Range assignment for energy efficient broadcasting in linear radio networks

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Received 24 August 2005; received in revised form 5 November 2005; accepted 19 November 2005

Communicated by J. Diaz

Abstract

Given a set $S$ of $n$ radio-stations located on a $d$-dimensional space, a source node $s \in S$ and an integer $h$ ($1 \leq h \leq n - 1$), the $h$-hop broadcast range assignment problem deals with assigning ranges to the members in $S$ so that $s$ can communicate with all other members in $S$ in at most $h$-hops, and the total power consumption is minimum. The problem is known to be NP-hard for $d \geq 2$. We propose an $O(n^2)$ time algorithm for the one dimensional version ($d = 1$) of the problem. This is an improvement over the existing result on this problem by a factor of $h$ [A.E.F. Clementi et al. The minimum broadcast range assignment problem on linear multi-hop wireless networks, Theoret. Comput. Sci. 299 (2003) 751–761].

Keywords: Range assignment; Broadcast; Mobile communication; Algorithm

1. Introduction

We consider the problem of assigning transmission ranges to the nodes of a linear radio-network to minimize power consumption while ensuring broadcast from a dedicated node (called source) to all other nodes in the network. A radio-network is a finite set $S$ of radio-stations located on a geographical region which can communicate each other by transmitting and receiving radio signals. Each radio-station $s \in S$ is assigned a range $\rho(s)$ (a non-negative real number) for communication with other stations. A radio-station $s$ (having $\rho(s) > 0$) can communicate (i.e., send a message) directly (i.e., in 1-hop) to any other station $t$, if the Euclidean distance between $s$ and $t$ is less than or equal to $\rho(s)$. If $s$ cannot communicate directly with $t$ due to its assigned range, then communication between them can be achieved using multi-hop transmission. If the maximum number of hops allowed ($h$) is small, then communication between a pair of radio-stations happen very quickly, but the power consumption of the entire radio-network will be high. On the other hand, if $h$ is large then the power consumption decreases, but communication delay takes place. The tradeoff between the power consumption of the radio-network and the maximum number of hops needed between a communicating pair of radio-stations are studied extensively in [6,7]. As in [5], we assume that $\text{power}(s) = (\rho(s))^2$. Thus the total power requirement (cost) of a range assignment $R = \{\rho(s) \mid s \in S\}$ is $\text{cost}(R) = \sum_{s \in S} \text{power}(s) = \sum_{s \in S} (\rho(s))^2$.

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The objective of h-hop broadcast range assignment problem is to assign transmission ranges \( \rho(t) \) to the radio-stations \( t \in S \) so that a dedicated radio-station (say \( s \in S \)) can transmit message to all other radio-stations using at most \( h \)-hops, and the total power consumption of the entire network is minimum. For \( h \geq 2 \), the problem is NP-hard even in 2D [2,3]. For the 1D version of the problem, a dynamic programming based algorithm is proposed in [5]. It runs in \( O(n h n^2) \) time, where \( n = |S| \). We improve the time complexity result of the problem proposed in [5]. Our algorithm is simple, and it runs in \( O(n^2) \) time and \( O(hn) \) space.

In spite of the fact that the model considered in this paper is simple, it is very much useful in studying road traffic information system where the vehicles follow roads and messages are to be broadcasted along lanes. Typically, the curvature of the road is small in comparison to the transmission range so that we can consider that the vehicles are moving on a line [4]. Linear radio networks have been observed to be important in several recent studies [4–7].

2. Structure of optimal broadcast range assignment

We assume that the radio-stations \( S = \{s_1, s_2, \ldots, s_n\} \) are ordered on the x-axis from left to right, with \( s_1 \) positioned at 0 (the origin). The position of \( s_j \) will be denoted by \( x(s_j) \). Thus, the distance between two radio-stations \( s_i \) and \( s_j \) is \( \delta(s_i, s_j) = |x(s_i) - x(s_j)| \). We will use \( C(S, s, h) \) to denote the minimum among the costs of the range assignments of the members in \( S \) for broadcasting message from the source station \( s (\in S) \) to all other radio-stations in \( S \) using at most \( h \)-hops. There may be several range assignments of \( S \) having cost \( C(S, s, h) \). We will use \( R(S, s, h) \) to denote one such range assignment, and will refer to it as optimal range assignment.

**Definition 1.** In a \( h \)-hop broadcast range assignment, a right-bridge \( \overline{s_\ell s_r} \) corresponds to a pair of radio-stations \( (s_\ell, s_r) \) such that \( s_\ell \) is to the left of \( s \), \( s_r \) is to the right of \( s \), and \( \delta(s_\ell, s_r) \leq \rho(s_r) < \delta(s_{\ell-1}, s_r) \). In other words, \( s_r \) can communicate with \( s_\ell \) in 1-hop due to its assigned range, but it cannot communicate with \( s_{\ell-1} \) in 1-hop.

**Definition 2.** In a \( h \)-hop broadcast range assignment, a right-bridge \( \overline{s_\ell s_r} \) (if exists) is called functional, if there exists a radio-station \( s_j \in S \) such that the minimum number of hops among all the paths from \( s \) to \( s_j \) that avoids the 1-hop communication \( \overline{s_\ell s_r} \), is greater than \( h \).

