \end{cases} \\
    u_1(t) & = \begin{cases} 
        u(t) + d & : \Delta u(t) > 0 \\
        u(t) - d & : \Delta u(t) < 0 \\
        u_1(t - 1) & : \Delta u(t) = 0
    \end{cases}
\]

The nonlinear inverse strategy of compensating the backlash effect is in principle to bump up the controller outputs while maintaining the direction of the input change (or without changing its input direction). This implies that the algorithm of backlash compensation is: (1) to detect the input move direction by observing its change (either positive, negative or zero), and (2) to bump up the input move by the dead-band size, \(d_j\), say \(j = 1\) for the positive direction and \(j = 2\) for the negative direction. If there is no change in the input move, apply the previous input move.

In compensating the effect of backlash nonlinearity in the context of MPC design framework, several approaches may be pursued. First is to re-tune the MPC controller by reducing the aggressiveness, i.e., imposing larger weights on the manipulated input moves. The deleterious effect of backlash may be suppressed by having larger weights on the manipulated inputs; however, with a consequent effect of sluggish response in the output. This approach also has an inherent difficulty in determining what is the best set of weights that need to be used, especially for highly interacting multivariable system. The re-tuning strategy is often leading to a nonconvex optimization problem, and has a nonlinear effect on the closed-loop performance.

Fig. 5. Computation of \(d_p(t)\) and \(d_a(t)\).
3.1. Hybrid formulation of input constraints

In the standard MPC design, the input constraint is simply given by:

\[ u_{\text{min}} \leq u(k) \leq u_{\text{max}} \]
\[ \Delta u_{\text{min}} \leq \Delta u_i(k) \leq \Delta u_{\text{max}} \]  

(17)

This formulation takes into account for the actuator saturation and the limits on the input rate of change, but not the actuator backlash.

To incorporate the backlash dynamics within the MPC design framework, we reconfigure the input constraint. One of the possible implementations is to include the approximate nonlinear inverse of the actuator backlash within the MPC design framework as illustrated in Fig. 6.

Let us introduce a set of logical variables \( \delta_{ij} \), where \( j = 1, 2, 3 \), represents the conditions of input change \( \Delta u_i(k) \) (e.g. positive, negative, or zero). The following propositional logics is imposed on \( \Delta u_i(k) \):

\[ \delta_1 = 1 \iff \Delta u_i(k) \geq d_i \]  

(18)

\[ \delta_2 = 1 \iff \Delta u_i(k) \leq -d_i \]  

(19)

\[ \delta_3 = 1 \iff \Delta u_i(k) = 0 \]  

(20)

where

\[ \sum_{j=1}^{3} \delta_{ij} = 1 \]  

(21)

and \( i = 1, 2, \ldots, m \), with \( m \) is the number of manipulated inputs in the system.

By combining (17) with (18)–(20), we obtain a set of linear constraint with propositional logics, where for the \( i \)th input it is given by:

\[ u_{\text{min}} \leq u_i(k) \leq u_{\text{max}} \]  

\[ \Delta u_{\text{min}} \leq \Delta u_i(k) \leq \Delta u_{\text{max}} \]  

\[ \delta_1 = 1 \iff \Delta u_i(k) \geq d_i \]  

\[ \delta_2 = 1 \iff \delta u_i(k) \leq -d_i \]  

\[ \delta_3 = 1 \iff \Delta u_i(k) = 0 \]  

(22)

(23)

(24)

(25)

(26)

The propositional logics specified in (22)–(26) will be transformed into a set of mixed-integer linear inequalities, i.e. linear inequalities involving both continuous variables \( u \in \mathbb{R}^m \) and logical variables \( \delta \in \{0, 1\} \) as follows. Let us define:

\[ \bar{u}_{\text{max}} = u_{\text{max}} - u(k - 1) \]  

(27)

\[ \bar{u}_{\text{min}} = u_{\text{min}} - u(k - 1) \]  

(28)

