Simple image set of linear mappings in a max–min algebra

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Abstract

For a given linear mapping, determined by a square matrix $A$ in a max–min algebra, the set $S_A$ consisting of all vectors with a unique pre-image (in short: the simple image set of $A$) is considered. It is shown that if the matrix $A$ is generally trapezoidal, then the closure of $S_A$ is a subset of the set of all eigenvectors of $A$. In the general case, there is a permutation $\pi$, such that the closure of $S_A$ is a subset of the set of all eigenvectors permuted by $\pi$. The simple image set of the matrix square and the topological aspects of the problem are also described.

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1. Introduction

Problems in many research areas, such as system theory, graph theory, scheduling, knowledge engineering, can be formulated in a compact way using the language of extremal algebras, in which the addition and multiplication of vectors and matrices is formally replaced by operations of maximum and minimum, or maximum and plus. Systematic approach in this field can be found in [5,6]. Many authors studied the questions of extremal algebras, similar to those of linear algebra, like solvability of the systems of linear equations, linear mappings, independence, regularity, eigenvectors and eigenvalues.

In [10], the following question was posed: given a fuzzy relation $R$ between medical symptoms expressing the action of a drug on patients in a given therapy, what is the greatest invariants of the system? The question leads to the problem of finding the greatest eigenvector of the max–min matrix $A$ corresponding to the fuzzy relation $R$.

The aim of this paper is to describe the set consisting of all vectors with a unique pre-image (in short: the simple image set) of a given max–min linear mapping. In the above interpretation, a vector with a unique pre-image corresponds to such a patient’s state, which results from a unique state preceding the treatment by a given drug. We present a close...
connection of the simple image set with the eigenspace of the corresponding matrix (the set of all fixed points of the mapping). The simple image set of the matrix square and the topological aspects of the problem are described in the last two sections. The questions considered in this paper are analogous to those in [1], where matrices and linear mappings in a max-plus algebra are studied.

2. Notions and notation

By a max–min algebra we mean a linearly ordered set \((\mathcal{B}, \leq)\) with the binary operations of maximum and minimum, denoted by \(\oplus\) and \(\otimes\). For any natural \(n > 0\), \(\mathcal{B}(n)\) denotes the set of all \(n\)-dimensional column vectors over \(\mathcal{B}\), and \(\mathcal{B}(m, n)\) denotes the set of all matrices of type \(m \times n\) over \(\mathcal{B}\). We shall use the notation \(M = \{1, 2, \ldots, m\}, N = \{1, 2, \ldots, n\}\). For \(x, y \in \mathcal{B}(n)\), we write \(x \leq y\), if \(x_i \leq y_i\) holds for all \(i \in N\), and we write \(x < y\), if \(x \leq y\) and \(x \neq y\). We say that a vector \(x \in \mathcal{B}(n)\) is increasing, if \(x_i \leq x_j\) holds for every \(i, j \in N, i \leq j\). The matrix operations over \(\mathcal{B}\) are defined with respect to \(\oplus, \otimes\), formally in the same manner as the matrix operations over any field. In general, \(\mathcal{B}\) need not be bounded. We shall denote by \(\mathcal{B}^*\) the bounded algebra derived from \(\mathcal{B}\) by adding the least element, or the greatest element (or both), if necessary. If \(\mathcal{B}\) itself is bounded, then \(\mathcal{B} = \mathcal{B}^*\). The least element in \(\mathcal{B}^*\) will be denoted by \(O\), the greatest one by \(I\). To avoid the trivial case, we assume \(O < I\). (This notation should not be mixed up with the notation for sets \(I_j(A, b)\) introduced below.)

Many authors considered the system of linear equations of the form

\[
A \otimes x = b,
\]

where the matrix \(A \in \mathcal{B}(m, n)\) and the vector \(b \in \mathcal{B}(m)\) are given, and the vector \(x \in \mathcal{B}(n)\) is unknown. It was shown in [4], that the consideration of the solvability of (1) may be reduced to the case when \(b_i > O\) for all \(i \in M\). In the following we shall use the notation from [7], where the questions of solvability and unique solvability were studied.

The solvability of the equation (1) is closely related to its greatest solution denoted by \(\bar{x}(A, b)\). The vector \(\bar{x}(A, b) \in \mathcal{B}^*(n)\) is defined by putting, for every \(j \in N\),

\[
M_j(A, b) := \{i \in M; a_{ij} > b_i\}, \quad \bar{x}_j(A, b) := \min_{\mathcal{B}^*} \{b_i; i \in M_j(A, b)\}.
\]

The vector \(\bar{x} = \bar{x}(A, b)\) is defined correctly, because the minimum in the definition is computed in the upper bounded algebra \(\mathcal{B}^*\). Therefore, every value \(\bar{x}_j\) is well-defined, even in the case, when \(M_j(A, b)\) is an empty set (then \(\bar{x}_j = \min_{\mathcal{B}^*} \emptyset = I \in \mathcal{B}^*\).

**Lemma 2.1 (Gavalec [7]).** Let \(x \in \mathcal{B}(n)\) be a solution of the equation \(A \otimes x = b\). Then \(x \leq \bar{x}\) and \(\bar{x}\) is a solution of the equation \(A \otimes x = b\) in \(\mathcal{B}^*(n)\).

**Theorem 2.2 (Gavalec [7]).** Let \(A \in \mathcal{B}(m, n), b \in \mathcal{B}(m)\). The equation \(A \otimes x = b\) has a solution in \(\mathcal{B}(n)\) if and only if \(\bar{x}\) is a solution of \(A \otimes x = b\) in \(\mathcal{B}^*(n)\).

The unique solvability of the equation (1) in a general max–min algebra \(\mathcal{B}\) is characterized in [7] by a necessary and sufficient condition. The following notation is used: for every \(j \in N\),

\[
I_j(A, b) := \{i \in M; a_{ij} \geq b_i = \bar{x}_j\}, \quad \mathcal{I}(A, b) := \{I_j(A, b); j \in N\},
\]

\[
K_j(A, b) := \{i \in M; a_{ij} = b_i = \bar{x}_j\}, \quad \mathcal{K}(A, b) := \{K_j(A, b); j \in N\}.
\]

If \(S\) is a set and \(\mathcal{C} \subseteq P(S)\) is a set of subsets of \(S\), we say that \(\mathcal{C}\) is a covering of \(S\), if \(\bigcup \mathcal{C} = S\), and we say that a covering \(\mathcal{C}\) of \(S\) is minimal, if \(\bigcup(\mathcal{C} - \{C\}) \neq S\) holds for every \(C \in \mathcal{C}\).