Similarly, one can define a left-bridge \( \overline{s_\ell s_r} \) and a functional left-bridge in a \( h \)-hops range assignment, where \( s_\ell \) and \( s_r \) are respectively to the left and right of \( s \).

**Theorem 1** (Clementi et al. [5]). Given a set of radio-stations \( S = \{s_1, s_2, \ldots, s_n\} \), a source node \( s \in S \), and an integer \( h \) (\( 1 \leq h \leq n - 1 \)), the optimal \( h \)-hop broadcast range assignment \( R(S, s, h) \) contains at most one functional bridge.

The algorithm proposed in [5] solves the problem in three phases. It computes optimal solutions having (i) no functional (left/right) bridge, (ii) one functional left-bridge only, and (iii) one functional right-bridge only. Finally, the one having minimum total cost is reported. Our algorithm is based on the same principle as in [5], but it considers the geometry of the range assignment for obtaining the optimal solution in each of the three cases mentioned in (i)–(iii) in a careful manner, and this leads to an algorithm with improved time complexity.

3. Geometric properties

**Lemma 1.** In a linear ordered set of radio-stations \( \{s_a, s_{a+1}, \ldots, s_b\} \subseteq S \), if the source station is at one end of the above set (say \( s_a \)), then for any \( 1 \leq \mu \leq b-a \), an optimum \( \mu \)-hop broadcast range assignment \( R(\{s_a, s_{a+1}, \ldots, s_b\}, s_a, \mu) \) should satisfy \( \sum_{k=a}^{b-1} \rho(s_k) = x(s_b) - x(s_a) \).

**Proof.** Consider the \( \mu \)-hop path for communication from \( s_a \) to \( s_b \) as shown in Fig. 1(a). Note that, one can reduce the total cost of range assignment \( (\sum_{k=a}^{b-1} \rho(s_k))^2 \) by setting \( \rho(s_{j+1}) = \rho(s_{j+2}) = \ldots = \rho(s_1) = 0 \) (see Fig. 1(b)). This maintains \( \mu \)-hop connections from \( s_a \) to all other nodes in the set. \( \square \)
Lemma 2. For a set of radio-stations $S = \{s_1, s_2, \ldots, s_n\}$, $C(S, s_1, \mu) = C(S, s_n, \mu)$.

Proof. Let $\{a_0, a_1, \ldots, a_{\mu-1}\} \subseteq S$ be the sequence of radio-stations having non-zero ranges in $\mathcal{R}(S, s_1, \mu)$. Here $a_0 = s_1$, and let us denote $a_{\mu} = s_n$. By Lemma 1, $\rho(a_i) = x(a_{i+1}) - x(a_i)$, for $i = 0, 1, \ldots, \mu - 1$. A feasible range assignment for communicating from $s_0$ to $s_1$ using $\mu$-hops is $\rho(a_i) = x(a_i) - x(a_{i-1})$, for $i = 1, 2, \ldots, \mu$, and its cost is same as $C(S, s_1, \mu)$. Thus $C(S, s_1, \mu) \leq C(S, s_n, \mu)$. Following the same method, it can be shown that $C(S, s_1, \mu) \leq C(S, s_n, \mu)$. Hence the result follows. □

Lemma 3. In an optimum $\mu$-hop broadcast range assignment $\mathcal{R}(S, s_1, \mu)$, if the range assigned to $s_1$ is $\rho(s_1) = \delta(s_1, s_j)$ for some $j > 1$, then there exists a $\mathcal{R}(S\{s_1\}, s_2, \mu)$, where $\rho(s_2) \geq \delta(s_2, s_j)$.

Proof. In $\mathcal{R}(S, s_1, \mu)$, $\rho(s_1) = \delta(s_1, s_j)$ implies that $\rho(s_2) = \rho(s_3) = \cdots = \rho(s_{j-1}) = 0$. Thus, if $C(S, s_1, \mu) = c$ then $C(S\{s_1, s_2, \ldots, s_{j-1}\}, s_j, \mu - 1) = c - (\delta(s_1, s_j))^2$. In other words, the range assignments of the radio-stations $S\{s_1, s_2, \ldots, s_{j-1}\}$ in $\mathcal{R}(S, s_1, \mu)$ are such that, it supports broadcasting from $s_j$ to all the radio-stations $\{s_{j+1}, \ldots, s_n\}$ in $(\mu - 1)$-hops with minimum cost.

Now, let us assume that the range assigned to $s_1$ is $\rho(s_2) = \delta(s_1, s_k)$. We need to prove that $k \geq j$.

Let us assume that $C(S\{s_1\}, s_2, \mu) = c'$. This implies, $C(S\{s_1, s_2, \ldots, s_{k-1}\}, s_k, \mu - 1) = c' - (\delta(s_2, s_k))^2$. Thus, $\{\delta(s_1, s_k), 0, 0, \ldots, 0, \mathcal{R}(S\{s_1, s_2, \ldots, s_{k-1}\}, s_k, \mu - 1)\}$ is a feasible range assignment (may not be optimum) for the $\mu$-hop broadcast from $s_1$ to all the nodes in $S\{s_1\}$, and its cost is equal to $(\delta(s_1, s_k))^2 + (c' - (\delta(s_2, s_k))^2) \geq c$. This implies, $c - c' \leq (\delta(s_1, s_2))^2 + 2\delta(s_1, s_2)\delta(s_2, s_k)$.

