

# A LECTURE ON THE LIOUVILLE VERTEX OPERATORS

J. TESCHNER

*Institut für theoretische Physik  
Freie Universität Berlin,  
Arnimallee 14  
14195 Berlin  
Germany*

We reconsider the construction of exponential fields in the quantized Liouville theory. It is based on a free-field construction of a continuous family of chiral vertex operators. We derive the fusion and braid relations of the chiral vertex operators. This allows us to simplify the verification of locality and crossing symmetry of the exponential fields considerably. The calculation of the matrix elements of the exponential fields leads to a constructive derivation of the formula proposed by Dorn/Otto and the brothers Zamolodchikov.

*Dedicated to A.A. Belavin on his 60<sup>th</sup> birthday*

## 1 Introduction

Thanks to conformal symmetry it suffices to know the three point functions of the exponential fields in quantum Liouville theory in order to characterize it completely<sup>1</sup>. An explicit formula for these three point functions was proposed in<sup>2,3</sup>. This proposal has successfully passed many checks, giving strong evidence for the truth of the conjecture. This formula can be considered to encode the exact solution of quantum Liouville theory once it has been shown to lead to a solution of the basic consistency requirements like the crossing symmetry<sup>4</sup>.

A problem of fundamental importance is to describe the monodromies of the conformal blocks in a conformal field theory. Knowing the monodromies not only allows one to formulate and hopefully solve the basic consistency conditions that the three point functions have to satisfy. It also forms the basis for a similar approach to the determination of the structure functions that characterize Liouville theory in the presence of a conformally invariant boundary<sup>5</sup>.

A program aimed at the determination of the monodromies of the conformal blocks in quantum Liouville theory was started in<sup>6</sup>. There we assumed that the monodromies can be described with the help of a generalization of the Moore-Seiberg formalism for rational conformal field theories. Within this formalism it suffices to know certain data, called fusion- and braid coefficients, in order to reconstruct the monodromies of conformal blocks in general. These data can be calculated directly in certain special cases. This information allows one to derive a closed system of functional equations for the general fusion- and braid coefficients from the basic consistency conditions of the Moore-Seiberg formalism. A solution of these functional equations was constructed in<sup>6,7</sup> from the representation theory of the modular double of  $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ <sup>8,9</sup>.

A more direct approach was initiated in<sup>4</sup>. Chiral vertex operators were constructed from a free chiral field. These vertex operators were shown to satisfy exchange relations

which lead to a direct calculation of the fusion- and braiding coefficients. The result was found to be equivalent to the one from <sup>6</sup>.

Unfortunately, the presentation in <sup>4</sup> was rather brief. The aim of the present paper will be to explain the constructions from <sup>4</sup> in more details and to present the derivations of some important results that were stated in <sup>4</sup> without a proof. In particular, we will derive a formula for the matrix elements of the chiral vertex operators. A simplified derivation of the exchange relations of the chiral vertex operators is presented here in some detail. With the help of these exchange relations we will furthermore construct the operator product expansion of the chiral vertex operators (fusion).

We may then construct the Liouville vertex operators  $V_\alpha(z, \bar{z})$  out of the chiral vertex operators. With the help of the exchange relations it becomes rather simple to prove the locality of the Liouville vertex operators. Our formula for the matrix elements of the chiral vertex operators finally allows us to compute the matrix elements of the  $V_\alpha(z, \bar{z})$ . Taken together our results represent a constructive derivation of the formula proposed by Dorn/Otto and the brothers Zamolodchikov for the three-point function in quantum Liouville theory.

The mathematically oriented reader will notice that our treatment of the vertex operators is somewhat formal. However, we will indicate in Section 2 that the problems associated with the field-theoretical nature of the chiral vertex operators are not worse than usual. Known techniques will be applicable. What is nonstandard in our case are the issues associated with the continuous spectrum of the zero modes, which lead e.g. to the first example for exchange relations that involve a continuous set of fields. These are the issues we will mainly be concerned with.

## 2 Definition and main properties of chiral vertex operators

### 2.1 The chiral free field

Given the possibility to map classical Liouville theory to a free field theory, see e.g. Section 13 of <sup>4</sup> for a review, it is natural to approach quantization of Liouville theory by first quantizing the free field theory and then trying to reconstruct the Liouville field operators in terms of operators in the free field theory.

Let us introduce the (left-moving) chiral free field  $\varphi(x_+) = \mathbf{q} + \mathbf{p}x_+ + \varphi_{<}(x_+) + \varphi_{>}(x_+)$ , with mode-expansion given by

$$\varphi_{<}(x_+) = i \sum_{n < 0} \frac{1}{n} \mathbf{a}_n e^{-inx_+}, \quad \varphi_{>}(x_+) = i \sum_{n > 0} \frac{1}{n} \mathbf{a}_n e^{-inx_+}, \quad (2.1)$$

The modes are postulated to have the following commutation and hermiticity relations

$$[\mathbf{q}, \mathbf{p}] = \frac{i}{2}, \quad \mathbf{q}^\dagger = \mathbf{q}, \quad \mathbf{p}^\dagger = \mathbf{p}, \quad [\mathbf{a}_n, \mathbf{a}_m] = \frac{n}{2} \delta_{n+m}, \quad \mathbf{a}_n^\dagger = \mathbf{a}_{-n}, \quad (2.2)$$

which are realized in the Hilbert-space

$$\mathcal{H}_L^F \equiv L^2(\mathbb{R}) \otimes \mathcal{F}, \quad (2.3)$$

where  $\mathcal{F}$  is the Fock-space generated by acting with the modes  $\mathbf{a}_n$ ,  $n < 0$  on the Fock-vacuum  $\Omega$  that satisfies  $\mathbf{a}_n \Omega = 0$ ,  $n > 0$ . We will mainly work in a representation where  $\mathbf{p}$  is diagonal.

## 2.2 Conformal symmetry

A representation of the conformal symmetry is defined on  $\mathcal{H}_L^{\mathcal{F}}$  by means of the standard free field realization of the Virasoro algebra. The action of the Virasoro algebra on  $\mathcal{H}_L^{\mathcal{F}}$  can be defined in terms of the generators  $L_n \equiv L_n(p)$ , where

$$\begin{aligned} L_n(p) &= (2p + inQ)a_n + \sum_{k \neq 0, n} a_k a_{n-k}, \quad n \neq 0, \\ L_0(p) &= p^2 + \frac{Q^2}{4} + 2 \sum_{k > 0} a_{-k} a_k. \end{aligned} \quad (2.4)$$

Equations (2.4) are well-known to yield a representation of the Virasoro algebra with central charge

$$c = 1 + 6Q^2. \quad (2.5)$$

We will mostly consider the case that  $Q > 2$  in the following, corresponding to central charge  $c > 25$ . It turns out, however, that the results that we obtain for this regime have an analytic continuation w.r.t. the parameter  $Q$  which allows one to cover the case  $c > 1$  as well.

Equations (2.4) define a one-parameter family of representations of the Virasoro algebra. This means that  $\mathcal{H}_L^{\mathcal{F}}$  decomposes as the direct integral of Fock-space representations of the Virasoro algebra:

$$\mathcal{H}_L^{\mathcal{F}} \simeq \int_{-\infty}^{\infty} dp \mathcal{F}_p, \quad (2.6)$$

where  $\mathcal{F}_p$  denotes the Virasoro representations defined on  $\mathcal{F}$  by means of the generators  $L_n(p)$  defined in (2.4). The representations  $\mathcal{F}_p$ , are known to be unitary highest weight representations of the Virasoro algebra, the weight being given in terms of  $p$  via  $\Delta_p = p^2 + \frac{1}{4}Q^2$ . We will denote the highest weight vector in  $\mathcal{F}_p$  by  $v_p$ .

The representations  $\mathcal{F}_p$  and  $\mathcal{F}_{-p}$  are unitarily equivalent. This follows immediately from the fact <sup>10</sup> that the monomials

$$L_{-m_1}(p) \dots L_{-m_l}(p)\Omega$$

form a basis for  $\mathcal{F}$  for any  $p \in \mathbb{R}$ . Replacing  $p \rightarrow -p$  therefore defines a unitary map  $S(p) : \mathcal{F}_p \rightarrow \mathcal{F}_{-p}$ . It is useful to collect the maps  $S(p)$ ,  $p \in \mathbb{R}$  into a single operator  $S \equiv S(p) : \mathcal{H}_L^{\mathcal{F}} \rightarrow \mathcal{H}_L^{\mathcal{F}}$ .

The representations are known to exponentiate to projective unitary representations of the group  $\text{Diff}(S_1)$  of orientation-preserving diffeomorphisms of the circle. Elements of  $\text{Diff}(S_1)$  can be parameterized by monotonic smooth functions  $h : \mathbb{R} \rightarrow \mathbb{R}$  that satisfy  $h(\sigma + 2\pi) = h(\sigma) + 2\pi$ . We will use the notation  $U_h$  for the operator that represent the diffeomorphism  $h$  in the representations of  $\text{Diff}(S_1)$  generated by the  $L_n$ .

## 2.3 Building blocks

The basic building blocks of all constructions will be the following objects:

NORMAL ORDERED EXPONENTIALS :

$$\begin{aligned} \mathbf{E}^\alpha(x_+) &\equiv \mathbf{E}_<^\alpha(x_+) \mathbf{E}_>^\alpha(x_+), & \mathbf{E}_<^\alpha(x_+) &= e^{\alpha q} e^{2\alpha\varphi_<^+(x_+)} e^{\alpha x + p} \\ & & \mathbf{E}_>^\alpha(x_+) &= e^{\alpha x + p} e^{2\alpha\varphi_>^+(x_+)} e^{\alpha q} \end{aligned} \quad (2.7)$$

SCREENING CHARGES:<sup>a</sup>

$$\mathbf{Q}(x) \equiv e^{-\pi b p} \int_0^{2\pi} d\sigma \mathbf{E}^b(x + \sigma) e^{-\pi b p}. \quad (2.8)$$

It will be quite important for our purposes to understand the mathematical nature of these objects a little better. Let us first of all recall that fields like the normal ordered exponentials do *not* represent true operators. Indeed, due to the usual short-distance singularities we can never make sense out of  $\|\mathbf{E}^\alpha(x_+)\psi\|^2$ . There are two standard ways to treat this nuisance: On the one hand one may systematically use *smear*ed fields, obtained by multiplying with some test-function and integrating over the cylinder. Alternatively one may analytically continue to negative Euclidean time, i.e. assume that  $\Im(x_+) > 0$ . Objects like  $\mathbf{E}^\alpha(\sigma - i\tau)$ ,  $\tau < 0$  will indeed represent densely defined, but unbounded operators. Combinations like

$$\Pi_p \mathbf{E}^\alpha(\sigma - i\tau) e^{-\epsilon L_0} : \mathcal{F}_{p+i\alpha} \rightarrow \mathcal{F}_p,$$

where  $\epsilon > 0 > \tau$  and  $\Pi_p$  denotes the projection onto  $\mathcal{F}_p$ , are even trace-class, as can be shown by means of a direct calculation of the trace in a coherent-state basis for  $\mathcal{F}$ .