A set of mixed-integer linear inequalities constraint for the \( i \)th input is established as:

\[ \Delta u_{\text{min}} \leq \Delta u_i(k) \leq \Delta u_{\text{max}} \]
\[ \delta u_i(k) - d_i \geq (\bar{u}_{\text{min}} - d_i)(1 - \delta_1) \]
\[ \delta u_i(k) - d_i \leq (\bar{u}_{\text{max}} - d_i)\delta_1 \]
\[ \delta u_i(k) + d_i \geq (\bar{u}_{\text{max}} + d_i)(1 - \delta_2) \]
\[ \delta u_i(k) + d_i \leq (\bar{u}_{\text{max}} + d_i)\delta_2 \]
\[ \delta u_i(k) \geq \bar{u}_{\text{max}}(1 - \delta_3) \]
\[ \delta u_i(k) \leq \bar{u}_{\text{min}}(1 - \delta_3) \]

Hence, the MPC optimization problem now has the hybrid set of input constraint:

\[ \Delta u_{\text{min}} \leq \Delta u_i(k) \leq \Delta u_{\text{max}} \]
\[ \delta u_i(k) - d_i \geq (\bar{u}_{\text{min}} - d_i)(1 - \delta_1) \]
\[ \delta u_i(k) - d_i \leq (\bar{u}_{\text{max}} - d_i)\delta_1 \]
\[ \delta u_i(k) + d_i \geq (\bar{u}_{\text{max}} + d_i)(1 - \delta_2) \]
\[ \delta u_i(k) + d_i \leq (\bar{u}_{\text{max}} + d_i)\delta_2 \]
\[ \delta u_i(k) \geq \bar{u}_{\text{max}}(1 - \delta_3) \]
\[ \delta u_i(k) \leq \bar{u}_{\text{min}}(1 - \delta_3) \]
\[ \sum_{j=1}^{3} \delta_{ij} = 1 \]  

(29)

(30)

The MPC design is then to solve a quadratic objective function of (16) subject to the reconfigured input constraints in (30) when determining its optimal inputs \( u^*(k) \) at each sampling time. By using (30) at every time step, the dead-band region is avoided by suppressing a further movement of the input that changes its direction. This is done by activating one and only one logical variable at a time.

Note that the proposed technique considers both the actuator backlash and saturation simultaneously when determining the MPC optimal inputs. Hence, we avoid the problem of producing the infinite input for compensating the backlash as is the case when the nonlinear inverse strategy is placed in series.

3.2. MIQP formulation

In this section, the MPC optimization problem in (16) is integrated with the hybrid set of input constraints. We show that this formulation belongs to an MIQP optimization problem. For clarity, we only show the case of \( m \)

![Fig. 6. An approximate nonlinear inverse strategy integrated within the MPC scheme.](image-url)
inputs and control horizon, $M = 1$. The linear inequality constraint in (30) can be re-arranged as follows:

$$
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
-1 & 0 & \Omega & 0 \\
-1 & \alpha & 0 & 0 \\
1 & 0 & \beta & 0 \\
0 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\delta_{m1} \\
\delta_{m2} \\
\delta_{m3}
\end{bmatrix}
+ 
\begin{bmatrix}
U_{\min} \\
-D \\
U_{\min}
\end{bmatrix}
\leq 0
$$

where

- $1 = \text{diag}[-1, \ldots, -1]$
- $I = \text{diag}[1, \ldots, 1]$
- $\theta = \text{diag}[-(u_{1\max} + d_1), \ldots, (u_{n\max} + d_n)]$
- $\Omega = \text{diag}[(u_{1\max} + d_1), \ldots, (u_{n\max} + d_n)]$
- $\alpha = \text{diag}[-(u_{1\max} - d_1), \ldots, -(u_{n\max} - d_n)]$
- $\beta = \text{diag}[(u_{1\max} + d_1), \ldots, (u_{n\max} + d_n)]$
- $\mu = \text{diag}[u_{1\min}, \ldots, u_{n\min}]
- \tau = \text{diag}[u_{1\min}, \ldots, u_{n\max}]

and

$$
\begin{bmatrix}
u_{1\min} \\
\vdots \\
u_{n\min}
\end{bmatrix},
\begin{bmatrix}
u_{1\max} \\
\vdots \\
u_{n\max}
\end{bmatrix}
$$