By a similar argument, $\{\delta(s_2, s_j), 0, 0, \ldots, 0, \mathcal{R}(S\{s_1, s_2, \ldots, s_{j-1}\}, s_j, \mu - 1)\}$ is a feasible range assignment for the $\mu$-hop broadcast from $s_2$ to all the nodes in $S\{s_1, s_2\}$, and its cost is equal to $(\delta(s_2, s_j))^2 + (c - (\delta(s_1, s_j))^2) \geq c'$. This implies, $c - c' \geq (\delta(s_1, s_2))^2 + 2\delta(s_1, s_2)\delta(s_2, s_j)$.

Combining these two inequalities, we have

$$(\delta(s_1, s_2))^2 + 2\delta(s_1, s_2)\delta(s_2, s_j) \leq c - c' \leq (\delta(s_1, s_2))^2 + 2\delta(s_1, s_2)\delta(s_2, s_k).$$

This implies $k \geq j$. □

In the following lemma, we prove that if we increase the number of allowable hops for broadcasting from a fixed radio-station, say $s_1$, to all the radio-stations to its right, then the gain in the cost obtained in two consecutive steps are monotonically decreasing.

Lemma 4. $C(S, s_1, \mu) - C(S, s_1, \mu + 1) \geq C(S, s_1, \mu + 1) - C(S, s_1, \mu + 2)$.

Proof. Let $A = \{a_0 = s_1, a_1, a_2, \ldots, a_{\mu-1}\}$ denote the subsequence (radio stations) of $S$ having non-zero ranges in $\mathcal{R}(S, s_1, \mu)$. We use $a_{\mu}$ to denote the radio-station $s_n$ and $\text{cost}(A)$ to denote $C(S, s_1, \mu)$. Here, the range assigned to $a_i \in A$ is $(x(a_{i+1}) - x(a_i))$ for $i = 0, 1, 2, \ldots, \mu - 1$. Again, let $B = \{b_0 = s_1, b_1, b_2, \ldots, b_{\mu+1}\}$ denotes the set of
Thus, the corresponding costs of the range assignments are $\mu$-hop broadcast range assignment from $s_1$ to $s_n$

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radio stations having non-zero ranges in $R(S, s_1, \mu + 2)$, i.e., $cost(B) = C(S, s_1, \mu + 2)$. As earlier, $s_n$ is denoted by $b_{\mu+2}$, and the ranges assigned to $b_i$ ($b \in B$) are $(x(b_{i+1}) - x(b_i))$ for $i = 0, 1, 2, \ldots, \mu + 1$. The two range assignments ($A$ and $B$) are shown in Fig. 2(a) using solid and dashed lines.

Observe that, $x(a_0) - x(b_1) < 0$, and $x(a_\mu) - x(b_\mu+1) > 0$. This implies, there exists at least one $i \in \{1, 2, \ldots, \mu - 1\}$ such that $x(a_i) - x(b_{i+1}) \leq 0$ and $x(a_{i+1}) - x(b_{i+2}) \geq 0$. We consider the smallest $i \geq 0$ such that $x(a_{i+1}) - x(b_{i+2}) \geq 0$, and construct two subsequences of radio stations $C = \{a_0 = b_0 = s_1, a_1, \ldots, a_i, b_{i+2}, b_{i+3}, \ldots, b_\mu+1\}$ and $D = \{a_0 = b_0 = s_1, b_1, b_2, \ldots, b_i+1, a_{i+1}, a_{i+2}, \ldots, a_\mu-1\}$, each of length $\mu + 1$. The ranges assigned to the members in $C$ and $D$ are, respectively,

- $\{x(a_1) - x(a_0), \ldots, x(a_i) - x(a_{i-1}), x(b_{i+2}) - x(a_i), x(b_{i+3}) - x(b_{i+2}), \ldots, x(b_\mu+2) - x(b_{\mu+1})\}$ (see Fig. 2(b)), and
- $\{x(b_1) - x(b_0), \ldots, x(b_{i+1}) - x(b_i), x(a_{i+1}) - x(b_{i+1}), x(a_{i+2}) - x(a_{i+1}), \ldots, x(a_\mu) - x(a_{\mu-1})\}$ (see Fig. 2(c)).

The corresponding costs of the range assignments are

\[
\text{cost}(C) = \sum_{j=0}^{j=i-1} (x(a_{j+1}) - x(a_j))^2 + (x(b_{j+2}) - x(a_j))^2 + \sum_{j=i+2}^{j=\mu+1} (x(b_{j+1}) - x(b_j))^2
\]

and

\[
\text{cost}(D) = \sum_{j=0}^{j=i} (x(b_{j+1}) - x(b_j))^2 + (x(a_{i+1}) - x(b_{i+1}))^2 + \sum_{j=i+1}^{j=\mu-1} (x(a_{j+1}) - x(a_j))^2.
\]

Thus,

\[
\text{cost}(C) + \text{cost}(D) = \left(\sum_{j=0}^{j=\mu-1} (x(a_{j+1}) - x(a_j))^2 - (x(a_{i+1}) - x(a_i))^2\right) + \left(\sum_{j=0}^{j=\mu+1} (x(b_{j+1}) - x(b_j))^2 - (x(b_{i+2}) - x(b_{i+1}))^2\right) + (x(b_{i+2}) - x(a_i))^2 + (x(a_{i+1}) - x(b_{i+1}))^2
\]

$= \text{cost}(A) + \text{cost}(B) + 2(x(a_i) - x(b_{i+1}))(x(a_{i+1}) - x(b_{i+2})) \leq \text{cost}(A) + \text{cost}(B)$ (due to the choice of $i$ as mentioned above).