The screening charges on the contrary make sense as true operators even for  $x \in \mathbb{R}$  as their definition already involves some smearing. More precisely: Let  $2b^2 < 1$ . The screening charges  $\mathbf{Q}(\sigma)$  then represent densely defined unbounded operators. To verify this statement let us note that

$$\|\mathbf{Q}(\sigma)\psi\|^2 = \int_\sigma^{2\pi+\sigma} d\sigma' d\sigma'' \langle \psi, \mathbf{Q}(\sigma') \mathbf{Q}(\sigma'') \psi \rangle.$$

The integrand develops for  $\sigma' \rightarrow \sigma''$  a singularity of the form  $|\sigma' - \sigma''|^{-2b^2}$  which is integrable for  $2b^2 < 1$ . Other values of  $b$  can be covered by means of analytic continuation w.r.t.  $b$ .

Let us note the following crucial property:

**POSITIVITY:** The screening charges are positive operators. Indeed, we have that

$$\langle \psi, \mathbf{Q}(\sigma)\psi \rangle = \int_\sigma^{2\pi+\sigma} d\sigma' \|\mathbf{E}_>^b(\sigma')\psi\|^2 > 0.$$

This property will be of fundamental importance for the rest since our construction of Liouville vertex operators will involve *complex* powers  $s$  of the screening charges. Positivity of  $\mathbf{Q}(\sigma)$  allows one to take arbitrary powers of these operators.

<sup>a</sup>In comparison with <sup>4</sup> we have modified our definition by the factors  $e^{-\pi b p}$ . This allows us to get rid of some annoying exponential prefactors in <sup>4</sup>.

## 2.4 Chiral vertex operators

Out of the building blocks introduced in the previous subsection we may now construct an important class of chiral fields.

$$h_s^\alpha(x_+) = E^\alpha(x_+) (Q(x_+))^s, \quad (2.9)$$

Positivity of  $Q$  allows us to consider these objects for *complex* values of  $s$  and  $\alpha$ .

The chiral fields  $h_s^\alpha(x_+)$  will have similar analytic properties as the normal ordered exponentials. As usual it is most convenient to work with the Euclidean fields obtained by analytically continuing to imaginary time. To be specific, let us define the Euclidean fields  $h_s^\alpha(w)$ ,  $w = \tau + i\sigma$  on the domain of  $e^{-\tau L_0}$  as

$$\begin{aligned} h_s^\alpha(w) &\equiv e^{(\tau+\epsilon)L_0} h_{s,\epsilon}^\alpha(\sigma) e^{-\tau L_0}, \\ h_{s,\epsilon}^\alpha(\sigma) &\equiv E^\alpha(\sigma + i\epsilon) e^{-\epsilon L_0} (Q(\sigma))^s, \end{aligned} \quad \tau < 0 < \epsilon < |\tau|. \quad (2.10)$$

One of the most basic properties of the  $h_s^\alpha(w)$  are the simple commutation relations with functions of  $\mathfrak{p}$ ,

$$h_s^\alpha(w) f(\mathfrak{p}) = f(\mathfrak{p} - i(\alpha + bs)) h_s^\alpha(w), \quad (2.11)$$

which follow from the fact that  $h_s^\alpha(w)$  depends on  $\mathfrak{q}$  only via an overall factor  $e^{2(\alpha+bs)\mathfrak{q}}$ .

It is often convenient to work with  $\Pi_p h_s^\alpha(w)$  rather than  $h_s^\alpha(w)$ , where  $\Pi_p$  denotes the projection onto an eigenspace  $\mathcal{F}_p \simeq \mathcal{F}$  of  $\mathfrak{p}$ . Equation (2.11) implies that  $\Pi_p h_s^\alpha(w)$  projects onto  $\mathcal{F}_q$ , where  $q = p + i(\alpha + bs)$ . In order to assure  $q \in \mathbb{R}$  we will mostly assume  $i(\alpha + bs) \in \mathbb{R}$  in the following. More general cases can afterwards be treated by analytic continuation.  $\Pi_p h_s^\alpha(w)$  therefore defines a family of operators

$$h_{p_2 p_1}^\alpha(w) : \mathcal{F}_{p_1} \rightarrow \mathcal{F}_{p_2}. \quad (2.12)$$

The operators  $h_{p_2 p_1}^\alpha(w)$  are called chiral vertex operators.

We would like to study products of these operators such as  $h_n(w_n) \dots h_1(w_1)$ ,  $h_k(w_k) \equiv h_{p_k p_{k-1}}^{\alpha_k}(w_k)$  and their matrix elements. In order to convince ourselves that these objects are well-defined let us observe that by projecting  $h_{s,\epsilon}^\alpha(\sigma)$ , cf. (2.10), onto  $\mathcal{F}_p$  one gets an operator  $h_{p_2 p_1}^{\alpha,\epsilon}(\sigma)$  that is the product of a trace-class operator with an operator that is bounded for  $s \in i\mathbb{R}$ . With the help of (2.10) one may then represent the matrix elements of  $h_n(w_n) \dots h_1(w_1)$ , in the following form:

$$\begin{aligned} e^{(\tau_n + \epsilon)\Delta_{p_n} - \tau_1 \Delta_{p_0}} \langle v_{p_n}, h_{n,\epsilon}(\sigma_n) e^{-(\tau_n - \tau_{n-1} - \epsilon)L_0} \dots \\ \dots h_{2,\epsilon}(\sigma_2) e^{-(\tau_2 - \tau_1 - \epsilon)L_0} h_{1,\epsilon}(\sigma_1) v_{p_0} \rangle, \end{aligned}$$

where we have set  $h_{k,\epsilon}(\sigma_k) \equiv h_{p_k p_{k-1}}^{\alpha_k, \epsilon}(\sigma_k)$ ,  $k = 1, \dots, n$ . These matrix elements are analytic w.r.t. the variables  $\tau_k - \tau_{k-1} \in \mathbb{H}_+$ , where  $\mathbb{H}_+ = \{z \in \mathbb{C}; \Re(z) > 0\}$ . Indeed, the positivity of  $L_0$  implies analyticity of the vector-valued function  $e^{-\tau L_0} \xi$ ,  $\xi \in \mathcal{H}_L^F$  on the right  $\tau$ -half-plane.

The convergence of the power series expansions in  $q_k = e^{\tau_k - \tau_{k-1}}$  finally allows one to establish the meromorphic continuation of the matrix elements to more general values of  $\alpha_k$ ,  $k = 1, \dots, n$  and  $p_l$ ,  $l = 0, 1, \dots, n$ , cf. <sup>4</sup>, Section 7.

## 2.5 Conformal covariance

The behavior of the normal ordered exponentials  $E^\alpha$  under conformal transformations can then be summarized by

$$U_h E^\alpha(\sigma) U_h^\dagger = (h'(\sigma))^{\Delta_\alpha} E^\alpha(h(\sigma)), \quad \Delta_\alpha = \alpha(Q - \alpha). \quad (2.13)$$

Let us now assume that the parameter  $b$  that enters the definition of the screening charge  $Q$  is related to the parameter  $Q$  via

$$Q = b + b^{-1}. \quad (2.14)$$

In this case (2.13) implies the following simple transformation law for  $Q(\sigma)$ :

$$U_h Q(\sigma) U_h^\dagger = Q(h(\sigma)). \quad (2.15)$$

Let us note that this implies the true invariance of  $Q(\sigma)$  under those elements of  $\text{Diff}(S_1)$  that satisfy  $h(\sigma) = \sigma$ . Equations (2.13) and (2.15) together finally imply that

$$U_h h_s^\alpha(\sigma) U_h^\dagger = (h'(\sigma))^{\Delta_\alpha} h_s^\alpha(h(\sigma)), \quad \Delta_\alpha = \alpha(Q - \alpha), \quad (2.16)$$

which is the standard transformation law for a covariant chiral field.

It is often convenient to trade the Euclidean cylinder parameterized by the coordinate  $w$  for the complex plane by making the conformal transformation  $z = e^w$ . The corresponding fields on the complex plane will simply be denoted by  $h_s^\alpha(z)$ . These fields are related to the  $h_s^\alpha(w)$  via

$$h_s^\alpha(w) = z^{\Delta_\alpha} h_s^\alpha(z). \quad (2.17)$$

The composition of two fields  $h_{s_2}^{\alpha_2}(z_2)h_{s_1}^{\alpha_1}(z_1)$  will as usual be well-defined if  $|z_2| > |z_1|$ .

Having discussed their mutual relations we shall in the following freely switch between distributional covariant fields  $h^\alpha(\sigma)$  on the (universal cover of the) unit circle  $S_1$ , the corresponding operators  $h^\alpha(w)$  on the Euclidean cylinder and their counterparts  $h^\alpha(z)$  on the punctured complex plane which are obtained via (2.17).

## 2.6 More advanced properties of $h_{p_2 p_1}^\alpha(w)$

Let us now summarize the properties of the fields  $h_{p_2 p_1}^\alpha(w)$  that will be crucial for their role as building blocks in the construction of Liouville vertex operators. Derivations will be given in the following sections.

MATRIX ELEMENTS —

$$\begin{aligned} e^{w(\Delta_{p_1} - \Delta_{p_2})} \langle v_{p_2}, h_{p_2 p_1}^\alpha(w) v_{p_1} \rangle &= \\ &= (\Gamma(b^2) b^{1-b^2})^s \frac{|\Gamma_b(Q - \alpha + i(p_2 + p_1))\Gamma_b(Q - \alpha + i(p_2 - p_1))|^2}{\Gamma_b(Q)\Gamma_b(Q - 2ip_2)\Gamma_b(Q - 2\alpha)\Gamma_b(Q + 2ip_1)} \end{aligned} \quad (2.18)$$

We have assumed  $\alpha \in \mathbb{R}$  in order to write the expression compactly. The definition and some relevant properties of the special function  $\Gamma_b(x)$ <sup>13</sup> are collected in the Appendix.