Hence, the new MPC optimization problem is expressed as:

$$
\min_{\tilde{z}} \quad z^T Q \tilde{z} + b^T \tilde{z}
$$

s.t. \quad C \tilde{z} + \dot{\tilde{z}} \leq 0

$$
\begin{bmatrix}
\tilde{z}_c \\
\tilde{z}_d
\end{bmatrix}
\in \mathbb{R}^{n_d}

\tilde{z}_d \in \{0, 1\}^{n_d}
$$

This is a mixed-integer quadratic programming (MIQP) optimization problem, where some efficient algorithms can be used [10]. The optimization problem involves a quadratic objective function and a set of mixed linear inequalities – the logical variables appear linearly with the optimization variables (i.e., there is no product terms between the logical variables, $\delta_{ij}$ and the optimization variables, $u$). Note that the matrix $Q$ in (37) is positive semi-definite – an important criteria that is necessary if such an optimization needs to be solved online for a global optimality.

**Remark.** In implementing this MIQP formulation, we are constraining the binary variable to be the same throughout the whole optimization control horizon. We acknowledge the fact that we may also allow the binary variables to vary from one control horizon to the next. However, as shown by the results that we will present in the later section, our current approach can manage to eliminate the backlash effect. Note that by adopting our current strategy, we may reduce to some extent the computational time in comparison to varying the binary variables over optimization horizon. Nevertheless, we think that it is an interesting topic for future work.

### 3.3. Stability analysis

Now, we analyze whether the resulting MIQP-based MPC is a stabilizing controller. For this, we follow the stability analysis of Bemporad and Morari [3] for the general system of mixed logical dynamical.

Define the states and inputs equilibrium as $(x_e, u_e)$, and let the logical variables be definitely admissible [27], where $\delta_c$ corresponds to the desired steady-state values for the logical variables. Let $r$ be the current time $t$, and $x(t)$ the current state. The MPC optimization problem (37) can be equivalently expressed in the general form of mixed logical dynamical system (MLD) as the following,

$$
\min_u \quad J = \sum_{k=0}^{M-1} \|u(k) - u_e\|_{Q_1}^2 + \|\delta(k) - \delta_c\|_{Q_2}^2 + \|x(k) - x_e\|_{Q_3}^2
$$

s.t. \quad x(P(t)) = x_e

$$
x(k+1) = Ax(k) + Bu(k)
y(k) = Cx(k)
E_2 \delta(k) \leq E_1 u(k) + E_3
$$

Let the matrix $Q(4m \times 4m)$ and $b(4m \times 1)$ be written as:

$$
Q = \begin{bmatrix}
H & 0 & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 0
\end{bmatrix},
B = \begin{bmatrix}
-G \\
0 \\
\vdots \\
0
\end{bmatrix}
$$

Let us define (31) as $C_z + \dot{z} \leq 0$ with $z = [z_c, z_d]^T$, where $z_c$ representing the continuous variables $u$, and $z_d$ are those of the logical variables $\delta_{ij}$. Also, for (16) let $H$ and $G$ be given by:

$$
H = \begin{bmatrix}
w_{11} & w_{12} & \ldots & w_{1m} \\
w_{21} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
w_{m1} & w_{m2} & \ldots & w_{mn}
\end{bmatrix},
G = \begin{bmatrix}
ev_1 \\
ev_2 \\
\vdots \\
ev_m
\end{bmatrix}
$$

Let the matrix $Q(4m \times 4m)$ and $b(4m \times 1)$ be written as:

$$
Q = \begin{bmatrix}
H & 0 & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 0
\end{bmatrix},
B = \begin{bmatrix}
-G \\
0 \\
\vdots \\
0
\end{bmatrix}
$$
where \( Q_1 > 0; Q_2 > 0; Q_3 > 0 \) and \( E_1, E_2 \) and \( E_3 \) can be straightforwardly derived from (31).

Assume that the optimal solution is given by \( U^* = \{u^*(0), u^*(1), \ldots, u^*(M-1)\} \), and the receding horizon strategy is applied as \( u(t) = u^*(0) \) before repeating the whole optimization procedure at time \( t+1 \). The control law (39) and (40) is referred to the mixed-integer predictive control (MIPC) law [3].