Let $O$ indicate a subsequence of $\mu + 1$ radio-stations with non-zero range assignments such that $s_1$ can send message to $s_n$ in $\mu + 1$ hops (or equivalently to all members in $S$ in at most $\mu + 1$ hops) and the cost of range assignment is minimum, i.e., $\text{cost}(O) = C(S, s_1, \mu + 1)$. Thus we have, $2 \times \text{cost}(O) \leq \text{cost}(C) + \text{cost}(D) \leq \text{cost}(A) + \text{cost}(B)$. □
4. Algorithms

Let $s_z \in S$ be the given source station (not necessarily the left-most/right-most in the ordering of $S$). Our algorithm for broadcasting from $s_z$ to all other radio-stations $s_j \in S$ consists of three phases. Phase 1 prepares four initial matrices. These are used in Phases 2 and 3 for computing optimal solution with no functional bridge, and exactly one functional bridge, respectively.

For notational convenience, if the source radio-station (say $s_a$) is at one end of a linearly ordered destination stations $\{s_1, s_{a+1}, \ldots, s_b\}$, then we will use $R(s_j, s_a, \mu)$ and $C(s_j, s_a, \mu)$ to denote the optimal range assignment $R(\{s_1, s_{a+1}, \ldots, s_b\}, s_a, \mu)$ and $C(\{s_a, s_{a+1}, \ldots, s_b\}, s_a, \mu)$, respectively.

4.1. Phase 1

In this phase, we prepare the following four initial matrices. These will be extensively used in Phases 2 and 3. Recall that $s_z$ is the source station.

$M_1$: It is a $h \times (x - 1)$ matrix. Its $(m, j)$th element $(1 \leq j < x)$ indicates the optimum cost of sending message from $s_j$ to $s_z$ (source station) using $m$ hops. In other words, $M_1[m, j] = C(s_j, s_z, m)$, where $1 \leq m \leq h$ and $1 \leq j < x$.

$M_2$: It is a $h \times (x - 1)$ matrix. Its $(m, j)$th element $(1 \leq j \leq x)$ indicates the optimum cost of sending message from $s_j$ to $s_1$ (left-most radio-station in $S$) using $m$ hops. In other words, $M_2[m, j] = C(s_1, s_j, m)$, where $1 \leq m \leq h$ and $1 \leq j \leq x$.

$M_3$: It is an $(n - x) \times (n - x)$ matrix. Its $(m, j)$th element $(x < j \leq n)$ indicates the optimum cost of sending message from $s_j$ to $s_z$ using $m$ hops. In other words, $M_3[m, j] = C(s_z, s_j, m)$, where $1 \leq m \leq h$ and $x < j \leq n$.

$M_4$: It is a $x \times (n - x)$ matrix. Its $(m, j)$th element $(x \leq j < n)$ indicates the optimum cost of sending message from $s_j$ to $s_n$ (right-most radio-station) using $m$ hops. In other words, $M_4[m, j] = C(s_n, s_j, m)$, where $1 \leq m \leq h$ and $x \leq j < n$.

Note that, the columns of $M_1$ are indexed as $[1, 2, \ldots, x - 1]$, whereas those in $M_2$ are indexed as $[2, 3, \ldots, x]$. Similarly, the columns of $M_3$ are indexed as $[x + 1, x + 2, \ldots, n]$, whereas those in $M_4$ are indexed as $[x, x + 1, \ldots, n - 1]$. We explain an incremental approach (in terms of hops) for constructing $M_1$. Similar procedure works for constructing the other three matrices.

Each entry of the matrix $M_1$ contains a tuple $(\chi, ptr)$, where the $\chi$ field of $M_1[m, j]$ contains $C(s_j, s_z, m)$, and its $ptr$ field is an integer which contains the index of the first radio-station (after $s_j$) on the $m$-hop path from $s_j$ to $s_z$. We will interchangeably use, $M_1[m, j]$ and $M_1[m, j]$. $\chi$ to denote $C(s_z, s_j, m)$. After computing up to row $m$ of the matrix $M_1$, the elements in the $(m + 1)$th row can easily be obtained as follows:

Consider an intermediate matrix $A$ of size $(x - 1) \times (x - 1)$. Its $(j, k)$th element contains the cost of $(m + 1)$-hop communication from $s_j$ to $s_k$ with first hop at $s_k$. Thus, $A[j, k] = (\delta(s_j, s_k))^2 + M_1[m, k]$. After computing the matrix $A$, we compute $M_1[m + 1, j] = Min_{k=1}^{\chi - 1} A[j, k]$, and $M_1[m + 1, j, ptr]$ will contain the value of $k$ for which $A[j, k]$ is contributed to $M_1[m + 1, j, ptr]$.