EXCHANGE RELATIONS —

$$h_{p_2 p_s}^{\alpha_2}(\sigma_2) h_{p_s p_1}^{\alpha_1}(\sigma_1) = \int_0^\infty dp_u B_E^\epsilon(p_s | p_u) h_{p_2 p_u}^{\alpha_1}(\sigma_1) h_{p_u p_1}^{\alpha_2}(\sigma_2), \quad (2.19)$$

where  $\epsilon \equiv \text{sgn}(\sigma_2 - \sigma_1)$ . In the notation for the kernel  $B_{\mathbb{E}}^\epsilon$  we have denoted the tuple of external parameters by  $\mathbb{E} = (\alpha_2, \alpha_1, p_2, p_1)$ . The explicit expression for  $B_{\mathbb{E}}^\epsilon(p_s | p_u)$  is of the following form:

$$\begin{aligned} B_{\mathbb{E}}^\epsilon(p_s | p_u) &= e^{\pi i \epsilon (\Delta_s + \Delta_u - \Delta_2 - \Delta_1)} B_{\mathbb{E}}(p_s | p_u), \\ B_{\mathbb{E}}(p_s | p_u) &= m(p_u) \int_{\mathbb{R}+i0} dt \prod_{i=1}^4 \frac{s_b(t + r_i)}{s_b(t + s_i)}, \end{aligned} \quad (2.20)$$

where  $m(p) = 4 \sinh 2\pi b p \sinh 2\pi b^{-1} p$ ,  $\Delta_{\natural} = p_{\natural}^2 + \frac{Q^2}{4}$ ,  $b \in \{1, 2, s, u\}$ , and the coefficients  $r_i$  and  $s_i$ ,  $i = 1, \dots, 4$  are defined by

$$\begin{aligned} r_1 &= i(\alpha_2 - \alpha_1) - p_2, & s_1 &= +p_u - i(Q - \alpha_2), \\ r_2 &= i(\alpha_2 - \alpha_1) + p_2, & s_2 &= -p_u - i(Q - \alpha_2), \\ r_3 &= +p_1, & s_3 &= +p_s - i\alpha_1, \\ r_4 &= -p_1, & s_4 &= -p_s - i\alpha_1. \end{aligned} \quad (2.21)$$

The special function  $s_b(x)$  can be defined by  $s_b(x) = \Gamma_b(\frac{Q}{2} + ix) / \Gamma_b(\frac{Q}{2} - ix)$ , see the Appendix for further information. For our purposes the most important property of the kernel  $B_{\mathbb{E}}^\epsilon(p_s | p_u)$  will be the unitarity relation

$$\int_0^\infty dp_s m(p_s) B_{\mathbb{E}}^\epsilon(p_s | p_u) (B_{\mathbb{E}}^\epsilon(p_s | p'_u))^* = m(p_u) \delta(p_u - p'_u). \quad (2.22)$$

The relations (2.19) and (2.22) will be the main ingredients that one needs to prove the locality of Liouville vertex operators.

### 3 Matrix elements

We shall present here the derivation of formula (2.18) for the matrix elements of the fields  $h_s^\alpha$ . Let us define the function  $M(\alpha, s | p)$  as

$$M(\alpha, s | p) \equiv \langle\langle p' | h_s^\alpha(1) | p \rangle\rangle, \quad p' \equiv p - i(\alpha + bs), \quad (3.23)$$

where we have used the notation

$$\langle\langle p_2 | \mathcal{O}(w) | p_1 \rangle\rangle = (v_{p_2}, \Pi_{p_2} \mathcal{O}(w) v_{p_1})_{\mathcal{F}}. \quad (3.24)$$

Let us emphasize that we are talking about a perfectly well-defined object, our task is just to make it more explicit. What we are going to assume, but not prove here, will be that  $M(\alpha, s | p)$  has a sufficiently large domain of analyticity in its dependence w.r.t.  $\alpha$  and  $s$ . Our strategy will be to derive a system of functional equations that will then characterize the function  $M(\alpha, s | p)$  completely.

#### 3.1 Auxiliary fields

As a useful technical device we shall employ the following set of fields:

$$\begin{aligned} h_0(z) &\equiv h_0^{-\frac{b}{2}}(z) = E^{-\frac{b}{2}}(z), \\ h_1(z) &\equiv h_1^{-\frac{b}{2}}(z) = e^{\frac{\pi i}{2} b^2} e^{-\pi b p} E^{-\frac{b}{2}}(z) Q(z) e^{-\pi b p}. \end{aligned} \quad (3.25)$$

We will need the following two properties of the fields  $h_r(z)$ ,  $r = 0, 1$ :

MATRIX ELEMENTS:

$$\begin{aligned} M_0 &\equiv \langle\langle p + i\frac{b}{2} | h_0(1) | p \rangle\rangle = 1 \\ M_1 &\equiv \langle\langle p - i\frac{b}{2} | h_1(1) | p \rangle\rangle = 2\pi \frac{\Gamma(1+b^2)}{\Gamma(1+b^2+2ibp)\Gamma(1-2ibp)}. \end{aligned} \quad (3.26)$$

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$$\partial^2 h_r = -b^2 : \mathbb{T} h_r : , \quad r = 0, 1, \quad (3.27)$$

where the normal ordering of the expression on the right hand side is defined as follows:

$$: \mathbb{T} h_r : = \sum_{n \leq -2} z^{-n-2} L_n h_r + \sum_{n \geq -1} h_r L_n z^{-n-2}. \quad (3.28)$$

In the rest of this subsection let us verify equation (3.26). The proof of (3.26) is trivial for  $h_0$ . In the case of  $h_1$  we shall use the standard normal ordering formula

$$\begin{aligned} E_{>}^{\alpha_2}(\sigma_2) E_{<}^{\alpha_1}(\sigma_1) &= \\ &= e^{i\alpha_2\alpha_1(\sigma_1-\sigma_2)} (1 - e^{i(\sigma_1-\sigma_2)})^{-2\alpha_2\alpha_1} E_{<}^{\alpha_1}(\sigma_1) E_{>}^{\alpha_2}(\sigma_2) \\ &= e^{-\pi i\alpha_2\alpha_1 \operatorname{sgn}(\sigma_2-\sigma_1)} |1 - e^{i(\sigma_1-\sigma_2)}|^{-2\alpha_2\alpha_1} E_{<}^{\alpha_1}(\sigma_1) E_{>}^{\alpha_2}(\sigma_2). \end{aligned} \quad (3.29)$$

The complex power function in the second line of (3.29) is defined as  $z^s = e^{s \ln(z)}$ , where  $\ln(z)$  denotes the principal value of the logarithm. Using (3.29) reduces the calculation of  $\langle\langle p - i\frac{b}{2} | h_1(0) | p \rangle\rangle$  to the integral

$$\langle\langle p - i\frac{b}{2} | h_1(0) | p \rangle\rangle = e^{\pi i b^2} e^{-2\pi b p} \int_0^{2\pi} d\sigma' e^{2b\sigma'(p-i\frac{b}{2})} (1 - e^{i\sigma'})^{b^2}. \quad (3.30)$$

By means of a simple contour deformation one easily reduces the evaluation of (3.30) to the integral that defines the Beta-function.

### 3.2 Strategy

Our strategy will be to consider the conformal blocks

$$\Psi_r(z_2, z_1) = \langle\langle p_2 | h_r(z_2) h_s^\alpha(z_1) | p_1 \rangle\rangle, \quad (3.31)$$

where  $p_2 = p_1 - i(\alpha + bs + \delta_r)$  with  $\delta_0 = -\frac{b}{2}$ ,  $\delta_1 = \frac{b}{2}$ . We will see that the leading singular behavior of  $\Psi_r(z_2, z_1)$  for  $z_2 \rightarrow z_1$  is of the form

$$\Psi_r(z_2, z_1) \underset{z_2 \rightarrow z_1}{\simeq} (z_2 - z_1)^{b\alpha} G_r(\alpha, s, p_1). \quad (3.32)$$

The coefficient  $G_r(\alpha, s, p)$  can be expressed in terms of  $M(\alpha, s | p)$  in two different ways:

On the one hand one may calculate the leading operator product singularity of  $h_r(z_2) h_s^\alpha(z_1)$  directly from the definition of these operators, with the result

$$h_r(z_2) h_s^\alpha(z_1) \underset{z_2 \rightarrow z_1}{\simeq} (z_2 - z_1)^{b\alpha} h_{s+r}^{\alpha-\frac{b}{2}}(z_1). \quad (3.33)$$



Indeed, to verify (3.33) it suffices to observe that

$$\begin{aligned} Q(\sigma_2)h_s^\alpha(\sigma_1) &= Q(\sigma_1)h_s^\alpha(\sigma_1) + (Q(\sigma_2) - Q(\sigma_1))h^\alpha(\sigma_1) \\ &\underset{\sigma_2 \rightarrow \sigma_1}{\sim} h_{s+1}^\alpha(\sigma_1) \end{aligned}$$

up to terms that are sub-leading for  $\sigma_2 \rightarrow \sigma_1$ . Inserting (3.33) into the definition (3.31) of  $\Psi_r(z_2, z_1)$  yields

$$G_r(\alpha, s, p_1) = M\left(\alpha - \frac{b}{2}, s + r \mid p_1\right). \quad (3.34)$$

On the other hand one may use the differential equation (3.27) to calculate  $\Psi_r(z_2, z_1)$  explicitly in terms of hypergeometric functions. Standard formulae for the asymptotic behavior of solutions to the hypergeometric differential equation will then give a second expression for  $G_r(\alpha, s, p)$ . Comparison of the two expressions yields the desired functional equations.

### 3.3 Calculation of the conformal blocks

We will find it convenient to use the notation

$$\begin{aligned} \alpha_2 &\equiv -\frac{b}{2}, & \beta_1 &\equiv \frac{Q}{2} + ip_1, \\ \alpha_1 &\equiv \alpha, & \beta_2 &\equiv \frac{Q}{2} + ip_2. \end{aligned} \quad (3.35)$$

Conformal invariance restricts  $\Psi_r(z_2, z_1)$  to have the form

$$\Psi_r(z_2, z_1) = z_1^\kappa \Psi_r(z), \quad z \equiv \frac{z_2}{z_1}, \quad \kappa \equiv \Delta_{\beta_2} - \Delta_{\beta_1} - \Delta_{\alpha_1} - \Delta_{\alpha_2}. \quad (3.36)$$

The differential equation (3.27) then implies that  $\Psi_r(z)$  must satisfy the differential equation

$$\left( \frac{1}{b^2} \frac{\partial^2}{\partial z^2} + \frac{2z-1}{z(1-z)} \frac{\partial}{\partial z} + \frac{\Delta_{\alpha_1}}{(1-z)^2} + \frac{\Delta_{\beta_1}}{z^2} - \frac{\kappa}{z(1-z)} \right) \Psi_r(z) = 0. \quad (3.37)$$

By making the ansatz

$$\Psi_r(z) = z^{b\beta_1} (1-z)^{b\alpha_1} F_r(z) \quad (3.38)$$

one finds the hypergeometric differential equation for  $F_r(z)$ . In order to determine the relevant solutions let us observe that

$$h_s^\alpha(z_1) v_p \underset{z_1 \rightarrow 0}{\simeq} z_1^{\Delta_q - \Delta_\alpha - \Delta_p} M(\alpha, s \mid p) v_q, \quad q = p - i(\alpha + bs). \quad (3.39)$$

Indeed, the limit  $z \rightarrow 0$  corresponds to taking  $\tau \rightarrow -\infty$  in (2.10) which obviously suppresses the contributions of states with higher  $L_0$ -eigenvalues. In the present case we need to take  $z \rightarrow \infty$  in order to apply (3.39). It follows that the asymptotic behavior of  $\Psi_r(z)$  for  $z \rightarrow \infty$  must be of the following form:

$$\begin{aligned} \Psi_r(z) &\underset{z \rightarrow \infty}{\simeq} M_r z^{b(\beta_2 - \alpha_2)} z^{rb(Q - 2\beta_2)}, \\ N_r &= M_r M(\alpha_1, s \mid p_1). \end{aligned} \quad (3.40)$$

There exist unique solutions of the hypergeometric differential equation that have the required asymptotic behavior (3.40), namely

$$F_r(z) = N_r z^{-u_r} F(u_r, v_r; w_r; \frac{1}{z}), \quad (3.41)$$

where

$$\begin{aligned} u_0 &= b(\alpha_1 + \alpha_2 + \beta_1 - \beta_2), & u_1 &= b(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - Q), \\ v_0 &= b(\alpha_1 + \alpha_2 + Q - \beta_1 - \beta_2), & v_1 &= b(\alpha_1 + \alpha_2 + \beta_2 - \beta_1), \\ w_0 &= 1 - b(2\beta_2 - Q), & w_1 &= 1 + b(2\beta_2 - Q). \end{aligned} \quad (3.42)$$

By combining equations (3.36),(3.38) and (3.41) one gets an explicit expression for  $\Psi_r(z_2, z_1)$  in terms of the matrix elements  $M_r$  and  $M(\alpha, s | p)$ .