**Lemma.** Let \((x_e, u_e)\) be an equilibrium pair and \( \delta_e \) definitely admissible. Assume that the initial states \( x(0) \) is such that a feasible solution of problem (39) exists at time \( t = 0 \). Then \( Q_1 > 0, Q_2 > 0, \) and \( Q_3 > 0 \), the MIPC law (39) and (40) stabilizes the system in the following sense:

\[
\lim_{t \to \infty} x(t) = x_e \\
\lim_{t \to \infty} u(t) = u_e \\
\lim_{t \to \infty} \|\delta(k|t) - \delta_e\|_{Q_e}^2 = 0
\]

while fulfilling the constraints (40).

**Proof.** The proof follows from standard Lyapunov arguments. Let \( U^* \) denote the optimal control sequence \( \{u^*(0), \ldots, u^*(M-1)\} \). Let \( V(t) \triangleq J(U^*, x(t)) \) denote the corresponding value attained by the performance index. Also, let \( U_1 = \{u^*(1), \ldots, u^*(M-2), u_e\} \). Then \( U_1 \) is feasible (but not optimal) at time \( t+1 \), along with the vectors \( \delta(k|t+1) = \delta(k+1|t) \) for \( k = 0, \ldots, M-1 \). Also \( \delta(M-1|t+1) = \delta_e \) since \( x(M-1|t-1) = x(M|t) = x_e \) and \( \delta_e \) definitely admissible. Hence,

\[
V(t+1) \leq J(U_1, x(t+1))
\]

\[
= V(t) - \|u(k) - u_e\|_{Q_1}^2 - \|\delta(k|t) - \delta_e\|_{Q_e}^2
\]

and \( V(t) \) is decreasing. Since \( V(t) \) is lower-bounded by zero, there exists \( V_\infty = \lim_{t \to \infty} V(t) \), which implies \( V(t+1) - V(t) \to 0 \). Therefore, each term of the sum

![Fig. 7. Key design parameters for MIQP-based MPC design.](https://example.com/fig7)

![Fig. 8. Grosdidier et al. [13] FCCU reproduced.](https://example.com/fig8)
\[ \|u(k) - u_c\|_{\Omega_1}^2 + \|\delta(k)\|_{\Omega_2}^2 + \|x(k) - x_c\|_{\Omega_3}^2 \leq V(t) - V(t-1) \]

converges to zero as well, which proves the theorem. □

Note that this stability analysis of the MIQP-based MPC is a subset of the stability result of the MIPC law [3], where in their case, the integer variable may appear in the state and input.

Table 1

<table>
<thead>
<tr>
<th>Variable description and abbreviation</th>
<th>Tag</th>
<th>Limit values</th>
<th>Control objective</th>
</tr>
</thead>
<tbody>
<tr>
<td>Combustion air flow (air)</td>
<td>( u_1 )</td>
<td>140–155 T/h</td>
<td>–</td>
</tr>
<tr>
<td>Hot gas oil flow (hot feed)</td>
<td>( u_2 )</td>
<td>190–110 m³/h</td>
<td>IRV</td>
</tr>
<tr>
<td>Combined cold gas and recycle oils (cold feed)</td>
<td>( u_3 )</td>
<td>230–250 °C</td>
<td>IRV</td>
</tr>
<tr>
<td>Feed preheat temperature (feed ( T ))</td>
<td>( u_4 )</td>
<td>320–535 °C</td>
<td>IRV</td>
</tr>
<tr>
<td>Riser outlet ( T ) (riser ( T ))</td>
<td>( u_5 )</td>
<td>320–80%</td>
<td>IRV</td>
</tr>
<tr>
<td>Recycle oil flow controller output (recycle)</td>
<td>( u_6 )</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Flue gas ( O_2 ) concentration (O( _2 ))</td>
<td>( j_1 )</td>
<td>0.9%</td>
<td>Set-point</td>
</tr>
<tr>
<td>Regenerator bed ( T ) (bed ( T ))</td>
<td>( j_2 )</td>
<td>3705–735 °C</td>
<td>Set-point</td>
</tr>
<tr>
<td>Fuel gas flow (fuel gas)</td>
<td>( j_3 )</td>
<td>Max 15 T/h</td>
<td>Set-point</td>
</tr>
<tr>
<td>Wet gas compressor suction pressure controller output (WGC)</td>
<td>( j_4 )</td>
<td>–</td>
<td>Set-point</td>
</tr>
<tr>
<td>Riser outlet temperature controller output (fresh cat VP)</td>
<td>( j_5 )</td>
<td>–</td>
<td>Set-point</td>
</tr>
<tr>
<td>Regenerated catalyst slide valve pressure differential (fresh cat ( dP ))</td>
<td>( j_6 )</td>
<td>–</td>
<td>Set-point</td>
</tr>
<tr>
<td>Spent catalyst slide valve pressure differential (spent cat ( dP ))</td>
<td>( j_7 )</td>
<td>–</td>
<td>Set-point</td>
</tr>
</tbody>
</table>