Straightforward application of the above method needs $O(x^2)$ time. But, Lemma 3 says that, if in the optimum $(m + 1)$-hop path from $s_j$ to $s_k$ with first hops at node $s_k$, then for any node $s_j'$ with $j' > j$, the optimum $(m + 1)$-hop path from $s_j'$ to $s_k$ with first hops at node $s_k'$ and $k' > k$. A simple method for computing the minimum of every row in the matrix $A$ (without enumerating all the entries in $A$) needs a total of $O(x \log x)$ time as follows:

Compute all the entries in the $\frac{x}{2}$th row of the matrix $A$, and find the minimum. Let it corresponds to $A[\frac{x}{2}, \beta]$. Next, compute the minimum entry in the $\frac{x}{4}$th row of $A$ by considering $\{A[\frac{x}{2}, j], j = 1, 2, \ldots, \beta\}$, and compute the minimum entry in the $\frac{3x}{4}$th row of $A$ by considering $\{A[\frac{3x}{4}, j], j = \beta, \beta + 1, \ldots, x - 1\}$. The process continues until all the rows of $A$ are considered.

Definition 3 (Aggarwal and Klawe [1]). A matrix $M$ is said to be monotone if for every $j, k, j', k'$ with $j < j', k < k'$, if $M[j, k] \geq M[j', k']$ then $M[j', k] \geq M[j, k']$.

Lemma 5. The matrix $A$ is a monotone matrix.
Proof. Given $A[j, k] \geq A[j, k']$, where $A[j, k] = (\delta(s_j, s_k))^2 + M_1[m, k]$ and $A[j, k'] = (\delta(s_j, s_{k'}))^2 + M_1[m, k']$. Thus, $M_1[m, k] - M_1[m, k'] \geq (\delta(s_j, s_{k'}))^2 - (\delta(s_j, s_k))^2$.

Now,

$$A[j', k] - A[j', k'] = (\delta(s_{j'}, s_k))^2 - (\delta(s_{j'}, s_{k'}))^2 + M_1[m, k] - M_1[m, k']$$

$$\geq (\delta(s_{j'}, s_k))^2 - (\delta(s_{j'}, s_{k'}))^2 + (\delta(s_j, s_{k'}))^2 - (\delta(s_j, s_k))^2 \geq 0$$

(on simplification). □

A recursive algorithm for monotone matrix searching is described in [1], which can compute the minimum entry in each row of a $x \times x$ monotone matrix in $O(x)$ time provided each entry of the matrix can be computed in $O(1)$ time. Using that algorithm, the matrix $M_1$ can be computed in $O(x \times h)$ time.

Lemma 6. Phase 1 needs $O(nh)$ time.

Proof. Follows from the fact that $M_1, M_2$ can be constructed in $O(x \times h)$ time, and $M_3, M_4$ needs $O((n - x) \times h)$ time. □

4.2. Phase 2

In this phase, we compute the optimal functional bridge-free solution for broadcasting message from $s_x$ to the other nodes in $S$. Here, the range to be assigned to $s_x$ is at least $\max(\delta(s_x, s_{x-1}), \delta(s_x, s_{x+1}))$.

Without loss of generality, we assume that $\delta(s_x, s_{x-1}) \leq \delta(s_x, s_{x+1})$. Thus, $\rho(s_x)$ is initially assigned to $\delta(s_x, s_{x+1})$, and let $s_k (k < x)$ be the farthest radio-station such that $s_x$ can communicate with $s_k$ in 1-hop (i.e., $\delta(s_k, s_x) \leq \delta(s_x, s_{x+1})) < \delta(s_{k-1}, s_x))$. If we use $R(S, s_x, h|\rho(s_x) = d)$ to denote the optimum range assignment for the $h$-hop broadcasting from $s_x$ to all the nodes in $S$ subject to the condition that the range assigned to $s_x$ is $d$, then

$$R(S, s_x, h|\rho(s_x) = \delta(s_x, s_{x+1})) = \{R((s_1, \ldots, s_k), s_k, h - 1), 0, 0, \ldots, 0, \delta(s_x, s_{x+1}), R(S \setminus \{s_1, \ldots, s_x\}, s_{x+1}, h - 1)\},$$

and its cost is

$$C^* = C(S, s_x, h|\rho(s_x) = \delta(s_x, s_{x+1})) = (\delta(s_x, s_{x+1}))^2 + M_2[h - 1, k] + M_4[h - 1, x + 1].$$

This can be computed in $O(1)$ time using the matrices $M_2$ and $M_4$. We use two temporary variables TEMP_Cost and TEMP_id to store $C^*$ and $s_{x+1}$.

Next, we increment $\rho(s_x)$ to $\min(\delta(s_x, s_{k-1}), \delta(s_x, s_{x+2}))$, and apply the same procedure to calculate the optimum cost of the $h$-hop broadcast from $s_x$. This may cause update of TEMP_Cost and TEMP_id. The same procedure is repeated by incrementing $\rho(s_x)$ to its next choice in the set $\{\delta(s_x, s_k), k = 1, 2, \ldots, x-1\} \cup \{\delta(s_x, s_j), j = k+1, \ldots, n\}$ so that it can communicate directly with one more node than its previous choice. At each step, the TEMP_Cost and TEMP_id are adequately updated. Thus, the procedure is repeated for $O(n)$ times, and the time complexity of this phase is $O(n)$.

