Analytical Performance of Rayleigh-Product MIMO Channels with Arbitrary-Power Co-channel Interference and Noise

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Abstract—This paper investigates the performance of Rayleigh-product multiple-input multiple-output (MIMO) channels in the presence of both co-channel interference (CCI) and thermal noise. We obtain closed form expressions for the cumulative distribution function and probability density function of the output signal-to-interference-plus-noise ratio (SINR) when optimum combining is employed. In contrast to prior results, our expressions apply for arbitrary numbers of interferers with arbitrary powers. Furthermore, the impact of noise is firstly addressed in our expressions. These are made possible based on the recent random matrix theory tools from which the new statistical properties of maximum eigenvalue of the resultant channel matrix can be derived. The new statistical results permit a general analysis for outage probability of the optimum combining system in Rayleigh-product MIMO channels. Simulation results are also provided to validate the analysis and to examine the effect of CCI and thermal noise on performance.

I. INTRODUCTION

The concept of frequency reuse is capitalized in practical communication systems to improve the overall spectrum efficiency. However, this introduces co-channel interference (CCI), which may degrade the performance and limit the quality of service offered to users [1]–[3]. One effective technique to suppress the effect of CCI is the adaptive arrays with linear combing. In particular, optimum combining performs optimally in terms of maximizing the output signal-to-interference-plus-noise ratio (SINR) [4]. This optimum combining scheme has been extended to multiple-input multiple-output (MIMO) antenna systems and widely-analyzed for Rayleigh [5]–[10] and Rician fading channels [11]–[13].

Most studies were based on the rich-scattering assumption that renders a full-rank channel matrix. Nevertheless, field measurements indicated that the channel in practice may exhibit a reduced-rank behavior due to the lack of scatterers around the transmitter and receiver terminals [14]. A more general double-scattering model which embraces this aspect in propagation environment has been proposed [15]. This model considers the rank deficiency as well as the spatial correlation, by characterizing the channel matrix as a matrix product involving three deterministic matrices (i.e., transmit, receiver, and scatter correlation matrices), and two statistically independent complex Gaussian matrices. In a recent work [16], the outage probability of the optimum combining system in MIMO Rayleigh-product channels, a particular form of double-scattering structure for which transmit antennas, receive antennas, and scatterers are spatially uncorrelated, was provided. For the mathematical tractability, it was assumed in [16] that (i) the systems are “interference-limited”, i.e., the impact of noise is neglected and the number of interferers is greater than or equal to the number of receiver antennas and (ii) the short-term average powers of interferers are equal.

In this paper, we release this two assumptions and study the performance of Rayleigh-product MIMO channels in the presence of arbitrary-power co-channel interference and thermal noise. Based on new exact results which we derive for the joint eigenvalue distributions of a certain product of finite-dimensional random matrices, we obtain a general expression for the cumulative distribution function (c.d.f) of the maximum eigenvalue of the resultant channel matrix. Our expression includes the effect of noise and is valid for arbitrary numbers of interferers with arbitrary powers. The new derived expression allows the analysis for c.d.f (equivalently the system outage probability) and probability density function (p.d.f) of the output SINR. Finally, we present some numerical examples to confirm the theoretical results and study the effect of CCI plus thermal noise pattern on system performance.

II. SYSTEM MODEL

We consider a MIMO system where the desired user has \( N_t \) antennas at the transmitter and \( N_r \) antennas at the receiver. The desired signal is corrupted by \( L \) co-channel interferers. Then, the \( N_r \times 1 \) received vector can be expressed as

\[
r = \sqrt{\Omega_0} H_w d_0 + \sum_{j=1}^{L} \sqrt{\Omega_j} h_j d_j + n,
\]

where \( \sqrt{\Omega_j} \) and \( d_j \) are the transmit power and transmit signal for the desired user \( (j = 0) \) and \( j \)th co-channel interferer \( (j = 1, \ldots, L) \) with \( \mathbb{E} [ ||d_j||^2 ] = 1 \), \( w_T \) is the \( N_t \times 1 \) beamforming vector of the desired user with \( ||w_T||^2 = 1 \) and \( n \) is the \( N_r \times 1 \) additive complex Gaussian noise vector with zero mean and