Note: 1100 m³/h, 2100 m³/h, 3240 °C, 4525 °C, 550%, 6715 °C, 713 T/h, 865%, 975%, 1025 kPa, 1122 kPa; *outputs are unconstrained.
3.4. Design parameters and implementation

The developed design method requires us to give an estimate of backlash width \( d \) and to know where and when the backlash is active in the control loop. For this purpose, we can use the available backlash detection techniques [28, 6]. Since the backlash effect is significant when the controller outputs change its direction in the proximity of their steady-state values, we may predict when the backlash is active by estimating the closed-loop time constant. In the MPC design, the prediction horizon \( P \) is typically selected such that it is larger than the closed-loop time constant of the process for stability reasons [22].

Note that if we activate the MIQP-MPC too early, the designed MPC would produce a sluggish response as the feasible inputs are limited by the reconfigured constraint (Fig. 7).

In implementing the MIQP-based MPC design, we may initially set \( d_i = 0 \) with the assumption that the backlash is not active in the control loop so that the MPC design follows the standard formulation. We can accommodate this condition as an additional logical variable \( d_i \) into (29) to get (17), which is the standard constraint formulation in the MPC design. This can be done as follows:

\[
\Delta u_{\text{min}} \leq \Delta u_i(k) \leq \Delta u_{\text{max}}
\]

\[
\Delta u_i(k) - d_i \geq (\bar{u}_{\text{min}} - d_i)(1 - \delta_1)
\]

\[
\Delta u_i(k) - d_i \leq (\bar{u}_{\text{min}} - d_i)\delta_1 + \delta_3 u_{\text{max}}
\]

\[
\Delta u_i(k) + d_i \geq (\bar{u}_{\text{min}} + d_i)\delta_2 + \delta_4 u_{\text{max}}
\]

\[
\Delta u_i(k) + d_i \leq (\bar{u}_{\text{min}} + d_i)(1 - \delta_2)
\]

\[
\Delta u_i(k) \geq \bar{u}_{\text{min}}(1 - \delta_3)
\]

\[
\Delta u_i(k) \leq \bar{u}_{\text{max}}(1 - \delta_3)
\]

\[
\sum_{j=1}^{4} \delta_j = 1
\]

The algorithm of the new design method is then implemented as:

1. Initialize the MIQP optimization with \( d_i = 0 \).
2. Solve the standard MPC optimization until the backlash is active.
3. When the backlash is active, supply an estimate of backlash size \( d_i = d_{\text{est}} \).
4. Solve the MIQP with \( d_{\text{est}} \).

Fig. 10. Output responses: (solid line) no backlash; (dashed line) with backlash.
Remark. We acknowledge that in real applications, disturbances may occur continuously and backlash may be active anywhere throughout the trajectory. Clearly, in such situations, a systematic method may be needed in properly determining when the backlash is active.

4. Application to an FCCU process

We apply the developed MIQP-based design of MPC to an industrial FCCU process [13]. Fig. 8 shows the FCC unit, where the unit operates under a full combustion mode. In this study, the MPC is implemented in cascade with some PI controllers.