4.3. Phase 3

In this phase, we compute an optimal range assignment for the $h$-hop broadcasting from $s_x$ to all other nodes in $S$ where the solution contains a functional right-bridge. Similar method will be adopted to compute the optimal solution with one functional left-bridge. The one having minimum cost is chosen as the optimal solution obtained in this phase.
Let us consider a range assignment which includes a right-bridge $s_i, s_j$, $i < x < j$. Let $s_i$ be such that $\delta(s_j, s_k) \leq \delta(s_j, s_i) < \delta(s_j, s_{k+1}), k \geq j$. This can be realized in the following two ways:

**Scheme 1.** Assign $\rho(s_j) = \delta(s_j, s_i)$.

**Scheme 2.** If $\delta(s_j, s_k) < \delta(s_i, s_j) < \delta(s_j, s_{k+1}) < \delta(s_j, s_{i-1})$, then assign $\rho(s_j) = \delta(s_j, s_{k+1})$.

We assume that $s_j$ is reached from $s_x$ using $m$ hops. Thus, using Scheme 1, $h$-hops connection from $s_x$ to all the nodes in $S$ is achieved by (i) reaching $s_i$ from $s_x$ in $(h - m - 1)$ hops, and (ii) reaching $s_x$ from $s_k$ in $(h - m - 1)$ hops. Here the cost of the range assignment is $B_1 = C(s_j, s_x, m) + (\delta(s_j, s_x))^2 + C(s_i, s_x, h - m - 1) + C(s_i, s_k, h - m - 1)$.

In Scheme 2, $s_j$ can directly communicate to $s_{k+1}$ to the right, and $s_i$ to the left. Thus, the $h$-hops connection from $s_x$ to all the nodes in $S$ is established by (i) reaching $s_i$ from $s_x$ in $(h - m - 1)$ hops, and (ii) reaching $s_x$ from $s_{k+1}$ in $(h - m - 1)$ hops. Here the cost of the range assignment is $B_2 = C(s_j, s_x, m) + (\delta(s_j, s_{k+1}))^2 + C(s_i, s_x, h - m - 1) + C(s_x, s_k, h - m - 1)$.

Denoting by $B(s_i, s_j, m)$ the cost of the range assignment with a right-bridge $\hat{s}_i, \hat{s}_j$ where $s_j$ is reached from $s_x$ using $m$ hops, we have $B(\hat{s}_i, \hat{s}_j, m) = \text{Min}(B_1, B_2)$.

Apart from identifying $s_x, B(\hat{s}_i, \hat{s}_j, m)$ can be calculated in $O(1)$ time, because

(i) $C(s_j, s_x, m) = C(s_x, s_x, m) = M_3[m, j]$ (by Lemma 2),
(ii) $C(s_i, s_x, h - m - 1) = M_2[h - m - 1, i]$,
(iii) $C(s_x, s_k, h - m - 1) = M_4[h - m - 1, k]$, and
(iv) all these matrices are already available.

To get an optimal solution with a right-bridge, we need to find $\text{Min}_{j=1}^{n-1} \text{Min}_{j=2}^{n} \text{Min}_{m=1}^{h-1} B(\hat{s}_i, \hat{s}_j, m)$.

In our algorithm, we fix each $s_i$ and compute $\text{Min}_{j=2}^{n} \text{Min}_{m=1}^{h-1} B(\hat{s}_i, \hat{s}_j, m)$ using Lemma 7.

**Lemma 7.** If $s_j \in S \setminus \{s_1, s_2, \ldots, s_x\}$, then

$$C(s_j, s_x, \mu - 1) - C(s_j, s_x, \mu) \leq C(s_{j+1}, s_x, \mu - 1) - C(s_{j+1}, s_x, \mu).$$

**Proof.** Let $A = \{a_0 = s_x, a_1, a_2, \ldots, a_{\mu - 1}\}$ denote the subsequence (radio stations) of $S$ having non-zero ranges in $R(s_{j+1}, s_x, \mu - 1)$. We use $a_{\mu - 1}$ to denote $s_{j+1}$. Thus, the range assigned to $a_i \in A$ is $x(a_{i+1}) - x(a_i)$ for $i = 0, 1, 2, \ldots, \mu - 2$. We use cost$(A)$ to denote $C(s_{j+1}, s_x, \mu - 1)$. Again, let $B = \{b_0, b_1, b_2, \ldots, b_{\mu - 1}\}$ denotes the set of radio-stations having non-zero ranges in $R(s_j, s_x, \mu)$, i.e., cost$(B) = C(s_j, s_x, \mu)$. The ranges assigned to $b_i (\in B)$ is $(x(b_{i+1}) - x(b_i))$ for $i = 0, 1, 2, \ldots, \mu - 1$.