**Theorem 2.3 (Gavalec [7]).** Let \(A \in \mathcal{B}(m, n), b \in \mathcal{B}(m)\). The equation \(A \otimes x = b\) has a unique solution \(x \in \mathcal{B}(n)\) if and only if the system \(\mathcal{I}(A, b)\) is a minimal covering of the set \(M - \bigcup \mathcal{K}(A, b)\).

The following equivalent formulation of Theorem 2.3 will be used in Section 4.
Theorem 2.4. Let $A \in \mathcal{B}(m, n), b \in \mathcal{B}(m)$. The equation $A \otimes x = b$ has a unique solution $x \in \mathcal{B}(n)$ if and only if there is a mapping $\varphi : M \to N$ and a subset $M' \subseteq M$ such that:

(i) restriction $\varphi|M'$ is a bijective mapping $M' \to N$;
(ii) for every $i \in M'$, $i \in I_{\varphi(i)}(A, b) - \bigcup \{I_j(A, b) \cup K_j(A, b); j \in N, j \neq \varphi(i)\}$;
(iii) for every $i \in M$, $i \in I_{\varphi(i)}(A, b) \cup K_{\varphi(i)}(A, b)$.

If $m = n$, then $M' = M$ and the condition (iii) can be left out.

3. Strongly regular and trapezoidal matrices

A square matrix in a max–min algebra is strongly regular if the matrix represents a uniquely solvable system of linear equations, for some right-hand side vector. Strong regularity was considered in [2–4] for special cases of max–min algebras, and relations to the trapezoidal property were described. The general case was studied in [9] and the results will be used in Section 4.

For $x \in \mathcal{B}$, the general successor GS$(x)$ of $x$ is defined by

$$\text{GS}(x) := \max\{y \in \mathcal{B}; x \leq y \land \neg(\exists z \in \mathcal{B})x < z < y\}.$$ 

In the case of a discrete max–min algebra $\mathcal{B}$, the above definition of general successor gives the same notion as in [3]. If $\mathcal{B}$ is dense, then GS$(x) = x$ for every $x \in \mathcal{B}$. However, the definition applies also in the cases when $\mathcal{B}$ is neither discrete nor dense. The general successor of a vector $b \in \mathcal{B}(n)$ is the vector $c = \text{GS}(b) \in \mathcal{B}(n)$ with $c_i = \text{GS}(b_i)$, for all $i \in N$.

For vectors $b, c \in \mathcal{B}(n)$, we write $b \sqsupset c$, when the strict inequality $b_i > c_i$ is fulfilled for every $i \in N$. Further, we write $b \sqsupseteq c$ when, for every $i \in N$, $b_i > c_i$ or $b_i = I$ holds true.

For $A \in \mathcal{B}(n, n)$, the diagonal vector $d(A) \in \mathcal{B}(n)$ and the overdiagonal maximum vector $a^*(A) \in \mathcal{B}(n)$ are defined by

$$d_i(A) := a_{ii}, \quad a^*_i(A) := \bigoplus_{k=1}^{i} \bigoplus_{j=k+1}^{n} a_{kj}.$$ 

When there is no danger of confusion, we use a shorter notation $d, a^*$, instead of $d(A), a^*(A)$.

Further, the overdiagonal delimiter $\Delta(A)$ is defined as the least vector $\Delta \in \mathcal{B}(n)$ with the properties

(i) $\Delta \geq \text{GS}(a^*)$,
(ii) $i \leq j \Rightarrow \Delta_i \leq \Delta_j$,
(iii) $j < i, \Delta_j \leq a_{kj} \Rightarrow \text{GS}(\Delta_j) \leq \Delta_i$

for all $i, j \in N$. We say that the overdiagonal delimiter $\Delta(A)$ is strict in $A$, if for any $j, k \in N, j \neq k$, the equalities $\Delta_j(A) = \Delta_k(A) = I$ imply $a_{jk} < I$. Similarly as above, the shorter notation $\Delta$, instead of $\Delta(A)$, is sometimes used. It is easy to verify that $\Delta(A)$ can be computed by recursion as follows:

$$\Delta_1 := \text{GS}(a^*_1),$$
$$\Delta_i := \Delta_{i-1} \oplus \text{GS}(a^*_i) \oplus \max\{\text{GS}(\Delta_k); k < i, \Delta_k = \Delta_{i-1} \leq a_{ik}\}, \quad \text{for } i > 1.$$ 

We say that a matrix $A \in \mathcal{B}(n, n)$ is generally trapezoidal, if the overdiagonal delimiter $\Delta(A)$ is strict in $A$ and $d(A) \sqsupset \Delta(A)$. We remark that, for the case of a discrete max–min algebra $\mathcal{B}$, the overdiagonal delimiter $\Delta(A)$ was defined without any special name in [3], and the notion of a generally trapezoidal matrix was denoted there as strongly trapezoidal. For a dense algebra $\mathcal{B}$, the overdiagonal delimiter is equal to the overdiagonal maximum vector $\Delta(A) = a^*(A)$ and our definition of a generally trapezoidal matrix coincides with the definition of a trapezoidal matrix in a dense algebra [2]. Thus, the above definition is a generalization of both cases and applies also in the cases when $\mathcal{B}$ is neither discrete nor dense.

If $A \in \mathcal{B}(n, n)$ and $b \in \mathcal{B}(n)$, then we say that $A$ is $b$-normal, if $i \in I_1(A, b)$ for every $i \in N$. 
Theorem 3.1 (Gavalec [9]). Let \( A \in \mathcal{B}(n, n) \). Then the following statements are equivalent:

(i) \( A \) is generally trapezoidal;
(ii) there is an increasing vector \( b \in \mathcal{B}(n) \) such that \( A \) is \( b \)-normal and the system \( A \otimes x = b \) is uniquely solvable.