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covariance matrix $N_0 I_{N_r}$, $h_j$ is the $N_r \times 1$ random matrix denoting the channel gain for the $j$th co-channel interferer, $H$ denotes the $N_r \times N_t$ channel gain matrix for the desired user. We follow [16] to model $H$ as the Rayleigh-product MIMO channel between the transmitter and the desired receiver given by

$$H = \frac{1}{\sqrt{N_s}} H_1 H_2,$$  \hspace{1cm} (2)

where $H_1$ and $H_2$ are $N_r \times N_s$ and $N_s \times N_t$ random matrices, respectively, and $N_s$ denotes the number of scatterers in the environment [15]. In addition, the random entries of $H_1$, $H_2$ and $\{h_j\}$ are independent and follow $CN(0,1)$. Then, the received signal vector $r$ in (1) can be rewritten as

$$r = \sqrt{\Omega_0} \text{H}_w^T d_0 + H_1 \Sigma_d^\frac{1}{2} d_l + n,$$ \hspace{1cm} (3)

where $H_1 = [h_1, ..., h_L]$, $\Sigma_d = \text{diag} \{\Omega_1, ..., \Omega_L\}$ and $d_l = [d_1, ..., d_L]^T$. Without loss of generality, we assume that $\Omega_1 < \Omega_2 < ... < \Omega_T$ are the $T$ distinct diagonal elements of the matrix $\Sigma_d$, with corresponding multiplicities $m_1, ..., m_T$ such that $\sum_{j=1}^T m_j = L$. We define $c_j$ as the integer such that $m_1 + ... + m_{c_j-1} < j \leq m_1 + ... + m_{c_j}$, $l_j = \sum_{i=1}^{c_j} m_k - j$ and $\mu_j = 1/\Omega_j$.

At the receiver, the weighting vector that maximizes the output SINR for a given $\text{H}_w$ is [11]

$$w_R = \alpha \text{R}^{-1} \text{H}_w^T,$$ \hspace{1cm} (4)

where $\text{R} = (H_1 \Sigma_d H_1^H + N_0 I_L)$ and $\alpha$ is an arbitrary constant which does not affect the SINR. According to Rayleigh-Ritz theorem in [17], the optimum weighting vector $w_R$ is the eigenvector corresponding to the largest eigenvalue of the result channel matrix, $\Psi = H_1^H \text{R}^{-1} H_1$, which as a result gives the maximum output SINR as

$$\gamma_{\text{SINR}} = \Omega_0 \lambda_{\text{max}},$$ \hspace{1cm} (5)

where $\lambda_{\text{max}}$ is the largest eigenvalue of $\Psi$.

III. DISTRIBUTIONS OF $\lambda_{\text{max}}$

In this section, we deduce the c.d.f of $\lambda_{\text{max}}$ in closed form for analyzing the outage probability of the MIMO optimum combining system later. Due to space constraints, we assume $N_t \leq N_s \leq N_r$ here. The derivations of other $(N_t, N_s, N_r)$ combinations are similar and will be presented in an extended journal version of this paper. For convenience of presentation, we define the following notations: $M = \text{max}(N_r, L)$, $N = \text{min}(N_r, L)$, $\kappa_1 = \text{max}(0, L - N_r)$, $\kappa_2 = N_r - N_s$, and $\kappa_3 = N_s - N_t$.