Three proportional-integral (PI) controllers were applied in the flows of the combustion air, the hot gas oils and the combined cold gas and recycle oils \((u_1, u_2, u_3)\). Two PI controllers were to control the feed preheat \((u_4)\) and the riser outlet temperature \((u_5)\). Recycle flow \((u_6)\) is regulated by adjusting the output of a hand controller. As studied by Grosdidier et al. [13], MPC is used to control seven variables, denoted as \(y_1, \ldots, y_7\) by manipulating six variables \(u_1, \ldots, u_6\) — see Table 1. Each variable is constrained by their associated high and low limits. All manipulated variables are initially at their ideal resting values (IRV), which is assumed to be their steady state, except for combustion air flow \((u_1)\) which is allowed to move freely.

The MPC is designed as follows:

1. The weighting of inputs and outputs were set as:

\[
S = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 50 & 0 & 0 & 0 & 0 \\
0 & 0 & 50 & 0 & 0 & 0 \\
0 & 0 & 0 & 50 & 0 & 0 \\
0 & 0 & 0 & 0 & 50 & 0 \\
200 & 0 & 0 & 0 & 0 & 0 \\
0 & 100 & 0 & 0 & 0 & 0 \\
0 & 0 & 100 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 0 & 0 \\
0 & 0 & 0 & 0 & 100 & 0 \\
0 & 0 & 0 & 0 & 0 & 100
\end{bmatrix}
\]

(47)

\[
R = \begin{bmatrix}
200 & 0 & 0 & 0 & 0 & 0 \\
0 & 100 & 0 & 0 & 0 & 0 \\
0 & 0 & 100 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 0 & 0 \\
0 & 0 & 0 & 0 & 100 & 0 \\
0 & 0 & 0 & 0 & 0 & 100
\end{bmatrix}
\]

(48)

Note that the weight on \(u_1\) is set to a small value of 1 as it must be allowed to move freely to maintain the oxygen

Fig. 11. Performance of the MIQP-MPC as compared to the standard MPC.
level in the flue exit, $y_1$. The oxygen level in the flue exit must be tightly controlled to avoid the conditions of partial burning. Also, it is uneconomic to have excess oxygen in the flue gas since unnecessary amount of energy is expended to blow air into the regenerator.

2. We chose the control horizon $M = 1$, the prediction horizon $P = 20$, and the sampling time of 1 min.

Unless stated differently, a step change of $+0.5 \text{ m}^3/\text{h}$ in the hot feed flow rate, $u_2$ was considered in the simulation study to compare the controller performance.

The performance of the MIQP-based design of MPC will be compared not only with the standard MPC design, but also with the nonlinear inverse strategy. Furthermore, the proposed MIQP-based MPC design is evaluated for not only the single backlash case (i.e. in the valve, $u_2$), but also the multiple backlash case (i.e. in the valves, $u_2$ and $u_3$).

4.1. Effect of backlash

In this study, we insert a backlash size of 0.25 in the valve $u_2$. Fig. 9 shows the phase plane plots of the inputs. Clearly, the backlash is active in the valve, $u_2$ as indicated by a limit cycle of $u_2$ input map (see Fig. 9(b)). Also, we observe a dead-time whenever the controller output ($u_{\text{MPC}}$) changes its direction. Due to the interacting nature of the FCCU process, the effect is also propagated to $u_3$. Such sustained oscillations in the manipulated inputs would unquestionably affect the outputs.

Fig. 10 shows the corresponding output responses, which are oscillating. This oscillating dynamics can be explained as follows: As the cracking needs to be maintained at a constant temperature, any changes in the flow of the hot feed would dictate a corresponding change in the amount of catalysts required for the cracking process. Hence, fluctuations in the hot feed flow rate would definitely cause the oscillating of the fresh cat VP values, which then forces the regenerated catalyst slide valve to oscillate as well.