The proof of Theorem 3.1 in [9] uses the following lemma, which will be useful in the next section.

Lemma 3.2 (Gavalec [9]). Let \( A \in \mathcal{B}(n, n) \) be generally trapezoidal matrix, let \( b \in \mathcal{B}(n) \) be an increasing vector \( b \in \mathcal{B}(n) \) such that

(i) \( d(A) \supseteq b \supseteq a^*(A) \);
(ii) \( (\forall i, j \in N) [(j < i \land b_j \leq a_{ij}) \Rightarrow b_j < b_i] \).

Then the system \( A \otimes x = b \) is uniquely solvable.

If \( A \in \mathcal{B}(n, n) \) is a square matrix and \( \varphi, \psi \) are permutations on \( N \), then \( A_{\varphi \psi} \in \mathcal{B}(n, n) \) denotes the result of applying the permutation \( \varphi \) to the rows and the permutation \( \psi \) to the columns of the matrix \( A \). We say that matrices \( A, B \) are equivalent if there are permutations \( \varphi, \psi \), such that \( B = A_{\varphi \psi} \), i.e. \( a_{ij} = b_{\varphi(i) \psi(j)} \) for every \( i, j \in N \).

Theorem 3.3 (Gavalec [9]). Let \( A \in \mathcal{B}(n, n) \). Then the following statements are equivalent:

(i) \( A \) is strongly regular;
(ii) \( A \) is equivalent to a generally trapezoidal matrix, i.e. there are permutations \( \varphi, \psi \) such that \( A_{\varphi \psi} \) is generally trapezoidal.

A general algorithm for deciding whether a given max–min matrix is strongly regular, was presented in [8].

4. Simple image set

For a square matrix \( A \in \mathcal{B}(n, n) \) and for a permutation \( \pi : N \to N \), we denote

\[
S_A := \{ b \in \mathcal{B}(n); (\exists x \in \mathcal{B}(n)) A \otimes x = b \},
\]

\[
\mathcal{F}_\pi(A) := \{ b \in \mathcal{B}(n); A \otimes b_\pi = b \},
\]

where \( b_\pi \) is created from \( b \) by permutation \( \pi \). If \( \pi \) is the identity permutation on \( N \), then we write simply \( \mathcal{F}(A) \) instead of \( \mathcal{F}_\pi(A) \). The set \( S_A \) is called the simple image set of the matrix \( A \), in short: the simple image set. The unique solvability of the equation \( A \otimes x = b \) for a given vector \( b \in S_A \) is described by the next theorem, which is based on the formulation of Theorem 2.4 for the case \( n \times n \). The lemma characterizes the corresponding unique solution \( x \).

Theorem 4.1. Let \( A \in \mathcal{B}(n, n) \), \( b \in \mathcal{B}(n) \). The equation \( A \otimes x = b \) has a unique solution \( x \in \mathcal{B}(n) \) if and only if there is a permutation \( \varphi : N \to N \) such that:

(i) for every \( i \in N, i \in I_{\varphi(i)}(A, b) \);
(ii) for every \( i, j \in N, i \neq j, i \notin I_{\varphi(j)}(A, b) \cup K_{\varphi(j)}(A, b) \).

Remark 4.1. By definition, the sets \( I_j(A, b) \) and \( K_j(A, b) \) are disjoint, for every \( j \in N \). Hence, if conditions (i) and (ii) in Theorem 4.1 are satisfied, then all sets \( K_j(A, b) \) for \( j \in N \) are empty.

Lemma 4.2. Let \( A \in \mathcal{B}(n, n) \), \( b \in \mathcal{B}(n) \). If the equation \( A \otimes x = b \) has a unique solution \( x \in \mathcal{B}(n) \), and if a permutation \( \varphi : N \to N \) fulfills the assertions (i) and (ii) in Theorem 4.1, then \( x = \tilde{x}(A, b) \) and

\[
a_{i \varphi(i)} \geq b_i = x_{\varphi(i)} \quad \text{for every } i \in N, \tag{2}
\]

\[
a_{i \varphi(j)} \otimes x_{\varphi(j)} < b_i \quad \text{for every } i, j \in N, i \neq j. \tag{3}
\]
**Proof.** The equality \( x = \bar{x}(A, b) \) follows from Lemma 2.1. Then the assertion (2) is a direct consequence of the definition of \( I_j(A, b) \) and of the assertion (i) in Theorem 4.1. The assertion (3) follows from two observations. First, the equality \( a_{\varphi(j)} \otimes x_{\varphi(j)} = b_j \) cannot hold for \( i \neq j \), in view of assertion (ii) in Theorem 4.1. Second, the reversed inequality \( a_{\varphi(j)} \otimes x_{\varphi(j)} > b_i \) would imply that \( x \) does not fulfill the \( i \)th equation in \( A \otimes x = b \). □

For generally trapezoidal matrices, we get further inequalities.

**Lemma 4.3.** If a matrix \( A \in \mathcal{B}(n, n) \) is generally trapezoidal, then

\[
\begin{align*}
    a_{ii} & > a_{ij}, \\
    a_{il} & > a_{ij}
\end{align*}
\]

for any \( i, j, l \in N, i < j, l \).

**Proof.** By the definition of general trapezoidality, we have for all \( i \in N \ a_{ii} = I, \) or \( a_{ii} > \text{GS}(a_i^*) > a_{ij} \), and \( a_{il} = I, \) or \( a_{il} > \text{GS}(a_i^*) > a_{ij} \).

For the cases \( a_{ii} = I, a_{il} = I, \) we notice that \( a_{ij} < I \) always holds true, because \( a_{ij} = I \) would imply \( \varphi^{-1}(\mathbb{I}) = \varphi^{-1}(A) = I, \) which would be in contradiction to the condition that \( \varphi(A) = I \) is strict in \( A \). □

**Theorem 4.4.** Let \( A \in \mathcal{B}(n, n) \) be generally trapezoidal. Then \( S_A \subseteq \mathcal{I}(A) \).

**Proof.** Let \( b \in S_A \), let \( x \in \mathcal{B}(n) \) be the unique solution of \( A \otimes x = b \). By Theorem 4.1 and Lemma 4.2, there is a permutation \( \varphi : N \to N \) such that the inequalities (2) and (3) hold true.