### 3.4 Functional equations

The asymptotic behavior of  $\Psi_r(z_2, z_1)$  can now be calculated with the help of the well-known formula

$$\begin{aligned} F(u, v; w; z) &= \frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)} F(u, v; 1+u+v-w; 1-z) \\ &+ (1-z)^{w-u-v} \frac{\Gamma(w)\Gamma(u+v-w)}{\Gamma(u)\Gamma(v)} F(w-u, w-v; 1+w-u-v; 1-z). \end{aligned}$$

The result is of the form (3.33), with

$$G_r(\alpha, s, p) = M_r M(\alpha, s | p) \frac{\Gamma(w_r)\Gamma(w_r - u_r - v_r)}{\Gamma(w_r - u_r)\Gamma(w_r - v_r)}. \quad (3.43)$$

By comparing (3.34) and (3.43) and plugging in (3.26) one deduces the two functional equations

$$M(\alpha - \frac{b}{2}, \rho + rb | \beta) = \chi_b^r(\alpha, \rho | \beta) M(\alpha, \rho | \beta) \quad (3.44)$$

where we have traded the parameter  $s$  for  $\rho = bs$  and the parameter  $p$  for  $\beta = \frac{Q}{2} + ip$ . The coefficients  $\chi_b^r(\alpha, \rho | p)$ ,  $r = 0, 1$  are given by the following expressions:

$$\begin{aligned} \chi_b^0(\alpha, \rho | \beta) &= \frac{\Gamma(2 - b(2\alpha + 2\beta + 2\rho - 2b))\Gamma(1 - b(2\alpha - b))}{\Gamma(2 - b(2\alpha + 2\beta + \rho - 2b))\Gamma(1 - b(2\alpha + \rho - b))}, \\ \chi_b^1(\alpha, \rho | \beta) &= \frac{2\pi\Gamma(1 + b^2)\Gamma(1 - b(2\alpha - b))}{\Gamma(b(\rho + Q))\Gamma(b(2\beta + \rho))\Gamma(1 - b(2\beta + 2\alpha + 2\rho - Q))}. \end{aligned} \quad (3.45)$$

Let us furthermore observe that the whole argument leading to (3.44) can be repeated with the alternative set  $\tilde{h}_r(z)$ ,  $r = 0, 1$  of auxiliary fields that is obtained by replacing  $b \rightarrow b^{-1}$  in the definition (3.25). This becomes possible due to the fact that the fields  $\tilde{h}_r(z)$ ,  $r = 0, 1$  satisfy

$$\partial^2 \tilde{h}_r = -b^{-2} : \tilde{h}_i : , \quad r = 0, 1. \quad (3.46)$$

As the result one finds a second set of functional equations:

$$M(\alpha - \frac{1}{2b}, \rho + \frac{r}{b} | p) = \chi_{1/b}^{(r)}(\alpha, \rho | p) M(\alpha, \rho | p). \quad (3.47)$$

The combined system of functional equations (3.44)(3.47) severely constrains the dependence of  $M(\alpha, \rho | p)$  w.r.t. the variables  $\alpha$  and  $\rho$ . However, it does not constrain the dependence w.r.t.  $p$ . This freedom is eliminated by noting that one must have

$$M(\alpha, \rho | p) \Big|_{\rho=0} \equiv 1 \quad (3.48)$$

for all  $\alpha, p$ . We therefore have a system of equations that can be expected to characterize  $M(\alpha, \rho | p)$  completely. It certainly does if  $b$  is irrational and if  $M(\alpha, \rho | p)$  is known to have a sufficiently large domain of analyticity w.r.t.  $\alpha$  and  $\rho$ , see e.g. <sup>5</sup>, Appendix C.

To conclude let us observe that by using (A.104)(A.106) it is straightforward to check that our expression (2.18) indeed solves the functional equations (3.44)(3.47). The condition (3.48) is trivially fulfilled by (2.18).

## 4 Braiding

### 4.1 Operator ordering, $I$

Let us now study the bilocal field  $h_{s_2}^{\alpha_2}(\sigma_2)h_{s_1}^{\alpha_1}(\sigma_1)$ . In the following we shall concentrate on the case  $\sigma_2 > \sigma_1$ , the case  $\sigma_1 < \sigma_2$  being completely analogous. In order to find a useful alternative representation for this field we shall reorder the ingredients that enter the definition of  $h_{s_2}^{\alpha_2}(\sigma_2)h_{s_1}^{\alpha_1}(\sigma_1)$  conveniently. To this aim let us introduce the intervals  $I = [\sigma_1, \sigma_2]$ ,  $I^c = [\sigma_2, \sigma_1 + 2\pi]$  and  $I' = [\sigma_1 + 2\pi, \sigma_2 + 2\pi]$  together with the corresponding operators

$$Q_I \equiv \int_I d\sigma E^b(\sigma), \quad Q_I^c \equiv \int_{I^c} d\sigma E^b(\sigma), \quad Q_{I'} \equiv \int_{I'} d\sigma E^b(\sigma). \quad (4.49)$$

By using the formulae

$$\begin{aligned} Q_I^c E_{<}^{\alpha_1}(\sigma_1) &= e^{-3\pi i b \alpha_1} E_{<}^{\alpha_1}(\sigma_1) Q_I^c, & E_{>}^{\alpha_2}(\sigma_2) Q_I^c &= e^{\pi i b \alpha_2} Q_I^c E_{>}^{\alpha_2}(\sigma_2), \\ Q_I^c E_{<}^{\alpha_1}(\sigma_1) &= e^{-\pi i b \alpha_1} E_{<}^{\alpha_1}(\sigma_1) Q_I^c, & E_{>}^{\alpha_2}(\sigma_2) Q_I &= e^{-\pi i b \alpha_2} Q_I E_{>}^{\alpha_2}(\sigma_2), \end{aligned}$$

which follow easily from equation (3.29) one may represent  $h_{s_2}^{\alpha_2}(\sigma_2)h_{s_1}^{\alpha_1}(\sigma_1)$  by

$$\begin{aligned} e^{\pi i \alpha_2 \alpha_1} |1 - e^{i(\sigma_1 - \sigma_2)}|^{2\alpha_2 \alpha_1} h_{s_2}^{\alpha_2}(\sigma_2) h_{s_1}^{\alpha_1}(\sigma_1) &= \\ = E_{<}^{\alpha_2}(\sigma_2) E_{<}^{\alpha_1}(\sigma_1) \left[ (Q_I^c + Q_{I'}^c e^{-2\pi i b \alpha_1})^{s_2} (Q_I^c + Q_I e^{-2\pi i b \alpha_2})^{s_1} \right] E_{>}^{\alpha_2}(\sigma_2) E_{>}^{\alpha_1}(\sigma_1). \end{aligned} \quad (4.50)$$

Let us next note that  $Q_I, Q_I^c$  and  $Q_{I'}$  satisfy algebraic relations of Weyl-type:

$$\begin{aligned} Q_I^c Q_I &= q^{-2} Q_I Q_I^c, & Q_I Q_{I'} &= q^4 Q_{I'} Q_I, \\ Q_I^c Q_{I'} &= q^{+2} Q_{I'} Q_I^c, \end{aligned} \quad (4.51)$$

where  $q = e^{\pi i b^2}$ . In order to continue we will have to learn how to deal with complex powers of sums of Weyl-type operators.

### 4.2 Interlude: Weyl-type operators

What we have to deal with are expressions like  $(U + V)^s$ , where  $U$  and  $V$  are positive self-adjoint operators that satisfy  $UV = q^2 VU$ . Thanks to positivity we can take logarithms of  $U$  and  $V$  to express them as

$$U = e^{2bx}, \quad V = e^{bx} e^{2\pi b p} e^{bx}, \quad \text{where } [x, p] = \frac{i}{2}.$$

We would like to ‘‘normal-order’’ the expression  $(U + V)^s$ , i.e. to write it in the form  $e^{bsx} f_s(p) e^{bsx}$ . The key ingredient for doing this will be the special function  $e_b(z)$

which can be defined in terms of the function  $s_b(x)$  (cf. the Appendix) by  $e_b(x) = e^{\frac{\pi i}{2}x^2} e^{-\frac{\pi i}{24}(2-Q^2)} s_b(x)$ . The following properties of  $e_b(x)$  will be crucial for our purposes.

$$(i) \text{ Functional equation: } e_b(x - i\frac{b}{2}) = (1 + e^{2\pi bx})e_b(x + i\frac{b}{2}). \quad (4.52)$$

$$(ii) \text{ Unitarity: } |e_b(x)| = 1 \quad \text{for } x \in \mathbb{R}. \quad (4.53)$$

(iii) Analyticity:  $s_b(x)$  is meromorphic,

$$\text{poles: } x = c_b + i(nb + mb^{-1}), n, m \in \mathbb{Z}^{\geq 0}. \quad (4.54)$$

$$\text{zeros: } x = -c_b - i(nb + mb^{-1}), n, m \in \mathbb{Z}^{\geq 0}.$$

With the help of these properties we may now calculate

$$\begin{aligned} (U + V)^s &= (e^{bx}(1 + e^{2\pi bp})e^{bx})^s \\ &\stackrel{(4.52)}{=} \left( e^{bx} \frac{e_b(p - i\frac{b}{2})}{e_b(p + i\frac{b}{2})} e^{bx} \right)^s = (e_b(p)e^{2bx}e_b^{-1}(p))^s \end{aligned} \quad (4.55)$$

$$\stackrel{(4.53)}{=} e_b(p)e^{2sbx}e_b^{-1}(p) = e^{sbx} \frac{e_b(p - is\frac{b}{2})}{e_b(p + is\frac{b}{2})} e^{sbx}.$$

On the right hand side we read off the desired normal ordering formula.

