Some integral inequalities of Simpson type for GA-$\varepsilon$-convex functions

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Abstract. We introduce a new concept “GA-$\varepsilon$-convex function” and establish some integral inequalities of Simpson type for GA-$\varepsilon$-convex functions.

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1 Introduction

Let us recall some definitions of several kinds of convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2. Let $f : I \subseteq \mathbb{R}^+ = (0, \infty) \rightarrow \mathbb{R}$. If the inequality

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y)$$

is valid for $x, y \in I$ and $t \in [0, 1]$, then $f(x)$ is said to be GA-convex on $I$.

Definition 1.3 ([7]). Let $X$ be a real linear space, the set $D \subseteq X$ be convex, and $f : D \rightarrow \mathbb{R}$ be a mapping. For some constant $\varepsilon \geq 0$, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon$$

holds for all $x, y \in D$ and $t \in [0, 1]$, then $f(x)$ is said to be $\varepsilon$-convex on $D$.

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It is well known that the classical inequalities for usually convex functions defined by Definition 1.1 are due to Hermite–Hadamard, which can be stated as the following theorem.

**Theorem 1.4.** If \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is a convex function on \( [a, b] \) with \( a, b \in I \) and \( a < b \), then we have

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

In the literature, Simpson’s inequality may be referred to as Theorem 1.5 below.

**Theorem 1.5** ([6]). If \( f : [a, b] \rightarrow \mathbb{R} \) is a four times continuously differentiable function on \( (a, b) \) and its fourth derivative on \( (a, b) \) is bounded by \( \| f^{(4)} \| = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \), then

\[
\left| \int_a^b f(x) \, dx - \frac{b - a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a + b}{2}\right) \right] \right| \leq \frac{(b - a)^5 \| f^{(4)} \|}{2880}.
\]

Let us reformulate some Hermite–Hadamard and Simpson type inequalities for the above convex functions.

**Theorem 1.6** ([5, Theorem 2.2]). Let \( a, b \in I^\circ \) with \( a < b \) and \( f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^\circ \). If \( |f'(x)| \) is convex on \( [a, b] \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{(b - a) (|f'(a)| + |f'(b)|)}{8}.
\]

**Theorem 1.7** ([12, Theorems 1 and 2]). Let \( a, b \in I \) with \( a < b \) and \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be differentiable on \( I^\circ \). If \( |f'(x)|^q \) is convex on \( [a, b] \) and \( q \geq 1 \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}
\]

and

\[
\left| \frac{f\left(\frac{a + b}{2}\right)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}.
\]

**Theorem 1.8** ([8, Theorems 2.3 and 2.4]). Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be differentiable on \( I^\circ \), \( a, b \in I^\circ \) with \( a < b \), and \( p > 1 \). If \( |f'(x)|^{p/(p-1)} \) is convex on \( [a, b] \), then

\[
\left| \frac{1}{b - a} \int_a^b f(x) \, dx - \frac{f\left(\frac{a + b}{2}\right)}{2} \right| \leq \frac{b - a}{16} \left( \frac{4}{p + 1} \right)^{1/p} \left\{ \left[ |f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)} \right]^{(p-1)/p} + \left[ 3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right]^{(p-1)/p} \right\}.
\]
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and

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f\left( \frac{a + b}{2} \right) \right| \leq \frac{b-a}{4} \left( \frac{4}{p + 1} \right)^{1/p} \left( |f'(a)| + |f'(b)| \right).
\]