We shall show that \( \varphi = \text{id}_N \). Let us assume, by contradiction, that \( \varphi \neq \text{id}_N \). We denote

\[
\begin{align*}
    i_1 & := \min\{i \in N; i \neq \varphi(i)\}, \\
    i_k & := \varphi(i_{k-1}) \quad \text{for} \ k > 1.
\end{align*}
\]

Let us take \( s \in N \) with \( i_{s+1} = i_1 \). Then \( \sigma = (i_1, i_2, \ldots, i_s) \) is a cycle in \( \varphi \), of length \( s \geq 2 \). In the rest of the proof, we shall use the notation \( a(i,j) := a_{ij} \) and \( b(i) := b_i \), to avoid the double indices.

**Claim 1.** \( b(i_1) > b(i_s) \).

**Proof of Claim 1.** By the minimality of \( i_1 \), we have \( i_1 < \varphi(i_1) \) and, by (4) we get

\[
am(i_1, i_1) > a(i_1, \varphi(i_1)).
\]

Using (2) and (3) for \( i = i_1, j = i_s = \varphi^{-1}(i_1) \), we get

\[
am(i_1, \varphi(i_1)) \geq b(i_1) > a(i_1, i_1) \otimes b(i_s).
\]

The inequalities (6), (7) imply

\[
am(i_1, i_1) > a(i_1, i_1) \otimes b(i_s)
\]

which gives

\[
am(i_1, i_1) \otimes b(i_s) = b(i_s).
\]

By (7), (8), we have \( b(i_1) > b(i_s) \).

**Claim 2.** For every \( k = 2, 3, \ldots, s \), if \( b(i_{k-1}) \geq b(i_1) \), then \( b(i_k) > b(i_1) \).

**Proof of Claim 2.** By (3), for \( i = i_k, j = i_{k-1} = \varphi^{-1}(i_k) \), and by the assumption of Claim 2, we have

\[
b(i_k) > a(i_k, i_k) \otimes b(i_{k-1}) \geq a(i_k, i_k) \otimes b(i_1).
\]
Theorem 4.5. Let
\[ a(i_k, i_k) > a(i_1, \varphi(i_1)) \]  
and, by (2), we get
\[ a(i_1, \varphi(i_1)) \geq b(i_1). \]  
The inequalities (10), (11) imply
\[ a(i_k, i_k) \otimes b(i_1) \geq a(i_1, \varphi(i_1)) \otimes b(i_1) = b(i_1) \]  
i.e., \( b(i_k) > b(i_1) \), in view of (9). The proof of Claim 2 is complete.

By applying Claim 2 recursively for \( k = 2, 3, \ldots, s \), we get \( b(i_s) > b(i_1) \), which is in contradiction to Claim 1. Therefore, \( \varphi = \text{id}_N \) and, by (2), \( x = b \) holds true. Thus, \( A \otimes b = b \), i.e. \( b \in F(A) \). \( \square \)

Theorem 4.5. Let \( A \in B(n, n) \). Then there is a permutation \( \pi : N \to N \), such that \( S_A \subseteq F_\pi(A) \).

Proof. If \( A \) is not strongly regular, then \( S_A = \emptyset \) and \( S_A \subseteq F_\pi(A) \) holds for every \( \pi : N \to N \). If \( A \) is strongly regular, then \( A \) is equivalent to a generally trapezoidal matrix \( C \), by Theorem 3.3. Thus, there are permutations \( \varphi, \psi \) such that \( A_{\varphi\psi} = C \). Let us denote \( \pi := \varphi \circ \psi^{-1} \). We shall prove that \( S_A \subseteq F_\pi(A) \).

Let \( b \in S_A \) and let \( x \in B(n) \) be the unique solution of the equation \( A \otimes x = b \). Applying the row permutation \( \varphi \) and the column permutation \( \psi \), we get \( A_{\varphi\psi} \otimes x_\psi = b_\varphi \), i.e. \( C \otimes x_\psi = b_\varphi \). Row permutations and column permutations do not change the cardinality of the solution set, therefore \( x_\psi \) is the unique solution of the equation \( C \otimes x = b_\varphi \). Thus, we have \( b_\varphi \in S_C \) and, in view of Theorem 4.4, \( b_\varphi \in F(C) \), i.e. \( x_\psi = b_\varphi \). This implies \( x = b_{\varphi\psi^{-1}} = b_\pi \). Therefore, \( A \otimes b_\pi = b \), i.e. \( b \in F_\pi(A) \). \( \square \)

The last theorem in this section describes the vectors in \( S_A \) in more detail.

Theorem 4.6. Let \( A \in B(n, n) \) be generally trapezoidal, let \( b \in B(n) \) be increasing. Then \( b \in S_A \) if and only if

1. \( d(A) \supseteq b \supseteq a^*(A) \),
2. \( \forall i, j \in N \) \( (j < i \land b_j \leq a_{ij} \) \( \Rightarrow b_j < b_i \) \).

Proof. If the conditions (i), (ii) are fulfilled, then the system \( A \otimes x = b \) is uniquely solvable, by Lemma 3.2. Thus, \( b \in S_A \).

For the converse implication, let us assume that \( b \in S_A \). By Theorem 4.4, we have \( b \in F(A) \), i.e. \( A \otimes b = b \). Then, in view of Lemma 4.2,
\[ a_{ii} \geq b_i = \bar{x}_i(A, b) \quad \text{for every } i \in N, \]  
\[ a_{ij} \otimes b_j < b_i \quad \text{for every } i, j \in N, i \neq j. \]

Claim 1. \( d(A) \supseteq b \).

Proof of Claim 1. Let \( i \in N \) be arbitrary, but fixed. In view of (13), we have \( a_{ii} > b_i \) or \( a_{ii} = b_i = \bar{x}_i(A, b) \). In the second case, the definition of the set \( M_i(A, b) \) gives \( i \notin M_i(A, b) \). Let us suppose that \( M_i(A, b) \neq \emptyset \). Then there exists \( j \in M_i(A, b), j \neq i \) with \( b_j = \bar{x}_i(A, b) \). This implies \( a_{ij} > b_j = b_i \) and \( a_{jj} \otimes b_j = b_j \), which is in contradiction to (14). Thus, \( M_i(A, b) \) must be empty and then \( a_{ii} = \bar{x}_i(A, b) = 1 \). We have proved that, for any \( i \in N, a_{ii} > b_i \) or \( a_{ii} = 1 \) hold true, i.e. \( d(A) \supseteq b \).