**Theorem 1:** if $N_t \leq N_s \leq N_r$, the c.d.f of $\lambda_{\text{max}}$ is given by

$$F_{\lambda_{\text{max}}}(x) = K \det (G(x)),$$ \hspace{1cm} (6)

where $K$ and $M \times M$ matrix $G(x)$ are given by (7) and (8) at the top of the next page in which $R_1(i,j)$ is given by

$$R_1(i,j) = \sum_{p=0}^{N_t} \left( \frac{N_r}{p} \right) N_0 \left( \frac{N_r - p - a_{i,j}(p)/2}{\mu_{i,j}(p) - 2/2} \right) \times \Gamma(p + l_j + 1) W_{b_{i,j}(p)/2, (1 - a_{i,j}(p))/2} \left( N_0 \mu_{i,j}(p) \right),$$ \hspace{1.5cm} (9)

where $a_{i,j}(p) = -(p + l_j) + (\kappa_1 + \kappa_2)$ and $b_{i,j}(p) = -(p + l_j) - (\kappa_1 + \kappa_2)$. $\tilde{\Omega}_i$ is defined as

$$\tilde{\Omega}_i = \Gamma(\kappa_4 - i + 1) \exp(N_0 \mu_{i,j}(p)/2) \times \sum_{p=0}^{N_s} \left( \frac{N_s}{p} \right) N_0 \left( \frac{N_r - p - c_{i,j}(p)/2}{\mu_{i,j}(p) - 2/2} \right) \times \Gamma(p + l_j + 1) W_{d_{i,j}(p)/2, (1 - c_{i,j}(p))/2} \left( N_0 \mu_{i,j}(p) \right),$$ \hspace{1.5cm} (10)

where $\kappa_4,i = M + N_s - N_t - i$, $c_{i,j}(p) = -(p + l_j) + (\kappa_4,i + 1)$, $d_{i,j}(p) = -(p + l_j) - (\kappa_4,i + 1)$, and $W_{\lambda_{\text{max}}}(z)$ is the Whittaker function [18, 9.220.4]. Also, $\phi_{2,i,j}(x)$, $\Theta_{1,i,j}$ and $\Theta_{2,i,j}(x)$ are given by (11) - (12) at the top of the next page, where $K_e(z)$ is the modified bessel function of second kind [18, 8.432]. $\Gamma(m,n) = \prod_{k=1}^{m} \Gamma(m,n + k)$ with $\Gamma(m) = \Gamma(m - 1)$, and $[\alpha]_k = \alpha - (a - k + 1)^{\alpha - k + 1}$.

**Proof:** See Appendix A.

From Theorem 1, we see that the impact of the noise on the c.d.f of $\lambda_{\text{max}}$ is clearly revealed in equation (8)-(12). Also, it is noted that equation (6)-(12) are concise formulas involving only standard functions, and therefore $F_{\lambda_{\text{max}}}(x)$ can be easily and efficiently evaluated numerically with software packages such as Maple and Mathematica.

An interference-limited environment where the number of interferers exceeds or is equal to the number of receiving antenna elements and the impact of noise is neglected has been widely investigated in digital microcellular system [2], [5], [16]. The analytical framework of Theorem 1 is also suitable for the interference-limited environment . For instance, we set $L \geq N_r$, $N_0 = 0$, and $T = 1$. Following the same steps from (24) to (29) in Appendix A, it can be shown after some lengthy algebraic manipulations that the c.d.f expression of $\lambda_{\text{max}}$ reduces to that of interference-limited case with equal interferers powers obtained in [16].

IV. PERFORMANCE ANALYSIS

A. OUTAGE PROBABILITY

The outage probability is an important statistical measure to assess the quality of service provided by the system. It is defined as the probability of failing to achieve an acceptable SINR. Therefore, the outage probability is simply the c.d.f of the output SINR evaluated at $\gamma_{\text{th}}$. According to Theorem 1, the outage probability of the optimum combining system in Rayleigh-product MIMO channels when $N_t \leq N_s \leq N_r$ is given by

$$P_{\text{out}} = \Pr(\gamma_{\text{SINR}} \leq \gamma_{\text{th}}) = K \det (G(\gamma_{\text{th}}/\Omega_0)).$$ \hspace{1cm} (13)