**Theorem 1.9** ([14]). Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be differentiable on \( I^\circ \), \( a, b \in I^\circ \) with \( a < b \), and \( f' \in L([a,b]) \), where \( L([a,b]) \) denotes the set of all Lebesgue integrable functions on the interval \([a,b] \). If \( |f'(x)|^q \) for \( q \geq 1 \) is convex on \([a,b] \), then

\[
\left| \frac{1}{6} \left[ f(a) + f(b) + 4f\left( \frac{a + b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{12} \left[ \frac{2q+1 + 1}{3(q+1)} \right]^{1/q}
\]

\[
\times \left[ \left( \frac{3}{4} |f'(a)|^q + |f'(b)|^q \right)^{1/q} + \left( \frac{1}{4} |f'(a)|^q + 3 |f'(b)|^q \right)^{1/q} \right]
\]

and

\[
\left| \frac{1}{6} \left[ f(a) + f(b) + 4f\left( \frac{a + b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{5(b-a)}{72}
\]

\[
\times \left[ \left( \frac{61 |f'(a)|^q + 29 |f'(b)|^q}{90} \right)^{1/q} + \left( \frac{29 |f'(a)|^q + 61 |f'(b)|^q}{90} \right)^{1/q} \right].
\]

For more information on Hermite–Hadamard and Simpson type inequalities for various convex functions, we refer the reader to the recently published articles [1–4, 8–11, 13, 15–25] and the closely related references therein.

In this paper, we will introduce a new concept “GA-\( \varepsilon \)-convex function” and establish some integral inequalities of Simpson type for GA-\( \varepsilon \)-convex functions.

## 2 A definition and a lemma

The so-called GA-\( \varepsilon \)-convex functions can be defined as follows.

**Definition 2.1.** Let \( f : I \subseteq \mathbb{R}^+ \to \mathbb{R} \) be a mapping and \( \varepsilon \geq 0 \) be a constant. If the inequality

\[
f(x^t y^{1-t}) \leq t f(x) + (1-t) f(y) + \varepsilon
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \), then \( f(x) \) is said to be a GA-\( \varepsilon \)-convex function on \( I \).

**Remark 2.2.** If \( f(x) \) is increasing and \( \varepsilon \)-convex on \( I \subseteq \mathbb{R}^+ \), then it is GA-\( \varepsilon \)-convex on \( I \). If \( f(x) \) is decreasing and GA-\( \varepsilon \)-convex on \( I \subseteq \mathbb{R}^+ \), then it is \( \varepsilon \)-convex on \( I \).
Example 2.3. Let \( f(x) = x^r \) for \( x \in I = (0, 1) \) and \( r \neq 0, 1 \). Then

\[
f(x^t y^{1-t}) = (x^r)^t (y^r)^{1-t} \leq tx^r + (1-t)y^r \\
= tf(x) + (1-t)f(y) \leq tf(x) + (1-t)f(y) + \varepsilon
\]

for all \( x, y \in I \) and \( t \in [0, 1] \) and for any constant \( \varepsilon \geq 0 \). This implies that \( f(x) \) is GA-convex and GA-\( \varepsilon \)-convex on \( I \). Furthermore,

(i) when \( r > 1 \) or \( r < 0 \),
   a. the function \( f(x) \) is convex on \( I \),
   b. for any constant \( \varepsilon \geq 0 \), it is \( \varepsilon \)-convex on \( I \);

(ii) when \( 0 < r < 1 \),
   a. the function \( f(x) \) is concave on \( I \),
   b. for any constant \( \varepsilon \geq 1 \), it is \( \varepsilon \)-convex on \( I \).

Example 2.4. Let \( f(x) = \frac{1}{2x} \) for \( x \in I = (0, 1) \).

It is easy to see that \( f(x) \) is convex on \( I \).

When putting \( x = 0.1, y = 0.9, \) and \( t = 0.5 \), since

\[
f(0.1^{0.5} \times 0.9^{1-0.5}) - 0.5f(0.1) - (1 - 0.5)f(0.9) > 0.07,
\]

the function \( f(x) \) is not GA-convex on \( I \).

For \( \varepsilon \geq \frac{1}{2} \), the function \( f(x) \) is GA-\( \varepsilon \)-convex on \( I \).

To establish some new Simpson type inequalities for GA-\( \varepsilon \)-convex functions, we need the following lemma.