Claim 2. \( b \supseteq a^*(A) \).

Proof of Claim 2. Let \( i, j, l \in N \) with \( j \leq i \) and \( j < l \). By the monotonicity of \( b \) and by (14), we have \( a_{jl} \otimes b_l < b_j \leq b_l \), which implies \( a_{jl} = a_{jl} \otimes b_l \) and \( a_{jl} < b_j \leq b_l \). Therefore, \( b_i > a^*_i(A) \) for any \( i \in N \), i.e. \( b \supseteq a^*(A) \).
By Claims 1 and 2, the condition (i) is satisfied. The condition (ii) is a direct consequence of (14). □

We close Section 4 by a simple example showing that for some $A \in B(n, n)$ the difference $F(A) - S_A$ can be non-empty, i.e. the inclusion $S_A \subseteq F(A)$ proved in Theorem 4.4 for all generally trapezoidal matrices, can be strict.

Example 1. Let $B$ be the open real interval $(0, 1)$, let $n = 3$ and let the matrix $A \in B(n, n)$ and vectors $b, b', c \in B(n)$ have the following values

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad b' = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \quad c = \begin{bmatrix} 1.5 \\ 2 \\ 2.5 \end{bmatrix}.$$

It can be immediately computed that $A \otimes b = b$, i.e. $b \in F(A)$. On the other hand, $A \otimes b' = b'$ holds true. Hence, $b$ has two pre-images in the max–min linear mapping corresponding to the matrix $A$ and, as a consequence, $b \notin S_A$.

Remark 4.2. In view of Theorem 3.1, the matrix $A$ in the above example is generally trapezoidal.

5. Simple image set of the matrix square

The results on the simple image set of a given max–min matrix are extended to matrix powers in this section.

Theorem 5.1. Let $A \in B(n, n)$ be generally trapezoidal. Then

(i) $d(A^2) = d(A)$,
(ii) $a^*(A^2) = a^*(A)$,
(iii) $x(A^2) = x(A)$,
(iv) $x(A^2)$ is strict in $A^2$,
(v) $A^2$ is generally trapezoidal.

Proof. (i) By assumption, the matrix $A \in B(n, n)$ is generally trapezoidal, i.e. the overdiagonal delimiter $x(A)$ is strict in $A$ and $x(A) \subseteq d(A)$. Let $i \in N$ be arbitrary, but fixed. By definition,

$$d_i(A^2) = \bigoplus_{k=1}^{n} (a_{ik} \otimes a_{ki}).$$

By the definition of $a^*(A)$, we have $a_{ki} \leq a^*_{i}(A)$, for $k < i$, and $a_{ik} \leq a^*_{i}(A)$, for $k > i$. As a consequence, for any $k \in N$, $k \neq i$, the inequalities

$$a_{ik} \otimes a_{ki} \leq a^*_{i}(A) \leq GS(a^*_{i}(A)) \leq x_i(A) \leq d_i(A) = a_{ii} = a_{ii} \otimes a_{ii}$$

hold true. Hence,

$$a_{ii} \otimes a_{ii} \leq \bigoplus_{k=1}^{n} (a_{ik} \otimes a_{ki}) \leq a_{ii} \otimes a_{ii}$$

which implies $d_i(A^2) = d_i(A)$. As $i \in N$ is arbitrary, we have $d(A^2) = d(A)$.

(ii) We shall denote by $a_{ij}^{(2)}$ the elements of the matrix $A^2$. Let us choose an arbitrary but fixed index $i \in N$. By the definition of $a^*(A^2)$, we have

$$a^*_{i}(A^2) = \bigoplus_{k=1}^{i} \bigoplus_{j=k+1}^{n} a_{kj}^{(2)}.$$
Let us take now two arbitrary, but fixed indices \( k, j \in N \) with \( 1 \leq k \leq i, k < j \leq n \). By the matrix multiplication rule,

\[
a_{kj}^{(2)} = \bigoplus_{p=1}^{n} (a_{kp} \otimes a_{pj}).
\]

If \( p < k \), then \( p < i \) and \( p < j \), which implies \( a_{kp} \otimes a_{pj} \leq a_{kj} \leq a_{kj}^*(A) \). For \( p = k \) we have \( a_{kk} \otimes a_{kj} = d_k(A) \otimes a_{kj} = a_{kj} \leq a_{kj}^*(A) \), and for \( p > k \) we have \( a_{kp} \otimes a_{pj} \leq a_{kp} \leq a_{kj}^*(A) \). Therefore, \( a_{kj}^{(2)} \leq a_{kj}^*(A) \). As \( k, j \in N \) are arbitrary with \( 1 \leq k \leq i, k < j \leq n \), we get \( a_{kj}^*(A^2) \leq a_{kj}^*(A) \).

On the other hand, since \( k < j \)

\[
a_{kj} = d_k(A) \otimes a_{kj} = a_{kk} \otimes a_{kj} \leq \bigoplus_{p=1}^{n} (a_{kp} \otimes a_{pj}) = a_{kj}^{(2)}
\]

which implies \( a_{kj}^{(2)} \geq a_{kj} \). As a consequence, \( a_{kj}^*(A^2) \geq a_{kj}^*(A) \). Thus, we have proved \( a_{kj}^*(A^2) = a_{kj}^*(A) \). This implies \( a^*(A^2) = a^*(A) \), because the index \( i \) has been chosen arbitrarily in \( N \).

(iii) By the recursive formula for the overdiagonal delimiter we have, in view of the statement (ii),

\[
x_1(A^2) = \text{GS}(a_1^*(A^2)) = \text{GS}(a_1^*(A)) = x_1(A).
\]

Proceeding by the mathematical induction with respect to \( i \), let us assume that \( i > 1, i \in N \) and \( x_k(A^2) = x_k(A) \) holds for \( k = 1, 2, \ldots, i - 1 \). We shall prove that under this assumption, \( x_i(A^2) = x_i(A) \) holds true, as well.