B. P.D.F OF OUTPUT SINR

From Theorem 1, we can obtain the p.d.f of the output SINR in (5) which is presented in the following theorem.
\[ K = \frac{(-1)^{(N-1)N+(N_r-1)N_r+2N_r(N_r-N_t)}/2}{\Gamma(N_t) N_0^{N_r(N_r-N_t)N_r}} \prod_{i=1}^{T} \frac{m_i}{m_i} \prod_{1 \leq i < j \leq T} (\mu(i) - \mu(j))^{m_i m_j} \cdot (7) \]

\[ \{ G(x) \}_{i,j} = \begin{cases} [k_t - l_j, \mu(e_{c,j})]_{k_t - l_j}, & 1 \leq i \leq \kappa_1, j \leq L \\ [k_t - l_j, \mu(e_{c,j})]_{k_t - l_j}, & 1 \leq i \leq \kappa_1, j > L \\ (-1)^{l_j} \sum_{p=0}^{M-i} \binom{M}{i-j} N_0^{M-i-p} - (p+1)_j \Gamma(p + l_j + 1), & \kappa_1 < i \leq \kappa_1 + \kappa_2, j \leq L \\ \frac{[k_t - l_j, \mu(e_{c,j})]_{k_t - l_j}}{\left(\frac{1}{M-j} \sum_{l=1}^{\infty} \frac{N_s}{N_0} \Gamma(p + l_j + 1) \sum_{k=0}^{M-i} \binom{M-i}{k} \sum_{h=0}^{N_r-N_s} \binom{N_r-N_s}{h} \mu(e_{c,j})^{k} \mu(e_{c,j})^{k} \phi_h(p+t)_j(\mu(e_{c,j})). \right)} & \kappa_1 < i \leq \kappa_1 + \kappa_2, j > L \\ \frac{[k_t - l_j, \mu(e_{c,j})]_{k_t - l_j}}{\left(\frac{1}{M-j} \sum_{l=1}^{\infty} \frac{N_s}{N_0} \Gamma(p + l_j + 1) \sum_{k=0}^{M-i} \binom{M-i}{k} \sum_{h=0}^{N_r-N_s} \binom{N_r-N_s}{h} \mu(e_{c,j})^{k} \mu(e_{c,j})^{k} \phi_h(p+t)_j(\mu(e_{c,j})). \right)} & \kappa_1 + \kappa_2 < i \leq \kappa_1 + \kappa_2 + \kappa_3, j \leq L \\ (2N_r-N_r-i-j) \Gamma(N_r + i - j) N_0^{-(N_r+i-j)}, & \kappa_1 + \kappa_2 < i \leq \kappa_1 + \kappa_2 + \kappa_3, j > L \\ (2N_r-N_r-i-j) \Gamma(N_r + i - j) N_0^{-(N_r+i-j)} & \kappa_1 + \kappa_2 + \kappa_3 < i \leq M, j \leq L \\ (2N_r-N_r-i-j) \Gamma(N_r + i - j) N_0^{-(N_r+i-j)} & \kappa_1 + \kappa_2 + \kappa_3 < i \leq M, j > L \end{cases} \quad (8) \]

\[ \phi_{2,i,j}(x) = \frac{\sum_{p=0}^{N_s} \binom{N_s}{p} N_0^{N_r-p} \Gamma(p + l_j + 1) \sum_{k=0}^{M-i} \binom{M-i}{k} \sum_{h=0}^{N_r-N_s} \binom{N_r-N_s}{h} \mu(e_{c,j})^{k} \mu(e_{c,j})^{k} \phi_h(p+l)_j(\mu(e_{c,j})). \right)}{\left(\frac{1}{M-j} \sum_{l=1}^{\infty} \frac{N_s}{N_0} \Gamma(p + l_j + 1) \sum_{k=0}^{M-i} \binom{M-i}{k} \sum_{h=0}^{N_r-N_s} \binom{N_r-N_s}{h} \mu(e_{c,j})^{k} \mu(e_{c,j})^{k} \phi_h(p+t)_j(\mu(e_{c,j})). \right)} \quad (11) \]