Lemma 2.5. Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R} \) be a differentiable function on \( I^o \) and \( a, b \in I^o \) with \( a < b \). If \( f' \in L([a, b]) \), then

\[
\frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx = \frac{\ln b - \ln a}{4} \\
\times \int_0^1 \left( t - \frac{1}{3} \right) \left[ a^{1-t/2} b^{t/2} f'(a^{1-t/2} b^{t/2}) - a^{t/2} b^{1-t/2} f'(a^{1/2} b^{1-t/2}) \right] \, dt.
\]

Proof. Integrating by parts and letting \( x = a^{1-t/2} b^{t/2} \) for \( 0 \leq t \leq 1 \) lead to

\[
\frac{\ln b - \ln a}{2} \int_0^1 \left( t - \frac{1}{3} \right) a^{1-t/2} b^{t/2} f'(a^{1-t/2} b^{t/2}) \, dt \\
= \int_0^1 \left( t - \frac{1}{3} \right) d \left[ f(a^{1-t/2} b^{t/2}) \right]
\]
\[
\begin{align*}
&= \left( t - \frac{1}{3} \right) f(a^{1-t/2}b^{1/2}) \bigg|_{t=0}^{t=1} - \int_0^1 f(a^{1-t/2}b^{1/2}) \, dt \\
&= \frac{2}{3} f(\sqrt{ab}) + \frac{1}{3} f(a) - \int_0^1 f(a^{1-t/2}b^{1/2}) \, dt \\
&= \frac{1}{3} f(a) + \frac{2}{3} f(\sqrt{ab}) - \frac{2}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx.
\end{align*}
\]

Similarly, we have
\[
\begin{align*}
\frac{\ln b - \ln a}{2} & \int_0^1 \left( t - \frac{1}{3} \right) a^{t/2}b^{1-t/2} f'(a^{t/2}b^{1-t/2}) \, dt \\
&= - \int_0^1 \left( t - \frac{1}{3} \right) d\left( f(a^{t/2}b^{1-t/2}) \right) \\
&= - \left( t - \frac{1}{3} \right) f(a^{t/2}b^{1-t/2}) \bigg|_{t=0}^{t=1} + \int_0^1 f(a^{t/2}b^{1-t/2}) \, dt \\
&= - \frac{2}{3} f(\sqrt{ab}) - \frac{1}{3} f(b) + \int_0^1 f(a^{t/2}b^{1-t/2}) \, dt \\
&= - \frac{2}{3} f(\sqrt{ab}) - \frac{1}{3} f(b) + \frac{2}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx. \quad \Box
\end{align*}
\]

3 Some new integral inequalities of Simpson type

Now we start out to establish some new integral inequalities of Simpson type for GA-$\varepsilon$-convex functions.

Theorem 3.1. Let $f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}$ be differentiable on $I^\circ$, $a, b \in I^\circ$ with $a < b$, and $f' \in L([a,b])$. If $|f'|^q$ is GA-$\varepsilon$-convex on $[a,b]$ for some constant $\varepsilon \geq 0$ and $q \geq 1$, then

\[
\left| \frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \frac{\ln b - \ln a}{4} \\
\times \left\{ M_1^{(q-1)/q}(a,b)[(M_1(a,b) - M_2(a,b))|f'(a)|^q + M_2(a,b)|f'(b)|^q \right. \\
+ \varepsilon M_1(a,b) \bigg]^{1/q} + M_1^{(q-1)/q}(b,a)\left[ M_2(b,a)|f'(a)|^q \\
+ (M_1(b,a) - M_2(b,a))|f'(b)|^q + \varepsilon M_1(b,a) \bigg]^{1/q} \right\},
\]
where

\[
M_1(u, v) = \frac{2}{3(\ln v - \ln u)} \left[ u^{5/6} L(u^{1/6}, v^{1/6}) + u^{1/2}(2v^{1/2} - u^{1/2}) - 2u^{1/2}v^{1/6}L(u^{1/3}, v^{1/3}) \right],
\]

\[
M_2(u, v) = \frac{2u^{1/2}}{3(\ln v - \ln u)^2} \left[ 4v^{1/6}L(u^{1/3}, v^{1/3}) + v^{1/2}(\ln v - \ln u) - 2u^{1/3}L(u^{1/6}, v^{1/6}) - 5v^{1/2} + 2u^{1/3}v^{1/6} + u^{1/2} \right],
\]

and

\[
L(u, v) = \begin{cases} 
\frac{u-v}{\ln u - \ln v}, & u \neq v, \\
u, & u = v
\end{cases}
\]

(\*)

is the logarithmic mean.