### 4.3 Operator ordering, II

Returning to the problem of the investigation of the conformal blocks let us now define operators  $x$  and  $s$  such that

$$\begin{aligned} Q_1^c &= e^{2bx}, & Q_1 &= e^{bx}e^{2\pi b(s-p)}e^{bx}, & [x, p] &= \frac{i}{2}, & [x, s] &= 0, \\ & & Q_1' &= e^{bx}e^{2\pi b(s+p)}e^{bx}. & & & [p, s] &= 0. \end{aligned} \quad (4.56)$$

It should be remarked that the operators  $x$  and  $s$  actually depend on  $\sigma_1$  and  $\sigma_2$ . We have suppressed this dependence for notational convenience. Generalizing the calculation in (4.55) slightly we find that

$$\begin{aligned} (Q_1^c + e^{-2\pi ib\alpha_1} Q_1')^{s_2} &= e^{s_2 bx} \frac{e_b(s - i\alpha_1 + p - is_2 \frac{b}{2})}{e_b(s - i\alpha_1 + p + is_2 \frac{b}{2})} e^{s_2 bx}, \\ (Q_1^c + e^{-2\pi ib\alpha_2} Q_1')^{s_1} &= e^{s_1 bx} \frac{e_b(s - i\alpha_2 - p - is_1 \frac{b}{2})}{e_b(s - i\alpha_2 - p + is_1 \frac{b}{2})} e^{s_1 bx}. \end{aligned} \quad (4.57)$$

Introducing the notation

$$\begin{aligned} E_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1) &\equiv e^{b(s_2 + s_1)x} E_{>}^{\alpha_2}(\sigma_2) E_{>}^{\alpha_1}(\sigma_1), \\ (E_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1))^\dagger &\equiv E_{<}^{\alpha_1}(\sigma_1) E_{<}^{\alpha_2}(\sigma_2) e^{b(s_2 + s_1)x} \end{aligned} \quad (4.58)$$

now leads to a representation for  $h_{s_2}^{\alpha_2}(\sigma_2)h_{s_1}^{\alpha_1}(\sigma_1)$  of the following form:

$$\begin{aligned} e^{\pi i \alpha_2 \alpha_1} |1 - e^{i(\sigma_1 - \sigma_2)}|^{2\alpha_2 \alpha_1} h_{s_2}^{\alpha_2}(\sigma_2) h_{s_1}^{\alpha_1}(\sigma_1) &= \\ &= (E_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1))^\dagger O_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1) E_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1), \end{aligned} \quad (4.59)$$

where the operator  $O_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1)$  is defined by

$$O_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1) \equiv \frac{e_b(s + p - i\alpha_1 - i\frac{b}{2}(s_2 + s_1)) e_b(s - p - i\alpha_2 - i\frac{b}{2}(s_1 + s_2))}{e_b(s + p - i\alpha_1 + i\frac{b}{2}(s_2 - s_1)) e_b(s - p - i\alpha_2 + i\frac{b}{2}(s_1 - s_2))} \quad (4.60)$$

We ultimately want to establish a relation between the products  $h_{s_2}^{\alpha_2}(\sigma_2)h_{s_1}^{\alpha_1}(\sigma_1)$  and  $h_{t_1}^{\alpha_1}(\sigma_1)h_{t_2}^{\alpha_2}(\sigma_2)$ . Applying the same ordering procedure to the latter yields

$$\begin{aligned} e^{\pi i \alpha_2 \alpha_1} |1 - e^{i(\sigma_1 - \sigma_2)}|^{2\alpha_2 \alpha_1} h_{t_1}^{\alpha_1}(\sigma_1) h_{t_2}^{\alpha_2}(\sigma_2) &= \\ &= (E_{t_2 t_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1))^\dagger P_{t_2 t_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1) E_{t_2 t_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1) \end{aligned} \quad (4.61)$$

where now

$$P_{t_2 t_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1) \equiv \frac{e_b(s - p + i\alpha_2 - i\frac{b}{2}(t_1 - t_2)) e_b(s + p + i\alpha_1 - i\frac{b}{2}(t_2 - t_1))}{e_b(s - p + i\alpha_2 + i\frac{b}{2}(t_2 + t_1)) e_b(s + p + i\alpha_1 + i\frac{b}{2}(t_2 + t_1))}. \quad (4.62)$$

We have put a lot of things into black boxes. The definition of the operators  $s$ , for example, requires taking logarithms of the screening charges. It may therefore seem far from clear that we have achieved anything useful so far. However, let us note that the operators  $E_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1)$  are in fact *symmetric* w.r.t. exchange of the labels 1 and 2. The sought-for exchange relation (2.19) is therefore equivalent to an identity relating the operators  $O_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1)$  and  $P_{t_2 t_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1)$ . Let us furthermore note that the two operators  $s$  and  $p$  commute with each other and may therefore be simultaneously diagonalized. This implies that (2.19) will be equivalent to an identity between ordinary meromorphic functions!

#### 4.4 Projecting onto $\mathcal{F}_{p_2}$

Let us now consider the projection  $\Pi_{p_2} h_{s_2}^{\alpha_2}(\sigma_2)h_{s_1}^{\alpha_1}(\sigma_1)$ . The operator  $p$  can then be replaced by its eigenvalue  $p_2 + \frac{i}{2}(\alpha_2 + \alpha_1 + bs_2 + bs_1)$ . For future convenience let us introduce  $s \equiv r + \frac{i}{2}(\alpha_2 - \alpha_1)$ . It will furthermore be convenient to trade the parameters  $(s_2, s_1, t_2, t_1, p)$  with  $t_1 + t_2 = s_2 + s_1$  for another set of parameters  $(p_2, p_1, p_s, p_u)$  that is defined by

$$\begin{aligned} p_2 &= p, & p_1 &= p_2 + i(\alpha_1 + \alpha_2 + bs_1 + bs_2), \\ p_s &= p_1 - i(\alpha_1 + bs_1), & p_u &= p_1 - i(\alpha_2 + bt_2). \end{aligned} \quad (4.63)$$

As a short notation for the external parameters let us finally introduce  $E = (\alpha_2, \alpha_1, p_2, p_1)$ .  $\Pi_{p_2} h_{s_2}^{\alpha_2}(\sigma_2)h_{s_1}^{\alpha_1}(\sigma_1)$  and  $\Pi_{p_2} h_{t_1}^{\alpha_1}(\sigma_1)h_{t_2}^{\alpha_2}(\sigma_2)$  can then be represented in the following form

$$\begin{aligned} e^{\pi i \alpha_2 \alpha_1} |1 - e^{i(\sigma_1 - \sigma_2)}|^{2\alpha_2 \alpha_1} \Pi_{p_2} h_{s_2}^{\alpha_2}(\sigma_2) h_{s_1}^{\alpha_1}(\sigma_1) &= \\ &= (E_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1))^\dagger O_E(p_s | r) E_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1), \\ e^{\pi i \alpha_2 \alpha_1} |1 - e^{i(\sigma_1 - \sigma_2)}|^{2\alpha_2 \alpha_1} \Pi_{p_2} h_{t_1}^{\alpha_1}(\sigma_1) h_{t_2}^{\alpha_2}(\sigma_2) &= \\ &= (E_{t_2 t_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1))^\dagger P_E(p_u | r) E_{t_2 t_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1), \end{aligned} \quad (4.64)$$

where the operators  $O_E(p_s | r)$  and  $P_E(p_u | r)$  are obtained from  $O_{s_2 s_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1)$  and  $P_{t_2 t_1}^{\alpha_2 \alpha_1}(\sigma_2, \sigma_1)$  respectively by making the substitutions described above.

Let us finally replace the special functions  $e_b(x)$  by their close relatives  $s_b(x)$

$$s_b(x) = e^{-\frac{\pi i}{2}x^2} e^{\frac{\pi i}{24}(2-Q^2)} e_b(x), \quad (4.65)$$

which satisfy an inversion relation of the simple form

$$s_b(x)s_b(-x) = 1. \quad (4.66)$$

We then find that  $O_E(p_s | r)$  and  $P_E(p_u | r)$  are given by expressions of the following form

$$\begin{aligned} O_E(p_s | r) &= C_E^s(p_s) e^{\pi b(s_1+s_2)r} F_E^s(p_s | r), \\ P_E(p_u | r) &= C_E^u(p_u) e^{\pi b(s_1+s_2)r} F_E^u(p_u | r), \end{aligned} \quad (4.67)$$

where  $F_E^b(p_s | r)$ ,  $b \in \{s, u\}$  are defined respectively by

$$\begin{aligned} F_E^s(p_s | r) &= \frac{s_b(r + p_2 + i(\alpha_2 - \alpha_1))s_b(r - p_1)}{s_b(r + p_s - i\alpha_1)s_b(r - p_s - i\alpha_1)} \\ F_E^u(p_u | -r) &= \frac{s_b(r + p_2 + i(\alpha_1 - \alpha_2))s_b(r - p_1)}{s_b(r + p_u - i\alpha_2)s_b(r - p_u - i\alpha_2)}. \end{aligned} \quad (4.68)$$

We have used the inversion relation (4.66). Otherwise we will only need to know the ratio  $C_E^s(p_s)/C_E^u(p_u)$  which is found to be of the form  $C_E(s)/D_E(t) = e^{\pi i(\Delta_s + \Delta_u - \Delta_1 - \Delta_2)}$  where  $\Delta_{\natural} \equiv p_{\natural}^2 + \frac{1}{4}Q^2$ ,  $\natural \in \{1, 2, s, u\}$ .