4.2. Performance of MIQP-MPC

We apply the MIQP-MPC to the FCCU case study to see how effective the new design method would be in reducing the effect of actuator backlash. We activate the

![Phase plane plots](image)
MIQP-based design at the same time as the disturbance acts to the closed-loop systems. Fig. 11 shows the performance of output responses and their associated error distributions (i.e. computed by comparing the output responses with and without backlash). Clearly, a significant improvement is achieved after applying the MIQP-MPC as compared to the standard MPC. Fig. 12 shows how the input $u_2$ of the MIQP-MPC moves. It is seen that the input intelligently reaches its terminal value so that a sustained oscillation can be reduced significantly.

4.3. Comparisons with nonlinear inverse strategy

We compare the performance of the MIQP-MPC with the nonlinear inverse strategy by considering two cases: (1) actuator backlash is active, but the actuator is not saturated; (2) both actuator saturation and backlash are active.

Figs. 13 and 14 show the comparison of the two compensation techniques. In this case the inputs do not hit its saturation limit. As a result, the nonlinear inverse strategy performs very well by perfectly compensating the backlash. In the MIQP-MPC, the input movement is suppressed after some time.

The actuator saturation becomes active when we introduce a set-point change in the $O_2$ flue gas concentration from 0.9% to 1.4% with the $u_2$ valve is operating near the saturation limit. In this case, the nonlinear inverse strategy fails to eliminate the backlash due to the saturation in the inputs (see Figs. 15 and 16). The MIQP-MPC, on the other hand, produces a better performance than the nonlinear inverse strategy as it anticipates the actuator saturation well ahead.

4.4. Multiple backlash case

Finally, the MIQP-MPC is tested for the case where the backlash occurs at multiple locations in the inputs. In this study, the backlash is fixed at $d = 0.5$ in two locations, namely in the valves $u_2$ and $u_3$. Three sets of disturbance...
changes are considered; (i) disturbance set $\mathcal{MD}_1 = [0, +0.5 \text{ m}^3/\text{h}, +0.5 \text{ m}^3/\text{h}, 0, 0, 0]$, (ii) disturbance set $\mathcal{MD}_2 = [0, +1 \text{ m}^3/\text{h}, +1 \text{ m}^3/\text{h}, 0, 0, 0]$, and (iii) disturbance set $\mathcal{MD}_3 = [0, +2 \text{ m}^3/\text{h}, +2 \text{ m}^3/\text{h}, 0, 0, 0]$. The design parameters for the MPC are $M = 3$, $P = 20$, and $t_{\text{act}} = 2P$ min. The similar weighting on inputs and outputs was applied.

Fig. 17 shows the performance of the MIQP-based MPC design for different disturbances. In all three cases, the oscillations caused by the backlash nonlinearity are significantly reduced; however, in the case of the smaller disturbance $\mathcal{MD}_1$, a sluggish response is observed for the MIQP-based MPC (see Fig. 17(a)).

Although an improved performance is significantly gained after applying the MIQP-MPC, we should note on the computation time required to solve the MIQP problem at each sampling time. As seen in Table 2, the computation time for the MIQP-based design is much higher than the
standard QP-based design. Nevertheless, for the FCCU case the average computation time is still lower than the sampling time of 1 min. Hence, such an increase in the computational time is still considered to be reasonably small.

5. Conclusions

In this paper, we have studied the effect of the actuator backlash on the MPC performance. In the presence of actuator backlash, the performance of MPC could deteriorate significantly, especially when the MPC was aggressively tuned. A tactical approach of reducing the effect of the actuator backlash has been developed by extending the standard MPC design to simultaneously and explicitly address both actuator saturation and backlash.

A hybrid formulation and design of MPC has been derived, where we used an MIQP optimization to generate...
optimal inputs in each sampling time. The MIQP-based approach was implemented to suppress the input move only at the proximity of the steady state. The MIQP-based approach could produce signals outside the dead-band (or keep the previous input value), which is similar to the nonlinear inverse strategy; however, such decision was made in a tactical way by respecting the input saturation limits. In its implementation, we could use a simple, online self-detection and estimation algorithm to automatically determine the activation time, the backlash width and the location of the backlash as done in Zabiri and Samyudia [28] and Zabiri [27].

The simulation results obtained from the application of the method to the FCCU case study demonstrated that in the presence of both actuator backlash and saturation the developed MIQP-MPC could produce a better closed-loop performance than the traditional design approaches.

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References