**Proof.** Since \(|f'|^q\) is a GA-\(\varepsilon\)-convex function on \([a, b]\), by Lemma 2.5 and by Hölder’s integral inequality, we have

\[
\left| \frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \\
\leq \frac{\ln b - \ln a}{4} \left\{ \left( \int_0^1 \left| t - \frac{1}{3} \right| a^{1-t/2} b^{t/2} \left| f'(a^{1-t/2} b^{t/2}) \right| \, dt \right)^{(q-1)/q} \\
+ \left( \int_0^1 \left| t - \frac{1}{3} \right| a^{t/2} b^{1-t/2} \left| f'(a^{t/2} b^{1-t/2}) \right| \, dt \right)^{(q-1)/q} \right\} \\
\leq \frac{\ln b - \ln a}{4} \left\{ \left( \int_0^1 \left| t - \frac{1}{3} \right| a^{1-t/2} b^{t/2} \left| f'(a^{1-t/2} b^{t/2}) \right|^q \, dt \right)^{1/q} \\
+ \left( \int_0^1 \left| t - \frac{1}{3} \right| a^{t/2} b^{1-t/2} \left| f'(a^{t/2} b^{1-t/2}) \right|^q \, dt \right)^{1/q} \right\}
\]

Note 2: We deleted all formula labels except for this one.
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\[ + \left( \int_{0}^{1} \left| t - \frac{1}{3} a^{t/2} b^{1-t/2} \right|^{(q-1)/q} dt \right) \]

\[ \times \left( \int_{0}^{1} \left| t - \frac{1}{3} a^{t/2} b^{1-t/2} \left[ \frac{t}{2} |f'(a)|^q + \left(1 - \frac{t}{2}\right) |f'(b)|^q + \varepsilon \right] dt \right)^{1/q} \right).\]

where

\[ \int_{0}^{1} \left| t - \frac{1}{3} a^{t/2} b^{1-t/2} \right| dt \]

\[ = \int_{0}^{1/3} \left( \frac{1}{3} - t \right) a^{t/2} b^{1-t/2} dt + \int_{1/3}^{1} \left( t - \frac{1}{3} \right) a^{t/2} b^{1-t/2} dt \]

\[ = \frac{2[\ln b - \ln a)(2\sqrt{ab} - a) - 6(a^{1/2}b^{1/2} - 2a^{5/6}b^{1/6} + a)]}{3(\ln b - \ln a)^2} \]

\[ = M_1(a, b), \]

\[ \int_{0}^{1} \left| t - \frac{1}{3} a^{t/2} b^{1-t/2} \right| dt = M_1(b, a), \]

\[ \int_{0}^{1} \left| t - \frac{1}{3} a^{t/2} b^{1-t/2} \right| \left[ \left( \frac{t}{2} |f'(a)|^q + \left(1 - \frac{t}{2}\right) |f'(b)|^q + \varepsilon \right] dt \]

\[ = (M_1(a, b) - M_2(a, b)) |f'(a)|^q + M_2(a, b) |f'(b)|^q + \varepsilon M_1(a, b), \]

and

\[ \int_{0}^{1} \left| t - \frac{1}{3} a^{t/2} b^{1-t/2} \right| \left[ \left( \frac{t}{2} |f'(a)|^q + \left(1 - \frac{t}{2}\right) |f'(b)|^q + \varepsilon \right] dt \]

\[ = M_2(b, a) |f'(a)|^q + (M_1(b, a) - M_2(b, a)) |f'(b)|^q + \varepsilon M_1(b, a). \]

The proof of Theorem 3.1 is thus complete.

\[ \Box \]