By the recursive formula, we have for \( i > 1 \)

\[
x_i(A^2) = x_{i-1}(A^2) \oplus \text{GS}(a_i^*(A^2))
\]

\[
\oplus \max\{\text{GS}(x_k(A^2)); k < i, x_k(A^2) = x_{i-1}(A^2) \leq a_{i-1}^{(2)} \}.
\]

In the notation

\[
\mathcal{H}_1 := \{\text{GS}(x_k(A)); k \in N, k < i, x_k(A) = x_{i-1}(A) \leq a_{i-1} \},
\]

\[
\mathcal{H}_2 := \{\text{GS}(x_k(A^2)); k \in N, k < i, x_k(A^2) = x_{i-1}(A^2) \leq a_{i-1}^{(2)} \}
\]

we can write

\[
x_i(A) = x_{i-1}(A) \oplus \text{GS}(a_i^*(A)) \oplus \max\mathcal{H}_1,
\]

\[
x_i(A^2) = x_{i-1}(A^2) \oplus \text{GS}(a_i^*(A^2)) \oplus \max\mathcal{H}_2.
\]

In view of the statement (ii), the equalities \( x_{i-1}(A^2) = x_{i-1}(A) \) and \( \text{GS}(a_i^*(A^2)) = \text{GS}(a_i^*(A)) \) hold true for the first two summands in the above formulas. To prove \( x_i(A^2) = x_i(A) \), the inequalities for the third summands described in Claims 1, 2 will be sufficient.

**Claim 1.** \( \max\mathcal{H}_1 \leq \max\mathcal{H}_2 \).

**Proof of Claim 1.** Let \( k \in N, k < i \) such that \( x_k(A) = x_{i-1}(A) \leq a_{i-1} \), i.e. let \( \text{GS}(x_k(A)) \in \mathcal{H}_1 \). The matrix \( A \) is generally trapezoidal, therefore \( x_k(A) \leq d_k(A) = a_{kk} \). Hence

\[
x_k(A) \leq a_k \otimes a_{kk} \leq \bigoplus_{p=1}^{n} (a_{kp} \otimes a_{pk}) = a_{ik}^{(2)}.
\]

Thus, by the induction assumption we get \( x_k(A^2) = x_{i-1}(A^2) \leq a_{i-1}^{(2)} \). As a consequence, \( \text{GS}(x_k(A)) = \text{GS}(a_k(A^2)) \in \mathcal{H}_2 \).

**Claim 2.** \( x_i(A) \geq \max\mathcal{H}_2 \).
Proof of Claim 2. Let $GS(z_k(A)^2) \in \mathcal{H}_2$, i.e. let $k \in N$, $k < i$ such that $z_k(A^2) = z_{i-1}(A^2) \leq a_{ik}^{(2)}$. By the induction assumption we get
\[
z_k(A) = z_{i-1}(A) \leq a_{ik}^{(2)} = \bigoplus_{p=1}^{n} (a_{ip} \otimes a_{pk})
\]
Then there is $p \in N$ such that $a_{ik} < I$, which implies $z_k(A) \leq a_{ip}$ and $z_k(A) \leq a_{pk}$.

Case 1: If $p < k$, then we have
\[a_{pk} \leq a^*_p(A) \leq GS(a^*_p(A)) \leq a_p(A) \leq z_k(A) \leq a_{pk},\]
where all inequalities are satisfied as equalities. Hence, we get $z_p(A) = z_k(A) \leq a_{ip}$, i.e. $GS(z_p(A)) = GS(z_k(A)) = GS(z_k(A^2)) \in \mathcal{H}_1$.

Case 2: If $k < p < i$, then
\[z_k(A) \leq a_p(A) \leq z_{i-1}(A) = z_i(A)
\]
implies $z_p(A) = z_k(A) \leq a_{ip}$. Hence, $GS(z_p(A)) = GS(z_k(A^2)) \in \mathcal{H}_1$.

Case 3: If $p = i$, then we have $z_k(A) \leq a_{ik}$. This directly implies $GS(z_k(A)) = GS(z_k(A^2)) \in \mathcal{H}_1$.

Case 4: If $p > i$, then $z_k(A) \leq a_{ip} \leq a^*_i(A) \leq GS(a^*_i(A)) \leq z_i(A)$. We shall distinguish two subcases: (a) $z_k(A) < z_i(A)$, (b) $z_k(A) = z_i(A)$.

Subcase (a): By the definition of the general successor, $GS(z_k(A)) \leq z_i(A)$ and Claim 2 follows.

Subcase (b): The inequalities in the above formula are all satisfied as equalities, i.e. $z_k(A) = a^*_i(A) = GS(a^*_i(A)) = z_i(A)$. Therefore, $z_k(A) = GS(z_k(A)) = z_i(A)$. Then, by the induction assumption, we have
\[z_k(A) = z_k(A^2) = GS(z_k(A^2)) = z_i(A).
\]

The assertions of Claims 1 and 2 imply that $z_i(A^2) = z_i(A)$. By this, the proof of the statement (iii) is complete.

(iv) Let $j, k \in N$, $j < k$ and let $z_j(A^2) = z_k(A^2) = I$. By the statement (iii), we have $z_j(A) = z_k(A) = I$. For every $p \in N$, $p \geq j$, this implies $z_p(A) = I$ and, in view of the fact that $z(A)$ is strict in $A$, at least one of the inequalities $a_{jp} < I$, $a_{pk} < I$, and at least one of the inequalities $a_{kp} < I$, $a_{pk} < I$ holds true.

On the other hand, for every $p \in N$, $p < j$, the inequalities $a_{pj} < I$, $a_{pk} < I$ are fulfilled. Namely, if we assume $a_{pj} = I$, or $a_{pk} = I$, then we get $a^*_p(A) = I$, which implies $z_p(A) = I$. Then a contradiction arises, as $z(A)$ is strict in $A$. Our consideration shows that
\[
a_{ik}^{(2)} = \bigoplus_{p=1}^{n} (a_{jp} \otimes a_{pk}) = \bigoplus_{p=1}^{j-1} (a_{jp} \otimes a_{pk}) \oplus \bigoplus_{p=j}^{n} (a_{jp} \otimes a_{pk}) < I,
\]
\[
a_{kj}^{(2)} = \bigoplus_{p=1}^{n} (a_{kp} \otimes a_{pj}) = \bigoplus_{p=1}^{j-1} (a_{kp} \otimes a_{pj}) \oplus \bigoplus_{p=j}^{n} (a_{kp} \otimes a_{pj}) < I.
\]

Hence, $z(A^2)$ is strict in $A^2$.