\[ \varphi_{v,\beta}(x,\omega) = \begin{cases} \int_{0}^{\infty} (t + \omega)^{v-1} \exp(-xN_s/t + N_0 t)dt, & \beta = 1 \\ \int_{0}^{\infty} (t + \omega)^{v-1} \exp(-xN_s/t + N_0 t)dt, & \beta = 2 \end{cases} \quad (11) \]

\[ \Theta_{1,i,j} = \frac{\Gamma(e_{c,j}) N_0^{-(e_{c,j})}}{\Theta_{2,i,j}(x) = 2 \sum_{k=0}^{M-i} \binom{M-i}{k} \binom{N_r}{e_{c,j}+k/2} N_0^{-(e_{c,j})/2} K_{(e_{c,j})}(2\sqrt{xN_0})}{M-k} \quad (12) \]

**Theorem 2:** When \( N_t \leq N_s \leq N_r \), the p.d.f of \( \gamma_{\text{SINR}} \) is given by

\[ f_{\gamma_{\text{SINR}}}(x) = \frac{K}{\Omega_0} \sum_{l=M-N+1}^{M} \det(G_l(x/\Omega_0)), \quad (14) \]

where \( G_l(x) \) is the \( M \times M \) matrix with elements \( \{ G_l(x) \}_{i,j} = \begin{cases} \{ G_l(x) \}_{i,j}, & i \neq l \\ (-1)^{l_j+1} N_s \left( \phi_{2,i,j+1,i,j} - \phi_{2,i,j+2,i,j} \right), & i = l, j \leq L \\ (-1)^{M-j+1} \left[ \frac{1}{M-j} \sum_{l=1}^{\infty} \left( e_{c,j} \Theta_{2,i,j}(x) - N_0 \Theta_{2,l,j+1}(x) \right) \right], & i = l, j > L \end{cases} \quad (15) \]

in which we make the definition \( \phi_{2,M+1,j,i} = 0 \) and \( \Theta_{2,M+1,j,i} = 0 \).

**Proof:** According to the relation \( f_{\lambda_{\text{max}}}(x) = \frac{dF_{\lambda_{\text{max}}}(x)}{dx} \), applying a well-known result for the derivative of a determinant, we have the p.d.f of \( \lambda_{\text{max}} \) given by

\[ f_{\lambda_{\text{max}}}(x) = K \sum_{l=M-N+1}^{M} \det(G_l(x)), \quad (16) \]

where \( G_l(x) \) is the \( M \times M \) matrix with elements

\[ \{ G_l(x) \}_{i,j} = \begin{cases} \{ G_l(x) \}_{i,j}, & i \neq l \\ (-1)^{l_j+1} N_s \left( \phi_{2,i,j+1,i,j} - \phi_{2,i,j+2,i,j} \right), & i = l, j \leq L \\ (-1)^{M-j+1} \left[ \frac{1}{M-j} \sum_{l=1}^{\infty} \left( e_{c,j} \Theta_{2,i,j}(x) - N_0 \Theta_{2,l,j+1}(x) \right) \right], & i = l, j > L \end{cases} \quad (17) \]

Combined with (5), the theorem is proved by taking the derivative and simplifying.

**V. NUMERICAL RESULTS**

In this section, some numerical examples are presented to validate the theoretical analysis. We define \( SNR = \Omega_0/N_0 \), and the normalized signal-to-interference ratio \( SIR = L \Omega_0 / \left( \sum_{j=1}^{L} \Omega_j \right) \). We set \( N_s = 3, N_t = 2 \), and consider the case with \( L = 5 \) co-channel interferers of a total interference power of 25 units in three different scenarios: (1) with distinct-power interferers \([1 2 5 8 9]\), (2) with arbitrary-power interferers \([1 5 5 5 9]\), and (3) with equal-power interferers \([5 5 5 5 5]\).