**Corollary 3.2.** Under the conditions of Theorem 3.1, when \( q = 1 \), we have

\[ \left| \frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_{a}^{b} \frac{f(x)}{x} dx \right| \]

\[ \leq \frac{\ln b - \ln a}{4} \left\{ \left[ M_1(a, b) - M_2(a, b) + M_2(b, a) \right] |f'(a)| + \left[ M_1(a, b) - M_2(b, a) \right] |f'(b)| + \varepsilon \left[ M_1(a, b) - M_1(b, a) \right] \right\}. \]
Theorem 3.3. Let $f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}$ be differentiable on $I^\circ$, $a, b \in I^\circ$ with $a < b$, and $f' \in L([a, b])$. If $|f'|^q$ for $q > 1$ is GA-$\varepsilon$-convex on $[a, b]$ for some constant $\varepsilon \geq 0$, then

$$\left| \frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|$$

$$\leq \frac{\ln b - \ln a}{4} \left( \frac{q - 1}{2q - 1} \left[ \left( \frac{2}{3} \right)^{(2q-1)/(q-1)} + \left( \frac{1}{3} \right)^{(2q-1)/(q-1)} \right] \right)^{1-1/q}$$

$$\times \left\{ \left[ a^{q/2} L(a^{q/2}, b^{q/2}) - M_3(a^{q/2}, b^{q/2}) \right] |f'(a)|^q$$

$$+ M_3(a^{q/2}, b^{q/2}) |f'(b)|^q + \varepsilon a^{q/2} L(a^{q/2}, b^{q/2}) \right\}^{1/q}$$

$$+ \left[ M_3(a^{q/2}, b^{q/2}) |f'(a)|^q + \left[ b^{q/2} L(a^{q/2}, b^{q/2})$$

$$- M_3(b^{q/2}, a^{q/2}) \right] |f'(b)|^q + \varepsilon b^{q/2} L(a^{q/2}, b^{q/2}) \right\}^{1/q} \right\},$$

where $L(u, v)$ is defined by (*) and

$$M_3(u, v) = \frac{u[v - L(u, v)]}{2(\ln v - \ln u)}$$

for positive numbers $u \neq v$.

Proof. By Lemma 2.5, Hölder’s inequality, and the GA-$\varepsilon$-convexity of $|f'|^q$ on $[a, b]$, we derive

$$\left| \frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|$$

$$\leq \frac{\ln b - \ln a}{4} \int_0^1 \left| t - \frac{1}{3} \left[ a^{1-t/2} b^{t/2} f'(a^{1-t/2} b^{t/2}) \right] \right|$$

$$+ a^{t/2} b^{1-t/2} |f'(a^{t/2} b^{1-t/2})| \, dt$$

$$\leq \frac{\ln b - \ln a}{4} \left( \int_0^1 \left| t - \frac{1}{3} \left[ \frac{q}{q-1} \right] \right| \, dt \right)^{1-1/q}$$

$$\times \left\{ \left[ \int_0^1 a^{q(1-t/2)} b^{qt/2} f'(a^{1-t/2} b^{t/2})^q \, dt \right]^{1/q}$$

$$+ \left[ \int_0^1 a^{qt/2} b^{q(1-t/2)} f'(a^{t/2} b^{1-t/2})^q \, dt \right]^{1/q} \right\}.$$
\[ \frac{1}{4} \ln b - \ln a \left( \int_0^1 \left| t - \frac{1}{3} \right|^{q/(q-1)} \, dt \right)^{1-1/q} \times \left\{ \int_0^1 a^{q(1-t/2)} b^{qt/2} \left[ \left( 1 - \frac{t}{2} \right) |f'(a)|^q + \frac{t}{2} |f'(b)|^q + \epsilon \right] \, dt \right\}^{1/q} + \left\{ \int_0^1 a^{qt/2} b^{q(1-t/2)} \left[ \frac{t}{2} |f'(a)|^q + \left( 1 - \frac{t}{2} \right) |f'(b)|^q + \epsilon \right] \, dt \right\}^{1/q}, \]