(v) The assumption that $A$ is generally trapezoidal and the statements (i), (iii) imply $z(A^2) = z(A) \subseteq d(A) = d(A^2)$. Further, the overdiagonal delimiter $z(A^2)$ is strict in $A^2$, by the statement (iv). Hence, $A^2$ is generally trapezoidal. \(\square\)

Lemma 5.2. Let $A \in \mathcal{B}(n, n)$ be generally trapezoidal, let $b \in SA$. Let $\varphi : N \rightarrow N$ be such a permutation that the vector $b_\varphi$ is increasing. Then

(i) $A_{\varphi \varphi}$ is generally trapezoidal,

(ii) $b_\varphi \in SA_{\varphi \varphi}$.

Proof. (i) Let $A \in \mathcal{B}(n, n)$, $b \in \mathcal{B}(n)$ and $\varphi : N \rightarrow N$ fulfill the assumptions of the lemma. It is easy to verify that for every $i \in N$, $\varphi(i) \in I_{\varphi(i)}(A_{\varphi \varphi}, b_\varphi)$. In other words, the matrix $A_{\varphi \varphi}$ is $b_\varphi$-normal. Then $A_{\varphi \varphi}$ is generally trapezoidal, by Theorem 3.1.
(ii) The statement can be verified by a direct computation. □

Theorem 5.3. Let \( A \in \mathcal{B}(n, n) \) be generally trapezoidal. Then \( S_A^2 = S_A \).

Proof. First let us prove the inclusion \( S_A^2 \supseteq S_A \). Let \( b \in S_A \), i.e. let the equation \( A \otimes x = b \) have a unique solution. In view of Theorem 4.4, \( b \) is the unique solution. Then,

\[
A^2 \otimes b = A \otimes (A \otimes b) = A \otimes b = b.
\]

On the other hand, let \( b' \in \mathcal{B}(n) \) be a solution of \( A \otimes x = b \). Then \( A^2 \otimes b' = A \otimes (A \otimes b') = b \). As \( b \) is the unique solution, we get first \( (A \otimes b') = b \) and then \( b' = b \). Hence, \( b \) is also the unique solution of the equation \( A^2 \otimes x = b \), i.e. \( b \in S_A^2 \).

To prove the converse inclusion \( S_A^2 \subseteq S_A \), let us assume that \( b \in S_A^2 \), i.e. we assume that \( b \) is the unique solution of the matrix equation \( A \otimes x = b \).

Claim 1. Let \( b \in S_A^2 \) be an increasing vector. Then \( b \in S_A \).

Proof of Claim 1. By Theorem 4.6, \( b \in S_A^2 \) implies

(i) \( d(A) \supseteq b \supseteq a^*(A^2) \),
(ii) \((\forall i, j \in N)[(j < i \land b_j \leq a_{ij}^{(2)}) \Rightarrow b_j < b_i]\).

The statement (i) implies \( d(A) \supseteq b \supseteq a^*(A) \), in view of Theorem 5.1(i), (ii). We shall show that \( b \) fulfills also the statement (ii), with respect to the matrix \( A \). Let \( i, j \in N, j < i, b_j \leq a_{ij} \). As \( A \) is generally trapezoidal, we have \( a_{ij} \leq d_i(A) \), which implies

\[
a_{ij} = d_i(A) \otimes a_{ij} = a_{ii} \otimes a_{ij} = \bigoplus_{p=1}^{n} (a_{ip} \otimes a_{pj}) = a_{ij}^{(2)}.
\]

Hence, \( b_j \leq a_{ij}^{(2)} \), which implies \( b_j < b_i \), by the above condition (ii). We have proved that \( b \in S_A \), in view of Theorem 4.6.

In the case, when the vector \( b \in S_A^2 \) is not increasing, we permute the components of \( b \) by such a permutation \( \varphi : N \to N \), that the permuted vector \( b_\varphi \) will be increasing. Applying Lemma 5.2 to \( A^2 \) and \( b \), we prove that

(i) \( A_{\varphi \varphi}^2 \) is generally trapezoidal,
(ii) \( b_\varphi \) is the unique solution of the equation \( A_{\varphi \varphi}^2 \otimes x = b_\varphi \).

Then, according to Claim 1, \( b_\varphi \) is the unique solution of the equation \( A_{\varphi \varphi} \otimes x = b_\varphi \). This implies that \( b \) is the unique solution of the equation \( A \otimes x = b \), i.e. \( b \in S_A \). □

6. Topological aspects

The inclusion \( S_A \subseteq \mathcal{F}(A) \) in Theorem 4.4 can be extended to the closure of \( S_A \). We shall consider the ordered set \( \mathcal{B} \) as a topological space with the interval topology, in which open intervals form a base of open sets. That means that every open set in \( \mathcal{B} \) is a union of some set of open intervals. Any open interval is either two-sided, of the form \( (b, c) := \{ x \in \mathcal{B}; b < x < c \} \), or one-sided, of the form \( (c) := \{ x \in \mathcal{B}; x < c \} \), or of the form \( (b) := \{ x \in \mathcal{B}; b < x \} \).

Further, the vector space \( \mathcal{B}(n) \), for a fixed \( n \), will be considered as a topological space with the product topology derived from the interval topology in \( \mathcal{B} \). In other words, the base of open sets in \( \mathcal{B}(n) \) is formed by the open sets of
Theorem 6.3. Let

\[ \mathcal{U} = \prod (\mathcal{U}_i; i \in N) := \{ x \in \mathcal{B}(n); (\forall i \in N) x_i \in \mathcal{U}_i \}, \]

where every \( \mathcal{U}_i \) is an open subset in \( \mathcal{B} \).