Fig. 1 shows analytical and Monte-Carlo simulated outage probability of the optimum combining system in Rayleigh-product MIMO channels, comparing different interference power distributions and \( N_r \). We see that in all cases the
This paper has presented the analytical performance results of MIMO optimum combining system in Rayleigh-product channels with CCI and thermal noise. We obtained the closed form expressions for the system outage probability and the p.d.f of the output SINR based on the new statistical results of the maximum eigenvalue of the resultant channel. Our results demonstrate the effect of noise and apply for arbitrary numbers of interferers with arbitrary powers. Numerical examples verify our theoretical analysis.

### APPENDIX A

**Proof of the Theorem 1**

In Theorem 1, the c.d.f of $\lambda_{\text{max}}$ includes two possible cases, $L \geq N_r$ and $L < N_r$, which will be proved below respectively.

#### A. $L \geq N_r$

Define $Z = H_1^* R^{-1} H_1$ and $W = H_2 \Sigma H_2^*$. First, we derive the eigenvalues distributions of $Z$. It is easy to observe that $Z$ and $W$ have $N_s$ and $N_r$ non-zeros ordered eigenvalues $\lambda_1 > \lambda_2 > ... > \lambda_{N_s} > 0$ and $\lambda_1 > \lambda_2 > ... > \lambda_{N_r} > 0$. We define $\alpha = [\alpha_1, \alpha_2, ..., \alpha_{N_s}]$ and $\lambda = [\lambda_1, \lambda_2, ..., \lambda_{N_r}]$. Based on the result in [20, Lemma 6], the joint p.d.f of non-zeros ordered eigenvalues of $Z$ conditioned on $W$ are given by

$$f_{Z|W}(\alpha | \lambda) = K_1(\lambda) \det(V_1(\alpha)) \det(G_1(\alpha, \lambda)),$$

where $V_1(\alpha)$ is the $N_s \times N_s$ Vandermonde matrix with elements $\{V_1(\alpha)\}_{i,j} = \alpha_j^{i-1}$,

$$K_1(\lambda) = \frac{1}{\Gamma(N_s)} \left(\frac{N_s}{N_r}\right)^{N_s} \prod_{1 \leq i,j \leq N_r} (\lambda_i + \lambda_j)^{N_s} (\lambda_i - \lambda_j)^j,$$

and the $N_r \times N_r$ matrix $G_1(\alpha, \lambda)$ has elements

$$\{G_1(\alpha, \lambda)\}_{i,j} = \begin{cases} (\lambda_j + N_0)^{\kappa_2-i}, & i \leq \kappa_2 \\ e^{-\lambda_j N_0} \beta_{i-(\kappa_2-2)}, & i > \kappa_2 \end{cases}$$

To derive the unconditional joint p.d.f of $\alpha$, we average (18) over the joint p.d.f of $\lambda$ which is again obtained by [20, Lemma 6]

$$h_1(\lambda) = K_2 \det(V_2(\lambda)) \det(G_2(\lambda)),$$

where $V_2(\lambda)$ denotes the $N_r \times N_r$ Vandermonde matrix with elements $\{V_2(\lambda)\}_{i,j} = \lambda_j^{i-1}$,

$$K_2 = \frac{1}{\Gamma(N_r)} \left(\frac{N_r}{N_r}\right)^{N_r} \prod_{i=1}^T \prod_{1 \leq i,j \leq T} (\mu(i) - \mu(j))^{m_i m_j},$$

and the $L \times L$ matrix $G_2(\lambda)$ is given by

$$\{G_2(\lambda)\}_{i,j} = \begin{cases} [\kappa_1 - i]^{-1}_{j} e^{-\mu(i) \lambda_j^{i+1}}, & i \leq \kappa_1 \\ (-\lambda_{i-\kappa_1})^j e^{-\mu(i) \lambda_{i-\kappa_1}}, & i > \kappa_1 \end{cases}$$
As a result, we get

\[
f_1(\alpha) = \int_{D_1} f_{Z|W,1}(\alpha | \lambda) h_1(\lambda) \, d\lambda
= K_3 \det (V_1(\alpha)) \times \int_{D_1} \prod_{i=1}^{N_r}(\lambda_i + N_0)^{N_r} \det (G_1(\alpha, \lambda)) \det (G_2(\alpha)) \, d\lambda
= K_3 \det (V_1(\alpha)) \det (G_3(\alpha)),
\]

where the multiple integral is over the domain \(D_1 = \{ \infty > \lambda_1 > \lambda_2 > \ldots > \lambda_{N_r} > 0 \} \) and \( K_3 = \frac{1}{(N_0)^{\frac{N_r}{2}}} \).