#### 4.5 The main identity

Let us momentarily assume that  $\alpha_k \in \frac{Q}{2} + i\mathbb{R}$ ,  $k = 1, 2$ . The functions  $F_E^b(p_s | r)$  have singularities as function of  $r \in \mathbb{R}$ . However, the  $F_E^b(p_s | r + i0)$  represent well-defined distributions that will be denoted as  $\langle d_p^b |$  for  $b = s, u$ . The crucial observation that finally leads us to the desired braid relations (2.19) is now the following:

**Theorem 1.** <sup>7</sup> *The following two sets of distributions*

$$\mathfrak{F}_E^b \equiv \{ | d_p^b \rangle ; p_s \in \mathbb{R}^+ \}, \quad b \in \{s, u\}$$

*form bases for  $L^2(\mathbb{R})$  in the sense of generalized functions. We have the relations*

$$\begin{aligned} \langle d_p^b | d_q^b \rangle &= m^{-1}(p) \delta(p - q) && \text{(Orthogonality),} \\ \int_{\mathbb{R}^+} dp m(p) | d_p^b \rangle \langle d_p^b | &= \text{id} && \text{(Completeness).} \end{aligned} \quad (4.69)$$

The proof of this result can be simplified considerably by noting that the functions in the numerators of (4.68) are pure phases due to the property  $|s_b(x)| = 1$  for  $x \in \mathbb{R}$  of the function  $s_b$ . Multiplying a function  $\psi(r) \in L^2(\mathbb{R})$  by these numerators therefore defines a unitary operator on  $L^2(\mathbb{R})$ . Moreover, the numerators in (4.68) do not depend on the labels  $p_s, p_u$  of the elements of  $\mathfrak{F}_E$ . Let us furthermore note that translation invariance of  $L^2(\mathbb{R})$  allows one to ignore the imaginary part of  $\alpha_i$  in the denominators. The statement of the theorem is therefore equivalent to the corresponding statement for the set of distributions  $\mathfrak{F} \equiv \{ f(p_s | r + i0) ; p_s \in \mathbb{R}^+ \}$ , where

$$f(p_s | r) = s_b^{-1}(r + p_s - c_b) s_b^{-1}(r - p_s - c_b),$$

where  $c_b = iQ/2$ . There is a short and direct proof for the completeness of the set  $\mathfrak{F}$  due to R. Kashaev <sup>12</sup>.

We are now in the position to complete the proof of (2.19). The completeness relation in (4.69) immediately implies that

$$\int_{\mathbb{R}^+} dq |d_q^u\rangle B_E^+(p|q) = |d_p^s\rangle, \quad B_E^+(p|q) \equiv m(q) \langle d_q^u | d_p^s \rangle. \quad (4.70)$$

This is equivalent to the identity

$$\int_{\mathbb{R}^+} dp_u B_E^+(p_s|p_u) P_E(p_u|r) = O_E(p_s|r), \quad (4.71)$$

which implies the desired braid-relation (2.19) thanks to (4.64). The generalization of this relation to more general values of  $\alpha_k, s_k, k = 1, 2$  can then be obtained by analytic continuation. The unitarity relations (2.22) also follow easily from the completeness relations in (4.69).

## 5 Fusion

In order to have a more uniform notation, we will in the following replace the labels  $\alpha$  of the chiral vertex operators  $h_{p_2 p_1}^\alpha(z)$  by  $q = c_b - i\alpha, c_b = i\frac{Q}{2}$ .

### 5.1 Descendants

It is often useful to generalize the chiral vertex operators  $h_{p_2 p_1}^q(z)$  by introducing a family of operators  $h_{p_2 p_1}^q(\xi|z)$  which are labeled by elements  $\xi \in \mathcal{F}_q$ . The operators  $h_{p_2 p_1}^q(\xi|z)$  are defined in terms of  $h_{p_2 p_1}^q(z)$  by means of the requirements

$$\begin{aligned} \text{(i)} \quad & h_{p_2 p_1}^q(L_{-n}\xi|z) = ((n-2)!)^{-1} : (\partial^{n-2} T(z)) h_{p_2 p_1}^q(\xi|z) : \text{ for } n > 1, \\ \text{(ii)} \quad & h_{p_2 p_1}^q(L_{-1}\xi|z) = \partial_z h_{p_2 p_1}^q(\xi|z), \\ \text{(iii)} \quad & h_{p_2 p_1}^q(v_p|z) = h_{p_2 p_1}^q(z), \end{aligned} \quad (5.72)$$

where the normal ordering in (i) is defined the same way as in (3.28). The definition of the descendants goes back to <sup>1</sup>, the present formulation is the one from <sup>4</sup>. For our purposes the main point to notice is that the definition of descendants introduces a second way to compose the chiral vertex operators which may be represented by the notation

$$h_{p_2 p_1}^{p_t} (h_{p_t q_1}^{q_z} (z_2 - z_1) v_{q_1} | z_1). \quad (5.73)$$

This expression is defined by expanding  $h_{p_t q_1}^{q_z} (z_2 - z_1) v_{q_1}$  as

$$h_{p_t q_1}^{q_z} (z_2 - z_1) v_{q_1} = (z_2 - z_1)^{\Delta_{p_t} - \Delta_{q_2} - \Delta_{q_1}} \sum_{n=0}^{\infty} (z_2 - z_1)^n \xi_{p_t}^{(n)}(q_1, q_2), \quad (5.74)$$

where  $\xi_{p_t}^{(n)}(q_1, q_2) \in \mathcal{F}_{p_t}$ . Terms like  $h_{p_2 p_1}^{p_t} (\xi_{p_t}^{(n)}(q_1, q_2) | z_1)$  are defined by (5.72). We will see later that the power series that is obtained by inserting (5.74) into (5.73) has a finite radius of convergence within matrix elements.

## 5.2 Fusion - the result

We shall now study the conformal blocks

$$\begin{aligned} \mathcal{F}_{E,p_s}(z_2, z_1) &\equiv \langle v_{p_2}, \mathbf{h}_{p_2 p_s}^{q_2}(z_2) \mathbf{h}_{p_s p_1}^{q_1}(z_1) v_{p_1} \rangle \\ &=: z_2^{\Delta_{p_2} - \Delta_{p_1} - \Delta_{q_2} - \Delta_{q_1}} \mathcal{F}_{E,p_s}(z), \quad z \equiv \frac{z_1}{z_2}. \end{aligned} \quad (5.75)$$

Our aim will be to describe the behavior of  $\mathcal{F}_{E,p_s}(z)$  near  $z = 1$ . More precisely, our aim is to show that  $\mathcal{F}_{E,p_s}(z)$  can be expanded as

$$\mathcal{F}_{E,p_s}^s(z) = \int_0^\infty dp_t \Phi_E(p_s | p_t) \mathcal{F}_{E,p_t}^t(z), \quad (5.76)$$

where the ‘‘t-channel’’ conformal blocks  $\mathcal{F}_{E,p_t}^t(z)$  are defined by

$$\mathcal{F}_{E,p_t}^t(z) = \langle v_{p_2}, \mathbf{h}_{p_2 p_1}^{p_t}(\mathbf{h}_{p_t q_1}^{q_2}(1-z)v_{q_1} | z) v_{p_1} \rangle, \quad (5.77)$$

which implies existence of an expansion of the following form

$$\mathcal{F}_{E,p_t}^t(z) = (1-z)^{\Delta_{p_t} - \Delta_{q_2} - \Delta_{q_1}} \sum_{n=0}^\infty (1-z)^n \mathcal{F}_{E,p_t}^t(n). \quad (5.78)$$

The ‘‘fusion coefficients’’  $\Phi_E(p_s | p_t)$  are given as

$$\Phi_E(p_s | p_t) = \frac{s_b(w_1) s_b(w_3) s_b(2q_1 - c_b)}{s_b(w_2) s_b(w_4) s_b(2p_u + c_b)} \int_{\mathbb{R}} dt \prod_{i=1}^4 \frac{s_b(t + u_i)}{s_b(t + v_i)}, \quad (5.79)$$

where the coefficients  $r_i$  and  $s_i$ ,  $i = 1, \dots, 4$  are defined respectively by

$$\begin{aligned} u_1 &= p_1 - q_2 - p_2, & v_1 &= +p_u - q_2 - c_b, & w_1 &= p_s + p_1 - q_1, \\ u_2 &= p_1 - q_2 + p_2, & v_2 &= -p_u - q_2 - c_b, & w_2 &= p_s + q_1 - p_1, \\ u_3 &= +q_1, & v_3 &= +p_s + p_1 - c_b, & w_3 &= p_2 + p_u - p_1, \\ u_4 &= -q_1, & v_4 &= -p_s + p_1 - c_b, & w_4 &= p_2 + p_1 - p_u. \end{aligned} \quad (5.80)$$

The ‘fusion coefficients’  $\Phi_E(p_s | p_t)$  are related to the b-Racah-Wigner coefficients  $\{\dots\}_b$  defined and studied in <sup>7</sup> via

$$\begin{aligned} \Phi_E(p_s | p_t) &= \frac{M(p_4, p_3, p_s) M(p_s, p_2, p_1)}{M(p_4, p_t, p_1) M(p_t, p_3, p_2)} \left\{ \begin{matrix} \alpha_1 & \beta_1 & \beta_s \\ \bar{\beta}_2 & \bar{\alpha}_2 & \bar{\beta}_t \end{matrix} \right\}_b \\ &= \frac{M(p_4, p_3, p_s) M(p_s, p_2, p_1)}{M(p_4, p_t, p_1) M(p_t, p_3, p_2)} \left\{ \begin{matrix} \beta_1 & \alpha_1 & \beta_s \\ \alpha_2 & \beta_2 & \beta_t \end{matrix} \right\}_b. \end{aligned} \quad (5.81)$$

where we used the notation  $\beta_b = \frac{Q}{2} + ip_b$ ,  $\bar{\beta}_b = \frac{Q}{2} - ip_b$  for  $b = 1, 2, s, t$ ,  $\alpha_k = \frac{Q}{2} + iq_k$ ,  $\bar{\alpha}_k = \frac{Q}{2} - iq_k$  for  $k = 1, 2$  and

$$M(p_3, p_2, p_1) = \frac{s_b(2p_2 - c_b)}{s_b(p_2 + p_3 - p_1)}. \quad (5.82)$$

The first line in (5.81) follows directly from the definitions, in order to go from the first to the second line we have used a symmetry of the b-Racah-Wigner coefficients found in <sup>5</sup>, Appendix B.2. This means that the b-Racah-Wigner coefficients describe the operator product expansion of the rescaled fields  $f_{p_3 p_1}^{p_2}(w) = (M(p_3, p_2, p_1))^{-1} \mathbf{h}_{p_3 p_1}^{p_2}(w)$ .