Let us now observe the pairs $(a_i, b_{i+1})$, for $i = 0, 1, 2, \ldots, \mu - 1$. Note that, $x(a_0) - x(b_1) < 0$, and $x(a_{\mu - 1}) - x(b_{i+1}) > 0$. This implies, there exists at least one $i \in [1, 2, \ldots, \mu - 1]$ such that $x(a_{i-1}) - x(b_i) \leq 0$ and $x(a_i) - x(b_{i+1}) \geq 0$. We consider the smallest $i \geq 1$ such that $x(a_i) - x(b_{i+1}) \geq 0$, and construct two subsequences of radio stations, namely $C = \{a_0 = b_0, a_1, a_2, \ldots, a_{i-1}, b_{i+1}, b_{i+2}, \ldots, b_{\mu - 1}\}$ and $D = \{a_0 = b_0, b_1, b_2, \ldots, b_i, a_i, a_{i+1}, \ldots, a_{\mu - 2}\}$, where length of $C$ is $\mu - 1$ and that of $D$ is $\mu$. The ranges assigned to the members in $C$ and $D$ are, respectively,

- $[x(a_1) - x(a_0), \ldots, x(a_{i-1}) - x(a_{i-2})]$, $[x(b_{i+1}) - x(a_1), x(b_{i+2}) - x(b_{i+1}), \ldots, x(b_i) - x(b_{i+1})]$,
- $[x(b_1) - x(b_0), \ldots, x(b_i) - x(b_{i+1}), x(a_1) - x(b_i), x(a_{i+1}) - x(a_i), \ldots, x(a_{\mu - 1}) - x(a_{i-2})]$.

The corresponding costs of these range assignments are

$$\text{cost}(C) = \sum_{j=0}^{j=i-2} (x(a_{j+1}) - x(a_j))^2 + (x(b_{j+1}) - x(a_{j-1}))^2 + \sum_{j=i+1}^{j=\mu-1} (x(b_{j+1}) - x(b_j))^2,$$

and

$$\text{cost}(D) = \sum_{j=0}^{j=i-1} (x(b_{j+1}) - x(b_j))^2 + (x(a_i) - x(b_i))^2 + \sum_{j=i}^{j=\mu-2} (x(a_{j+1}) - x(a_j))^2.$$

Thus,

$$\text{cost}(C) + \text{cost}(D) = \left( \sum_{j=0}^{j=\mu-2} (x(a_{j+1}) - x(a_j))^2 - (x(a_i) - x(a_{i-1}))^2 \right).$$
Lemma 9. For a given $i$, the right-bridge is no need to check the costs by increasing $m$.

Here, $B_{(s_1, s_2)}$ automatically imply in $B_{(s_1, s_2, h)}$. As stated in the lemma, $B_{(s_1, s_2)}$ implies $a_1 = a_2 - a_3 < 0$. Thus, $B_{(s_1, s_2)} = 0$ and $B_{(s_1, s_2, h)} = 0$. This causes the reduction in number of hops from $h = \mu - 2$ to $h = \mu - 3$ for reaching $a_1 = a_2 - a_3 < 0$. By Lemma 4, $a_1 < a_2$ and $a_3 = a_3$. Then,

$$B_{(s_1, s_2, h)} = 0 - B_{(s_1, s_2, h)} = 0 - a_3 = a_1 < a_2 - a_3 < 0.$$ 

Lemma 8 implies that while using the right-bridge $s_{i+j+1}$, we vary the number of hops $m$ to reach from $s_2$ to $s_j$, and compute the corresponding cost $B_{(s_1, s_j)}$. As soon as $m = \mu$ is reached such that $B_{(s_1, s_j)} = B_{(s_1, s_j, h)} + 1$, there is no need to check the costs by increasing $m$ beyond $\mu + 1$.

After computing the optimum range assignment with the right-bridge $s_{i+j+1}$, we proceed to compute the same with right-bridge $s_{i+j}$. The following lemma says that if the optimum $B_{(s_1, s_j)}$ is achieved for $m = \mu$ then while considering the right-bridge $s_{i+j+1}$, the optimum $B_{(s_1, s_j+1)}$ will be achieved for some $m > \mu$. Here, it needs to be mentioned that, we could not explore any relationship among the optimum costs of range assignments using $s_{i+j}$ and $s_{i+j+1}$.

Lemma 9. For a given $s_i \in S$, $i < \alpha$, if $\text{Min}_{m=1} B_{(s_1, s_j)}$ and $\text{Min}_{m=1} B_{(s_1, s_{j+1})}$ are achieved for $m = \mu$ and $\nu$, respectively, then $\nu > \mu$.

Proof. As $s_i$ is fixed, we compute the optimal range assignment $R(s_1, s_i, h - m - 1)$ to reach from $s_i$ to $s_i$.

While using $s_{i+j}$, $\rho(s_j) = \delta(s_j, s_i)$, and this enables $s_j$ to reach $s_i$ to its right (i.e. $\delta(s_j, s_i) \geq \delta(s_j, s_k)$). Similarly, while using $s_{i+j+1}$, $\rho(s_j) = \delta(s_j+1, s_i)$, and this enables $s_{j+1}$ to reach $s_i$ to its right (i.e. $\delta(s_{j+1}, s_i) \geq \delta(s_{j+1}, s_j)$). Here $j + 1 \leq k \leq \ell$.

In order to prove the lemma, we need only to show that $B_{(s_1, s_{j+1})} = B_{(s_1, s_{j+1}, h)}$. By Lemma 8, this will automatically imply $B_{(s_1, s_{j+1}, h)} = B_{(s_1, s_{j+1}, h)}$ for all $m \leq \mu$. Thus, if $\text{Min}(B_{(s_1, s_{j+1}, h)})$ is achieved for $m = \nu$, then $\nu > \mu$.