where

\[ \int_0^1 \left| t - \frac{1}{3} \right|^{q/(q-1)} \, dt = \frac{q - 1}{2q - 1} \left[ \frac{2}{3} \left( \frac{2q - 1}{q - 1} \right) \right] + \frac{1}{3} \left( \frac{2q - 1}{q - 1} \right). \]

\[ \int_0^1 a^{q(1-t/2)} b^{qt/2} \, dt = a^{q/2} L(a^{q/2}, b^{q/2}), \]

\[ \int_0^1 a^{qt/2} b^{q(1-t/2)} \, dt = b^{q/2} L(a^{q/2}, b^{q/2}), \]

and

\[ \int_0^1 a^{q(1-t/2)} b^{qt/2} \left[ \left( 1 - \frac{t}{2} \right) |f'(a)|^q + \frac{t}{2} |f'(b)|^q \right] \, dt = \frac{a^{q/2}}{q(\ln b - \ln a)} \left\{ \left[ L(a^{q/2}, b^{q/2}) + (b^{q/2} - 2a^{q/2}) \right] |f'(a)|^q + [b^{q/2} - L(a^{q/2}, b^{q/2})] |f'(b)|^q \right\} = [a^{q/2} L(a^{q/2}, b^{q/2}) - M_3(a^{q/2}, b^{q/2})] |f'(a)|^q + M_3(a^{q/2}, b^{q/2}) |f'(b)|^q, \]

\[ \int_0^1 a^{qt/2} b^{q(1-t/2)} \left[ \frac{t}{2} |f'(a)|^q + \left( 1 - \frac{t}{2} \right) |f'(b)|^q \right] \, dt = \frac{b^{q/2}}{q(\ln b - \ln a)} \left\{ \left[ L(a^{q/2}, b^{q/2}) - a^{q/2} \right] |f'(a)|^q + \left[ (2b^{q/2} - a^{q/2}) - L(a^{q/2}, b^{q/2}) \right] |f'(b)|^q \right\} = M_3(b^{q/2}, a^{q/2}) |f'(a)|^q + [b^{q/2} L(a^{q/2}, b^{q/2}) - M_3(b^{q/2}, a^{q/2})] |f'(b)|^q. \]

This completes the proof of Theorem 3.3.
Theorem 3.4. Let \( f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R} \) be differentiable on \( I^\circ \), \( a, b \in I^\circ \) with \( a < b \), and \( f' \in L([a, b]) \). If \( |f'|^q \) for \( q > 1 \) is GA-\( \varepsilon \)-convex on \([a, b]\) for some constant \( \varepsilon \geq 0 \), then

\[
\left| \frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \frac{\ln b - \ln a}{4} \left( \int_0^1 \left( 3^{-q+2} \frac{1}{(q + 1)(q + 2)} \right)^{1/q} \left[ a^{q/(2q-1)} L(a^{q/(2q-1)}, b^{q/(2q-1)}) \right]^{1-1/q} \times \left[ (2^{q+1} (3q + 8) + (6q + 11)) \right]^{1/q} + \left[ b^{q/(2q-1)} L(a^{q/(2q-1)}, b^{q/(2q-1)}) \right]^{1-1/q} \times \left[ (2^{q+1} (3q + 4) + 1) \right]^{1/q} + \left[ b^{q/(2q-1)} L(a^{q/(2q-1)}, b^{q/(2q-1)}) \right]^{1-1/q} \times \left[ (2^{q+1} (3q + 8) + (6q + 11)) \right]^{1/q} \right) \}
\]

where \( L(u, v) \) is defined by (*)..