### 5.3 Elementary braid relation

As a preparation we will need the limit of our general braid relation (2.19) where  $p_1 \rightarrow c_b$ . The representation  $\mathcal{F}_{c_b}$  is the vacuum representation that has vanishing highest weight. Let us observe that formula (2.18) implies that both sides generically vanish in this limit unless some pole from the  $\Gamma_b$ -functions in the numerator cancels the zero from the factor  $1/\Gamma_b(Q + 2ip_1)$ . This is the case if one has set  $p_s = p_1$  before taking the limit, as we shall assume from now on. We are going to show that in the limit  $p_1 \rightarrow c_b$  the braid relation (2.19) simplifies to

$$\mathbf{h}_{p_2 q_1}^{q_2}(\sigma_2) \mathbf{h}_{q_1 c_b}^{q_1}(\sigma_1) = \Omega^\epsilon(p_2, q_2, q_1) \mathbf{h}_{p_2 q_2}^{q_1}(\sigma_1) \mathbf{h}_{q_2 c_b}^{q_2}(\sigma_2), \quad (5.83)$$

where  $\epsilon = \text{sgn}(\sigma_2 - \sigma_1)$ ,

$$\Omega^\epsilon(p_2, q_2, q_1) = e^{\pi i \epsilon (\Delta_{q_2} + \Delta_{q_1} - \Delta_{p_2})} \frac{s_b(p_2 + q_1 - q_2) s_b(2q_2 - c_b)}{s_b(p_2 + q_2 - q_1) s_b(2q_1 - c_b)}. \quad (5.84)$$

In order to derive (5.83) let us write  $p_1$  as  $p_1 = c_b - i\epsilon$  for a small positive  $\epsilon$ . It is easy to see that the matrix element  $\langle v_{p_u}, \mathbf{h}_{p_u p_1}^{q_2}(1) v_{p_1} \rangle$  behaves for  $\epsilon \rightarrow 0$  as

$$\langle v_{p_u}, \mathbf{h}_{p_u p_1}^{q_2}(1) v_{p_1} \rangle \underset{\epsilon \rightarrow 0}{\sim} \frac{2\epsilon}{\epsilon + i(p_u - q_2)}. \quad (5.85)$$

A non-vanishing integrand in (2.19) is therefore only found if  $p_u = q_2$ . However, in order to extract the singular behavior for  $\epsilon \rightarrow 0$ ,  $p_u \rightarrow q_2$  we need to observe that the braid coefficients (2.20) develop a pole at these parameter values. This pole comes from the fact that the pole at  $t = c_b - r_3$  from the numerator and the pole at  $t = -c_b - s_1$  from the denominator of the integrand both approach the point  $t = 0$  in the limit  $\epsilon \rightarrow 0$ ,  $p_u \rightarrow q_2$ . In order to extract the resulting singular behavior of the integral one may deform the contour of integration into a contour that separates the pole at  $t = c_b - r_3$  from all the other poles in the upper t-half-plane, and a small circle around the point  $t = c_b - r_3$ . Only the contribution from the small circle shows singular behavior in the limit that we consider. By taking into account eqn. (A.114) we find that

$$\int_{\mathbb{R}} dt \prod_{i=1}^4 \frac{s_b(t + r_i)}{s_b(t + s_i)} \underset{\substack{p_u \rightarrow q_2 \\ \epsilon \rightarrow 0}}{\sim} \frac{s_b(p_2 + q_1 - q_2) s_b(2q_2 - c_b)}{s_b(p_2 + q_2 - q_1) s_b(2q_1 - c_b)} \frac{1}{2\pi i} \frac{-1}{p_u - q_2 + i\epsilon} \quad (5.86)$$

up to terms that are regular in this limit. Collecting the  $\epsilon$ -dependent factors one finds

$$\frac{1}{2\pi} \frac{2\epsilon}{(p_u - q_2)^2 + \epsilon^2} \underset{\epsilon \rightarrow 0}{\rightarrow} \delta(p_u - q_2).$$

To complete the verification of our claim (5.83) is now the matter of a straightforward calculation.

### 5.4 Fusion - the derivation

Let us consider the following matrix element:

$$\mathcal{F}_{E, p_s}(z_2, z_1, z_0) \equiv \langle v_{p_2}, \mathbf{h}_{p_2 p_s}^{q_2}(z_2) \mathbf{h}_{p_s p_1}^{q_1}(z_1) \mathbf{h}_{p_1 c_b}^{p_1}(z_0) v_0 \rangle, \quad (5.87)$$

where  $v_0 \equiv v_{c_b}$ . We will assume for the moment that  $z_k = e^{i\sigma_k}$ ,  $k = 0, 1, 2$  with  $\sigma_2 > \sigma_1 > \sigma_0$ .  $\mathcal{F}_{E, p_s}(z_2, z_1, z_0)$  is the boundary value of a function that is analytic for  $|z_2| >$

$|z_1| > |z_0|$ . We may then use the relations (5.83) and (2.19) to express  $\mathcal{F}_{E,p_s}(z_2, z_1, z_0)$  as

$$\begin{aligned} \mathcal{F}_{E,p_s}(z_2, z_1, z_0) &\equiv \\ &= \Omega^+(p_s, q_1, p_1) \int_0^\infty dp_u B_D^+(p_s | p_u) \langle v_{p_2}, h_{p_2 p_u}^{p_1}(z_0) h_{p_u q_1}^{q_2}(z_2) h_{q_1 c_b}^{q_1}(z_1) v_0 \rangle, \end{aligned} \quad (5.88)$$

where  $D = (q_2, p_1, p_2, q_1)$ . Let us note the simple relation  $e^{z_1 L - 1} \xi = h_{p_u s_b}^{p_u}(\xi | z_1) v_0$ , which implies that  $h_{p_u q_1}^{q_2}(z_2) h_{q_1 c_b}^{q_1}(z_1) v_0$  can be written as

$$\begin{aligned} h_{p_u q_1}^{q_2}(z_2) h_{q_1 c_b}^{q_1}(z_1) v_0 &= e^{z_1 L - 1} h_{p_u q_1}^{q_2}(z_2 - z_1) v_{q_1} \\ &= h_{p_u s_b}^{p_u}(h_{p_u q_1}^{q_2}(z_2 - z_1) v_{q_1} | z_1) v_0. \end{aligned} \quad (5.89)$$

By inserting (5.89) into (5.88) and using (5.83) (now with  $\epsilon = -1$ ) again we finally find the expression

$$\begin{aligned} \mathcal{F}_{E,p_s}(z_2, z_1, z_0) &\equiv \\ &= \int_0^\infty dp_u \Phi_E(p_s | p_u) \langle v_{p_2}, h_{p_2 p_1}^{p_u}(h_{p_u q_1}^{q_2}(z_2 - z_1) v_{q_1} | z_1) h_{p_u c_b}^{p_u}(z_0) v_0 \rangle, \end{aligned} \quad (5.90)$$

where the coefficients  $\Phi_E(p_s | p_u)$  are given as

$$\Phi_E(p_s | p_u) = \Omega^+(p_s, q_1, p_1) B_D^+(p_s | p_u) \Omega^-(p_2, p_1, p_u). \quad (5.91)$$

By assembling the pieces one verifies that this expression coincides with the one given in (5.79). The fact that the expansion in powers of  $\frac{z_2 - z_1}{z_1 - z_0}$  has a finite radius of convergence can be seen by returning to (5.88) and noting that

$$\langle v_{p_2}, h_{p_2 p_u}^{p_1}(z_0) h_{p_u q_1}^{q_2}(z_2) h_{q_1 c_b}^{q_1}(z_1) v_0 \rangle = \langle v_{p_2}, h_{p_2 p_u}^{p_1}(z_0 - z_1) h_{p_u q_1}^{q_2}(z_2 - z_1) v_{q_1} \rangle,$$

cf. our discussion in Subsection 2.4. It can finally be checked that the result is independent of our choice of the order of the variables  $\sigma_k$ ,  $k = 0, 1, 2$  if one excludes that case that  $\sigma_0$  lies between  $\sigma_1$  and  $\sigma_2$ . The latter case is not suitable for studying the behavior where  $\sigma_2$  approaches  $\sigma_1$ .

## 6 General exponential operators

### 6.1 Introducing the right-movers

Let us now introduce a second, right-moving chiral field  $\bar{\varphi}(x_-) = \bar{q} + \bar{p}x_- + \bar{\varphi}_<(x_-) + \bar{\varphi}_>(x_-)$ . The commutation relations for the modes are obtained from (2.2) by replacing  $(q, p, a_n) \rightarrow (\bar{q}, \bar{p}, \bar{a}_n)$ . The corresponding Hilbert-space will be denoted by  $\mathcal{H}_R^F$ . Normal ordered exponentials  $\bar{E}^\alpha(x_-)$  and screening charges  $\bar{Q}(x_-)$  are defined by the obvious substitutions. This will *not* be case for our definition of  $\bar{h}_s^\alpha(x_-)$ , however. Instead we will find it convenient to define  $\bar{h}_s^\alpha(x_-)$  as

$$\bar{h}_s^\alpha(x_-) = \bar{S}^{-1} \bar{E}^{\bar{\alpha}}(x_-) (\bar{Q}(x_-))^{\bar{s}} \bar{S}, \quad (6.92)$$

where  $\bar{\alpha} = Q - \alpha$ ,  $\bar{s} = -s - Q/b$ , and  $\bar{S}$  is the intertwining operator that is defined in the same way as we have defined  $S$  in Subsection 2.2. The definition is such that the fields  $\bar{h}_s^\alpha(\bar{w})$  have the shift property

$$\bar{h}_s^\alpha(\bar{w})f(\bar{p}) = f(\bar{p} - i(\alpha + bs))\bar{h}_s^\alpha(\bar{w}). \quad (6.93)$$

The formula for the matrix elements of the fields  $\bar{h}_s^\alpha(x_-)$  is obtained from (2.18) by the substitutions  $s \rightarrow -s - Q/b$ ,  $p_i \rightarrow -p_i$  for  $i = 1, 2$  as well as  $\alpha \rightarrow Q - \alpha$ . In order to find the exchange relations of the fields  $\bar{h}_s^\alpha(x_-)$  from (2.19) it is enough to replace  $\alpha_i \rightarrow Q - \alpha_i$  and  $\sigma_i \rightarrow -\sigma_i$  for  $i = 1, 2$ .

In the following we will only be interested in the diagonal subspace  $\mathcal{H}_D^F$  in  $\mathcal{H}^F = \mathcal{H}_L^F \otimes \mathcal{H}_R^F$  that is defined by the condition  $\mathbf{p} = \bar{\mathbf{p}}$ . We clearly have  $\mathcal{H}^F \simeq L^2(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{F}$ . We shall only consider combinations of the chiral fields like  $h_s^\alpha(x_+) \otimes \bar{h}_s^\alpha(x_-)$ . These fields preserve the diagonal  $\mathbf{p} = \bar{\mathbf{p}}$  thanks to the equality of shifts (2.11) and (6.93). The projection to  $\mathcal{H}_D^F$  is automatic if one considers matrix elements like  $\langle v_p \otimes \bar{v}_p, \mathbf{O} \psi \rangle_{\mathcal{H}^F}$ .