By applying the general integral identity [21, App.B] and [18, 3.381.4], with some mathematical manipulations, we have the \(L \times L\) matrix \(G_3(\alpha)\) in (25) at the top of next page. Now utilizing the results in [11], the c.d.f of the largest eigenvalue of \(\Psi\) conditioned on \(Z\) is given by

\[
F_{\lambda_{\max}|Z,1}(x | \alpha) = \frac{(-1)^{N_r(N_r-N_s)}}{\det(V_Z(\alpha))} \Gamma(N_t)(N_t),
\]

where the determinant of \(V_Z(\alpha)\) is given by

\[
\det(V_Z(\alpha)) = \left( \prod_{i=1}^{N_r} \alpha_i^{N_r+1-N_s} \prod_{1 \leq i < j \leq N_r} (\alpha_i - \alpha_j) \right),
\]

and \(G_\alpha(x)\) is an \(N_s \times N_s\) matrix with elements given by in (28) at the top of next page.

The unconditional c.d.f of \(\lambda_{\max}\) can be obtained by

\[
F_{\lambda_{\max}}(x) = \int_{D_2} F_{\lambda_{\max}|z,1}(x | \alpha) f_1(\alpha) \, d\alpha,
\]

where the multiple integral is over the domain \(D_2 = \{ \infty > \alpha_1 > \alpha_2 > \ldots > \alpha_{N_r} > 0 \}\). Substituting (24) and (26) into (29), the theorem is proved in the \(L \geq N_r\) case by applying the general integral identity [21], performing some subsequent simplifications to the resulting determinant using [18, 3.381.4].

**B. \(L < N_r\)**

In this case, we have the ordered eigenvalues of \(W\) given by \(\lambda_1 > \lambda_2 > \ldots > \lambda_{L} > \lambda_{L+1} = \ldots = \lambda_{N_r} = 0\). Based on [20, Lemma 6], we have the joint p.d.f of non-zeros ordered eigenvalues of \(Z\) conditional on \(W\) as follow

\[
f_{Z|W,2}(\alpha | \lambda) = K_4(\lambda) \det (V_1(\alpha)) \det (G_4(\alpha, \lambda)),
\]

where

\[
K_4(\lambda) = K_4' = \frac{\prod_{i=1}^{L}(\lambda_i + N_0)^{N_r}}{\prod_{1 \leq i < j \leq L}(\lambda_i - \lambda_j)^{N_r-L}}.
\]

and \(G_4(\alpha, \lambda)\) is the \(N_r \times N_r\) matrix with elements

\[
\{G_4(\alpha, \lambda)\}_{i,j} = \begin{cases} 
(\lambda_j + N_0)^{\kappa_2-j}, & \text{if } \kappa_2 > j \leq L, \\
N_r - j = N_r - j, & \text{if } \kappa_2, j > L, \\
e^{-\mu(\delta_j)}, & \text{if } \kappa_2, j > L, \\
(\lambda_i + N_0)^{\kappa_2-i}, & \text{if } \kappa_2, j > L.
\end{cases}
\]

We define \(\lambda' = [\lambda_1, \lambda_2, \ldots, \lambda_L]\). Similar as (18), the p.d.f of \(\lambda'\) is given by

\[
h_2(\lambda') = K_5 \det (V_3(\lambda')) \det (G_5(\lambda')) \prod_{i=1}^{L} \lambda_{N_r-L},
\]

where \(V_3(\lambda')\) and \(G_5(\lambda')\) are the \(L \times L\) Vandermonde matrix with elements \(\{V_3(\lambda')\}_{i,j} = \lambda_j^{i-1}\) and \(\{G_5(\lambda')\}_{i,j} = (-\lambda_j)^{i-1} \lambda_j^{i-1}\).