Proof. By Lemma 2.5, Hölder’s inequality, and the GA-\( \varepsilon \)-convexity of \( |f'|^q \) on \([a, b]\), we obtain

\[
\left| \frac{f(a) + 4f(\sqrt{ab}) + f(b)}{6} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \frac{\ln b - \ln a}{4} \left( \int_0^1 \left( 3^{-q+2} \frac{1}{(q + 1)(q + 2)} \right)^{1/q} \left[ a^{q/(2q-1)} b^{q/(2q-1)} \right]^{1-1/q} \right)
\]

\[
\times \left[ \int_0^1 \left| t - \frac{1}{3} \right|^q |f'(a^{1-t/2}b^{t/2})|^q dt \right]^{1/q} + \left[ \int_0^1 a^{q/(2q-1)} \right]^{1/q} \left[ \int_0^1 \left| t - \frac{1}{3} \right|^q |f'(a^{t/2}b^{1-t/2})|^q dt \right]^{1/q} \}
\]

\[
\leq \frac{\ln b - \ln a}{4} \left( \int_0^1 \left( 3^{-q+2} \frac{1}{(q + 1)(q + 2)} \right)^{1/q} \left[ a^{q/(2q-1)} b^{q/(2q-1)} \right]^{1-1/q} \times \left[ \int_0^1 \left| t - \frac{1}{3} \right|^q |f'(a^{1-t/2}b^{t/2})|^q dt \right]^{1/q} + \left[ \int_0^1 \left| t - \frac{1}{3} \right|^q \left( (1 - t/2) |f'(a)|^q + t/2 |f'(b)|^q - 1 \right) dt \right]^{1/q} \}
\]
Some integral inequalities of Simpson type

\[ + \left[ \int_0^1 a^{qt/(2(q-1))} b^{q(1-t/2)/(q-1)} \, dt \right]^{1-1/q} \times \left[ \int_0^1 \left| t - \frac{1}{3} \right|^q \left( \frac{t}{2} f'(a) \right|^q + \left( 1 - \frac{t}{2} \right) \left| f'(b) \right|^q + \varepsilon \right) \, dt \right]^{1/q}, \]

where

\[ \int_0^1 \left| t - \frac{1}{3} \right|^q \, dt = \frac{3^{-(q+1)}}{q + 1} \left( 2^q + 1 \right), \]
\[ \int_0^1 a^{q(1-t/2)/(q-1)} b^{q(1-t/2)/(q-1)} \, dt = a^{q/(2(q-1))} L(a^{q/(2(q-1))}, b^{q/(2(q-1))}), \]
\[ \int_0^1 a^{t/2(q-1)} b^{(1-t/2)/(q-1)} \, dt = b^{q/(2(q-1))} L(a^{q/(2(q-1))}, b^{q/(2(q-1))}), \]
\[ \int_0^1 \frac{t}{2} \left| t - \frac{1}{3} \right|^q \, dt = \frac{3^{-(q+2)}}{2(q+1)(q+2)} \left[ 2^q + 1 \right] (3q + 4) + 1], \]
\[ \int_0^1 \left( 1 - \frac{t}{2} \right) \left| t - \frac{1}{3} \right|^q \, dt = \frac{3^{-(q+2)}}{2(q+1)(q+2)} \left[ 2^q + 1 \right] (3q + 8) + (6q + 11). \]

The proof is complete. □

**Theorem 3.5.** Let \( f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}_0 \) be GA-e-convex on \([a, b]\) and \( f \in L([a, b]). \) Then

\[ \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2} + \varepsilon \]

and

\[ \int_a^b f(x) \, dx \leq [L(a, b) - a] f(a) + [b - L(a, b)] f(b) + \varepsilon (b - a), \]

where \( L(u, v) \) is defined by (\(*\))

**Proof.** Letting \( x = a^{1-t} b^t \) for \( 0 \leq t \leq 1 \) gives

\[ \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx = \int_0^1 f(a^{1-t} b^t) \, dt \leq \int_0^1 \left[ (1-t) f(a) + t f(b) + \varepsilon \right] \, dt = \frac{f(a) + f(b)}{2} + \varepsilon \]
and
\[
\int_a^b f(x) \, dx = (\ln b - \ln a) \int_0^1 a^{1-t} b^t f(a^{1-t}b^t) \, dt \\
\leq (\ln b - \ln a) \int_0^1 a^{1-t} b^t [(1-t)f(a) + tf(b) + \varepsilon] \, dt \\
= [L(a,b) - a]f(a) + [b - L(a,b)]f(b) + \varepsilon(b - a).
\]

Theorem 3.5 is thus proved.

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Bibliography


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