Lemma 6.1. Let \( x, y \in \mathcal{B}, x < y \). Then there are open subsets \( \mathcal{U}, \mathcal{V} \subseteq \mathcal{B} \) with \( x \in \mathcal{U}, y \in \mathcal{V} \), and with \( x' < y' \) for every \( x' \in \mathcal{U}, y' \in \mathcal{V} \).

Proof. If there is \( a \in \mathcal{B} \), such that \( x < a < y \), then we set \( \mathcal{U} := (a) \) and \( \mathcal{V} := (a) \). Otherwise, we set \( \mathcal{U} := (x) \) and \( \mathcal{V} := (x) \).

Lemma 6.2. If \( f, g : \mathcal{B}(n) \to \mathcal{B} \) are continuous mappings, then the mappings \( f \oplus g, f \otimes g \) are continuous, as well.

Proof. Let \( x \in \mathcal{B}(n) \), let us denote \( y := f(x), z := g(x) \), and let \( \mathcal{V} \subseteq \mathcal{B} \) be an open set with \( y \oplus z \in \mathcal{V} \). We shall show that there is an open subset \( \mathcal{U} \subseteq \mathcal{B}(n) \) with \( x \in \mathcal{U} \) and \( f(\mathcal{U}) \subseteq \mathcal{V} \). We shall distinguish two cases.

Case 1: If \( y = z \), then \( y \oplus z = y = z \in \mathcal{V} \). As mappings \( f, g \) are continuous, there are open subsets \( \mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{B}(n) \) with \( x \in \mathcal{U}_1, x \in \mathcal{U}_2 \) and \( \mathcal{f}(\mathcal{U}_1) \subseteq \mathcal{V}, \mathcal{g}(\mathcal{U}_2) \subseteq \mathcal{V} \). Denoting \( \mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2 \) we get \( x \in \mathcal{U} \) and \( f(\mathcal{U}) \subseteq \mathcal{V} \).

Case 2: If \( y \neq z \), then we may assume \( y < z \), without any loss of generality. Then, in view of Lemma 6.1, there are open subsets \( \mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{B} \), with \( y \in \mathcal{V}_1, z \in \mathcal{V}_2 \), and with \( y' < z' \) for every \( y' \in \mathcal{V}_1, z' \in \mathcal{V}_2 \).

The continuity of mappings \( f, g \) implies that there are open subsets \( \mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{B}(n) \), such that \( x \in \mathcal{U}_1, x \in \mathcal{U}_2 \), \( f(\mathcal{U}_1) \subseteq \mathcal{V}_1 \), and \( g(\mathcal{U}_2) \subseteq \mathcal{V}_2 \). If we denote \( \mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2 \), then we have \( x \in \mathcal{U} \) and \( f(\mathcal{U}) \subseteq \mathcal{V}_1 \), \( g(\mathcal{U}) \subseteq \mathcal{V}_2 \). Thus, for any \( x' \in \mathcal{U} \) we have \( f(x') \in \mathcal{V}_1, g(x') \in \mathcal{V}_2 \), which implies \( f(x') < g(x') \) and \( (f \oplus g)(x') = f(x') \oplus g(x') = g(x') \in \mathcal{V} \), i.e. \( (f \oplus g)(\mathcal{U}) \subseteq \mathcal{V} \).

We have proved that mapping \( f \oplus g \) is continuous. The continuity of \( f \otimes g \) is proved analogously.

Using Lemmas 6.1 and 6.2, the following theorem can be proved by a standard topological argument.

Theorem 6.3. Let \( A \in \mathcal{B}(n, n) \). Then the mapping \( f_A : \mathcal{B}(n) \to \mathcal{B}(n) \), defined by \( f_A(x) := A \otimes x \), is continuous.

Theorem 6.4. Let \( A \in \mathcal{B}(n, n) \). Then \( \mathcal{F}(A) \) is a closed subset of \( \mathcal{B}(n) \).

Proof. We shall prove that the complement \( \mathcal{B}(n) - \mathcal{F}(A) \) is open. Let \( b \in \mathcal{B}(n) - \mathcal{F}(A) \), i.e. \( A \otimes b \neq b \). By Lemma 6.1, there are disjoint open subsets \( \mathcal{U}, \mathcal{V} \subseteq \mathcal{B}(n) \) with \( b \in \mathcal{U}, A \otimes b \in \mathcal{V} \). By Theorem 6.3, mapping \( f_A : x \to A \otimes x \) is continuous. Therefore, there is an open subset \( \mathcal{W} \subseteq \mathcal{B}(n) \) with \( b \in \mathcal{W}, f_A(\mathcal{W}) \subseteq \mathcal{V} \). Without any loss of generality, we may assume that \( \mathcal{W} \subseteq \mathcal{U} \), i.e. \( \mathcal{W} \cap \mathcal{V} = \emptyset \).

Thus, for any \( x \in \mathcal{W} \), we have \( A \otimes x = f_A(x) \in \mathcal{V} \), which implies \( A \otimes x \neq x \), i.e. \( x \notin \mathcal{F}(A) \). We have proved that for any \( b \notin \mathcal{F}(A) \) there is an open neighbourhood \( \mathcal{W} \) with \( b \in \mathcal{W}, \mathcal{B}(n) - \mathcal{F}(A) \) is open and \( \mathcal{F}(A) \) is closed.

Theorem 6.5. Let \( A \in \mathcal{B}(n, n) \) be generally trapezoidal. Then \( \operatorname{cl}(S_A) \subseteq \mathcal{F}(A) \).

Proof. By Theorem 4.4, \( S_A \subseteq \mathcal{F}(A) \). In view of Theorem 6.4, \( \mathcal{F}(A) \) is closed, and therefore \( \operatorname{cl}(S_A) \) must be a subset of \( \mathcal{F}(A) \).

Remark 6.1. In general, the inclusion sign in Theorem 6.5 cannot be substituted by the sign of equality. E.g. the least vector \( o \in \mathcal{B}(n) \), with \( o_i = O \) for every \( i \in N \), belongs to \( \mathcal{F}(A) \). On the other hand, we can easily see by Theorem 4.6, that vector \( o \) is not in \( \operatorname{cl}(S_A) \), if \( A \) is generally trapezoidal and if there is at least one overdiagonal element \( a_{ij} > O \).
References