## 6.2 Construction

Let us introduce the following one-parameter family of fields

$$\boxed{\begin{aligned} V_\alpha(w, \bar{w}) &\equiv \lambda_b n(\alpha) N^{-1} W_\alpha(w, \bar{w}) N, \\ W_\alpha(w, \bar{w}) &\equiv \frac{1}{2} \int_{\mathbb{T}} ds m(\mathbf{p}) h_s^\alpha(w) \otimes \bar{h}_s^\alpha(\bar{w}). \end{aligned}} \quad (6.94)$$

Besides the notation  $\mathbb{T} \equiv -\frac{Q}{2} + i\mathbb{R}$  we have introduced the following objects in (6.94):

$$(i) \text{ Field normalization factors: } \begin{cases} n(\alpha) \equiv (\pi\mu\gamma(b^2)b^{2-2b^2})^{-\frac{\alpha}{b}} \frac{\Gamma_b(2\alpha - Q)}{\Gamma_b(2\alpha)}, \\ \lambda_b = 2\pi(\Gamma(b^2)b^{1-b^2})^{\frac{1+b^2}{b^2}}, \end{cases} \quad (6.95)$$

$$(ii) \text{ Similarity transformation: } N \equiv N(\mathbf{p}) \equiv (\pi\mu\gamma(b^2)b^{2-2b^2})^{-\frac{i}{b}\mathbf{p}} \frac{\Gamma_b(Q - 2i\mathbf{p})}{\Gamma_b(-2i\mathbf{p})} \quad (6.96)$$

The field  $V_\alpha(w, \bar{w})$  represents the exponentials of the Liouville field in the quantized theory.

## 6.3 Properties

The following properties of the fields  $V_\alpha(w, \bar{w})$  can be seen to capture the essence of the exact solution of the quantized Liouville theory <sup>4</sup>.

COVARIANCE —

$$\begin{aligned} [L_n, V_\alpha(w, \bar{w})] &= e^{nw}(\partial_w + \Delta_\alpha n)V_\alpha(w, \bar{w}), \\ [\bar{L}_n, V_\alpha(w, \bar{w})] &= e^{n\bar{w}}(\partial_{\bar{w}} + \Delta_\alpha n)V_\alpha(w, \bar{w}). \end{aligned} \quad (6.97)$$

MATRIX ELEMENTS —

$$\begin{aligned}
& e^{-(w+\bar{w})(\Delta_{p_2}-\Delta_{p_1})} \langle v_{p_2} \otimes \bar{v}_{p_2}, \mathbf{V}_\alpha(w, \bar{w}) v_{p_1} \otimes \bar{v}_{p_1} \rangle = \\
& = (\pi\mu\gamma(b^2)b^{2-2b^2})^s \frac{\Upsilon_0 \Upsilon_b(Q-2ip_2) \Upsilon_b(2\alpha) \Upsilon_b(Q+2ip_1)}{|\Upsilon_b(\alpha+i(p_2+p_1)) \Upsilon_b(\alpha+i(p_2-p_1))|^2} \quad (6.98)
\end{aligned}$$

We have again assumed  $\alpha \in \mathbb{R}$  in order to write the expression compactly. The special function  $\Upsilon_b(x)$  is defined<sup>b</sup> as  $\Upsilon_b^{-1}(x) = \Gamma_b(x)\Gamma_b(Q-x)$ , and  $\Upsilon_0 = 2\pi\Gamma_b^{-2}(Q)$ .

LOCALITY —

$$\Pi_D [\mathbf{V}_{\alpha_2}(\sigma_2), \mathbf{V}_{\alpha_1}(\sigma_1)] = 0 \quad \text{where } \mathbf{V}_\alpha(\sigma) \equiv \mathbf{V}_\alpha(w, \bar{w})|_{w=i\sigma}, \quad (6.99)$$

and  $\Pi_D : \mathcal{H}^F \rightarrow \mathcal{H}_D^F$  denotes the projection to the diagonal  $\mathfrak{p} = \bar{\mathfrak{p}}$ .

The calculation of the matrix elements (6.98) is straightforward. One just needs to combine the definitions of the previous subsection with formula (2.18), keeping in mind that one has to make the substitutions  $\alpha \rightarrow Q - \alpha$ ,  $p_i \rightarrow -p_i$  to get the matrix elements of the right-movers.

In order to verify locality it clearly suffices to consider the fields  $W_\alpha$ . Locality of these fields follows easily by first using the exchange relations (2.19) to change the order of the chiral components, and then simplifying the resulting expression with the help of (2.22). This works straightforwardly as long as  $\alpha_k \in \frac{Q}{2} + i\mathbb{R}$ ,  $k = 1, 2$ , since the exchange relations of the fields  $\bar{h}_s^\alpha$  then involve the complex conjugate of the kernel  $B_E^\epsilon$ . Validity of (6.99) for more general values of  $\alpha_k$  follows by analytic continuation. Let us furthermore notice that conformal covariance is manifest in our construction.

#### 6.4 Final remarks

It may be interesting to observe that our ansatz (6.94) yields local covariant fields for *arbitrary* choice of the functions  $n(\alpha)$  and  $N(p)$ . This freedom is eliminated by invoking further consistency conditions of the conformal bootstrap. In order to see this, let us recall some elements of the discussion in <sup>4</sup>, Part I. State-operator correspondence implies a relation between vacuum expectation values and matrix elements,

$$\begin{aligned}
& \langle v_{p_3} \otimes \bar{v}_{p_3}, \mathbf{V}_{\alpha_2}(z_2, \bar{z}_2) v_{p_1} \otimes \bar{v}_{p_1} \rangle = \\
& = \lim_{z_3 \rightarrow \infty} \lim_{z_1 \rightarrow 0} |z_3|^{4\Delta_{\alpha_3}} \langle 0 | \mathbf{V}_{\alpha_3}(z_3, \bar{z}_3) \mathbf{V}_{\alpha_2}(z_2, \bar{z}_2) \mathbf{V}_{\alpha_1}(z_1, \bar{z}_1) | 0 \rangle \quad (6.100) \\
& =: |z_2|^{2(\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1})} C(\alpha_3, \alpha_2, \alpha_1)
\end{aligned}$$

where  $\alpha_1 = \frac{Q}{2} + ip_1$  and  $\alpha_3 = \frac{Q}{2} - ip_3$ . Locality implies that the three point function  $C(\alpha_3, \alpha_2, \alpha_1)$  must be symmetric in its three arguments. Locality of the fields  $\mathbf{V}_\alpha$  together with symmetry of the three point functions then imply the crossing symmetry of the four-point functions, see <sup>4</sup>, Section 5.

It is easy to show that the requirement that  $C(\alpha_3, \alpha_2, \alpha_1)$  should be symmetric fixes the functions  $n(\alpha)$  and  $N(p)$  up to a constant, which may be re-absorbed into the definition of  $\lambda_b$ . The left-over freedom is on the one hand the possibility to redefine the prefactor  $\lambda_b$ , and on the other hand the freedom to replace the constant  $\mu$  by a function of  $b$ . This remaining freedom can be fixed by demanding

<sup>b</sup>This definition differs from the one used in <sup>3</sup> by a  $b$ -dependent factor that drops out in (6.98)

(i) that  $\lim_{\alpha \rightarrow 0} V_\alpha = \text{id}$ , which implies <sup>4</sup>, Section 4

$$\lim_{\alpha \rightarrow 0} C\left(\frac{Q}{2} - ip_2, \alpha, \frac{Q}{2} + ip_1\right) = 2\pi\delta(p_2 - p_1), \quad (6.101)$$

(ii) that the quantized Liouville field  $\phi(w, \bar{w}) = \left(\frac{1}{2}\partial_\alpha V_\alpha(w, \bar{w})\right)_{\alpha=0}$  satisfies a quantum version of the Liouville equation which takes the specific form <sup>4</sup> (Section 9)

$$\partial_w \bar{\partial}_{\bar{w}} \phi(w, \bar{w}) = \pi\mu b V_b(w, \bar{w}). \quad (6.102)$$

If we adopt these requirements we are indeed forced to choose  $n(\alpha)$ ,  $N(p)$  and  $\lambda_b$  as given in (6.95), (6.96) The resulting formula for the three point function  $C(\alpha_3, \alpha_2, \alpha_1)$  is exactly the one proposed in <sup>3</sup>.

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## Appendix

### A Special functions

#### A.1 The function $\Gamma_b(x)$

The function  $\Gamma_b(x)$  is a close relative of the double Gamma function studied in <sup>13,14</sup>. It can be defined by means of the integral representation

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left( \frac{e^{-xt} - e^{-Qt/2}}{(1 - e^{-bt})(1 - e^{-t/b})} - \frac{(Q - 2x)^2}{8e^t} - \frac{Q - 2x}{t} \right). \quad (A.103)$$

Important properties of  $\Gamma_b(x)$  are

$$(i) \text{ Functional equation: } \Gamma_b(x + b) = \sqrt{2\pi} b^{bx - \frac{1}{2}} \Gamma^{-1}(bx) \Gamma(x). \quad (A.104)$$

$$(ii) \text{ Analyticity: } \Gamma_b(x) \text{ is meromorphic,} \\ \text{poles: } x = -nb - mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}. \quad (A.105)$$

$$(iii) \text{ Self-duality: } \Gamma_b(x) = \Gamma_{1/b}(x). \quad (A.106)$$

#### A.2 The function $s_b(x)$

The function  $s_b(x)$  may be defined in terms of  $\Gamma_b(x)$  as follows

$$s_b(x) = \Gamma_b\left(\frac{Q}{2} + ix\right) / \Gamma_b\left(\frac{Q}{2} - ix\right). \quad (A.107)$$

This function, or close relatives of it like  $e_b(x) = e^{\frac{\pi i}{2} x^2} e^{-\frac{\pi i}{24}(2-Q^2)} s_b(x)$ , have appeared in the literature under various names like ‘‘Quantum Dilogarithm’’ <sup>15</sup>, ‘‘Hyperbolic G-function’’ <sup>16</sup>, ‘‘Quantum Exponential Function’’ <sup>17</sup> and ‘‘Double Sine Function’’, we refer

to the appendix of <sup>18</sup> for a useful collection of properties of  $s_b(x)$  and further references. An integral that represents  $\log s_b(x)$  is

$$\log s_b(x) = \frac{1}{i} \int_0^{\infty} \frac{dt}{t} \left( \frac{\sin 2xt}{2 \sinh bt \sinh b^{-1}t} - \frac{x}{t} \right). \quad (\text{A.108})$$

The most important properties for our purposes are

(i) Functional equation:  $s_b(x - i\frac{b}{2}) = 2 \cosh \pi b x s_b(x + i\frac{b}{2})$ . (A.109)

(ii) Analyticity:  $s_b(x)$  is meromorphic,

poles:  $x = c_b + i(nb - mb^{-1}), n, m \in \mathbb{Z}^{\geq 0}$ . (A.110)

zeros:  $x = -c_b - i(nb - mb^{-1}), n, m \in \mathbb{Z}^{\geq 0}$ .

(iii) Self-duality:  $s_b(x) = s_{1/b}(x)$ . (A.111)

(iv) Inversion relation:  $s_b(x)s_b(-x) = 1$ . (A.112)

(v) Unitarity:  $\overline{s_b(x)} = 1/s_b(\bar{x})$ . (A.113)

(vi) Residue:  $\text{res}_{x=c_b} s_b(x) = (2\pi i)^{-1}$ . (A.114)

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