To prove the above inequality, let $a_1 = C(s_j, s_{j-1}, h - m - 1) - C(s_j, s_{j-1}, h - m),$

$$a_1 = C(s_j, s_{j-1}, h - m - 1) - C(s_j, s_{j-1}, h - m),$$

$$a_2 = C(s_1, s_j, h - m - 1) - C(s_1, s_j, h - m),$$

$$a_3 = C(s_{j-1} + 1, s_j, h - m - 1) - C(s_{j-1} + 1, s_j, h - m)$$ and,

$$a_3 = C(s_{j-1} + 1, s_j, h - m - 1) - C(s_{j-1} + 1, s_j, h - m).$$
As $B(\overline{s_j}, \mu - 1) > B(\overline{s_j}, \mu)$, we have $a_1 - a_2 - a_3 > 0$. By Lemma 7, $a_1' > a_1$ and $a_3' < a_3$. Hence, the amount of gain in cost for increasing the number of hops from $\mu - 1$ to $\mu$ for reaching from $s_x$ to $s_{j+1}$ and then using the bridge $\overline{s_j}s_{j+1}$ for the broadcast to the other nodes in $S$ is equal to $B(\overline{s_j}s_{j+1}, \mu - 1) - B(\overline{s_j}s_{j+1}, \mu) = a_1' - a_2 - a_3 > a_1 - a_2 - a_3 \geq 0$. □

Given a source-station $s_x$ and another station $s_i$ $i < x$, the optimal range assignment of the members in $S$ consisting of a functional right-bridge incident at $s_i$, can be computed using the following algorithm:

**Algorithm Range_Assign_using_Right_Bridge($s_i$)**

**Step 1:** We initialize $OPT\_j = x$, $OPT\_cost = \infty$ and $k\_store = x$, and $\mu = 1$ (*$\mu$ stores the number of hops alloted to reach $s_j$ from $s_x$*).

Start with $m = 1$ and $j = x + 1$.

The role of $k\_store$ will be clear in the procedure compute invoked from this algorithm.

**Step 2:** At each $j$, we execute compute($B(\overline{s_j}s_j, m)$, $k\_store$) by incrementing $m$ from its current value upwards until

(i) $B(\overline{s_j}s_j, m) > B(\overline{s_j}s_j, m - 1)$ is achieved (see Lemma 8) or

(ii) $m$ attains its maximum allowable value $\text{Min}(h - 2, j - x)$.

**Step 3:** Update $OPT\_cost$ and $OPT\_j$ observing the value of $B(\overline{s_j}s_j, m - 1)$ or $B(\overline{s_j}s_j, m)$ depending on whether Step 2 has terminated depending on Case (i) or Case (ii).

**Step 4:** For the next choice of $j$, update $\mu$ by $m - 1$ or $m$ depending on whether Case (i) or (ii) occurred in Step 2 (see Lemma 9).

Procedure compute($B(\overline{s_j}s_j, m)$, $k\_store$)

- Initialize $k = k\_store$.
- Increment $k$ to identify the right-most radio-station such that $\delta(s_j, s_k) \leq \rho(s_j) (= \delta(s_j, s_i))$.
- Set $k\_store = k$ for further use. (*i.e., for next $j$, the search for $k$ will start from $k\_store$*)
- Compute $B(\overline{s_j}s_j, m) = (p(s_j))^2 + R(s_j, s_x, m) + R(s_i, s_k, h - m - 1) + R(s_n, s_k, h - m - 1)$; the last three terms are available in $M_3[m, j]$, $M_2[h - m - 1, i]$ and $M_4[h - m - 1, k]$ respectively.

**Theorem 2.** For a given $s_i$ ($i < x$), algorithm Range_Assign_using_Right_Bridge needs $O(n - x + h)$ time in the worst case.

**Proof.** Follows from Lemmas 8 and 9, and the role of $k\_store$ in the procedure compute for locating rightmost $s_k$ such that $\delta(s_j, s_k) \geq \rho(s_j)$. □

4.4 Complexity analysis

**Theorem 3.** Given a set of radio station $S$ and a source station $s_x \in S$, the optimum range assignment for broadcasting message from $s_x$ to all the members in $S$ using at most $h$-hops can be computed in $O(n^2)$ time and using $O(nh)$ space.

**Proof.** Phase 1 needs $O(nh)$ time for initializing the matrices. Optimum functional bridge-free solution can be obtained in $O(n)$ time as described in Phase 2. Finally in Phase 3, we fix $s_x$ to the left of $s_x$ and identify the optimum solution with a functional right-bridge incident at $s_i$ in $O(n - x + h)$ time (see Theorem 2). For $(x - 1)$ such $s_i$’s, the total time required in this phase is $O(x \times (n - x + h))$. Similarly, the worst case time required for finding the optimum range assignment with exactly one functional left-bridge is $O((n - x) \times (x + h))$. Thus, the result follows. □

**Note.** The interesting question is whether one can design an efficient algorithm for Phase 3 such that the time complexity can further be reduce to $O(nh \times \text{polylog}(h))$.

**References**