The unconditional p.d.f of \(Z\) in the \(L < N_r\) case can be obtained by

\[
f_2(\alpha) = \int_{D_3} f_{Z|W,2}(\alpha | \lambda) h_2(\lambda') \, d\lambda'
= K_6 \det (V_1(\alpha)) \det (G_6(\alpha)),
\]

where the multiple integral is over the domain \(D_3 = \{ \infty > \lambda_1 > \lambda_2 > \ldots > \lambda_L > 0\}\), \(K_6 = \frac{1}{(N_r-1)/2} K_4 K_5\) and \(G_6(\alpha)\) is the \(N_r \times N_r\) matrix with elements given by in (37) at the top of next page.

The theorem is proved in the \(L < N_r\) case by averaging the conditional c.d.f (26) over the joint p.d.f (36) similar as (29) and performing some subsequent simplifications to the resulting determinant using [18, 3.381.4] and [18, 3.471.9].

**REFERENCES**


\begin{align}
\{G_3 (\alpha)\}_{i,j} &= \left\{ \begin{array}{ll}
[1 \text{ if } i = j] & \mu_{(c_j)}^{N_j - \kappa - i - j}, \\
(-1)^j \sum_{p=0}^{L-i} \binom{L-i}{p} \Gamma (p+l_j+1) N_0^{L-i-p} \mu_{(c_j)}^{-(p+l_j+1)}, & \kappa_1 < i \leq \kappa_1 + \kappa_2 \\
(-1)^j e^{-N_0 \alpha_{(i-\kappa_2 - \kappa_1)}} N_s \sum_{p=0}^{N_s-1} \binom{N_s}{p} \Gamma (p+l_j+1) N_0^{N_s-p} \left( \mu_{(c_j)} + \alpha_{(i-\kappa_2 - \kappa_1)} \right)^{-p+l_j+1}, & \kappa_1 + \kappa_2 < i \leq L
\end{array} \right.
\end{align}

(25)

\begin{align}
\{G_\alpha (x)\}_{i,j} &= \left\{ \begin{array}{ll}
(-1/\alpha_j)^{\kappa_3 - i}, & i \leq \kappa_3 \\
(N_s - i)^{1/\alpha_j} N_r^{N_r-i+1} \left( 1 - e^{-\frac{\lambda N_s}{\alpha_j}} \sum_{k=0}^{N_r-i+1} \kappa_2 \right)^{\kappa_2 (N_r-i)}, & i > \kappa_3
\end{array} \right.
\end{align}

(28)

\begin{align}
\{G_0 (\alpha)\}_{i,j} &= \left\{ \begin{array}{ll}
[1 \text{ if } i = j] & \mu_{(c_j)}^{N_j - \kappa - i - j}, \\
(-1)^j \sum_{p=0}^{N_r-i} \binom{N_r-i}{p} \Gamma (p+l_j+1) N_0^{N_r-i-p} \mu_{(c_j)}^{-(p+l_j+1)}, & i \leq \kappa_2, j \leq L \\
(-1)^j e^{-N_0 \alpha_{(i-\kappa_2)}} N_s \sum_{p=0}^{N_s} \binom{N_s}{p} N_0^{N_s-p} \Gamma (p+l_j+1) \left( \mu_{(c_j)} + \alpha_{(i-\kappa_2)} \right)^{-p+l_j+1}, & i > \kappa_2, j \leq L \\
(-\alpha_{(i-\kappa_2)}^{N_r-j} e^{-N_0 \alpha_{(i-\kappa_2)}} N_s \sum_{p=0}^{N_s} \binom{N_s}{p} N_0^{N_s-p} \Gamma (p+l_j+1) \left( \mu_{(c_j)} + \alpha_{(i-\kappa_2)} \right)^{-p+l_j+1}, & i > \kappa_2, j > L
\end{array} \right.
\end{align}

(37)


