

Existence and construction of Hamiltonian paths and cycles on conforming tetrahedral meshes

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This paper addresses the existence and construction of Hamiltonian paths and Hamiltonian cycles on conforming tetrahedral meshes. The paths and cycles are constrained to pass from one tetrahedron to the next one through a vertex. For conforming tetrahedral meshes, under certain conditions which are normally satisfied in finite-element computations, we show that there exists a through-vertex Hamiltonian path between any two tetrahedra. The proof is constructive from which an efficient algorithm for computing Hamiltonian paths and cycles can be directly derived.

Keywords: adaptive finite element; tetrahedral meshes; Hamiltonian path; Hamiltonian cycle; mesh partitioning

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1. Introduction

Adaptive finite-element methods (AFEM) are very efficient for solving partial differential equations, especially when the solutions have local singularities, and are one of today's hot spots in scientific computing research. The main idea of AFEM is adjusting the computational meshes and/or the order of finite-element bases according to properties of the computed solution to achieve optimal computational complexity for the given problem. Mesh partitioning is one of the key issues in parallel implementations of AFEM on distributed memory parallel computers. It is used to initially partition the mesh into submeshes in order to distribute them to processes, as well as to repeatedly repartition the mesh in order to preserve good load balance during the mesh adaptation process. One important class of mesh-partitioning algorithms consists of first ordering all elements in the mesh as a one-dimensional list, then cutting the list into segments of approximately equal weight, and Hamiltonian path provides a useful way for ordering elements in mesh-partitioning algorithms [1,2,4,5].

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A Hamiltonian path is defined as a sequence of elements in which each element appears exactly once. Heber *et al.* [3] proved that Hamiltonian paths exist under very mild conditions for two-dimensional triangle meshes, where the paths enter an element through an edge or a vertex and leave an element through an edge or a vertex. The statement of the theorem of [3] implies that the meshes should have no local cut vertices and be conforming [6]. Later, Mitchell [6] studied Hamiltonian paths and cycles for two- and three-dimensional meshes. For tetrahedral meshes, since Hamiltonian paths and cycles that enter an element through a face and leave an element through a face may not exist, the through-vertex Hamiltonian path and cycle were introduced in [6], and it was proved that the through-vertex Hamiltonian paths and cycles exist if the mesh contains no local cut vertices or local cut edges.

Due to the same reason, the through-vertex Hamiltonian path and cycle are employed in this paper. We address the existence of through-vertex Hamiltonian paths and cycles on conforming tetrahedral mesh under the assumption that the dual graph of the mesh is connected. This assumption is usually satisfied by finite-element meshes and is one of the bases of the implementation of our parallel adaptive finite-element package Parallel Hierarchical Grid (PHG) [8]. We will show that for any conforming tetrahedral mesh whose dual graph is connected, there exists a through-vertex Hamiltonian path between any two elements. The improvement of our result over the result of [6] is a stronger existence theorem (existence of a through-vertex Hamiltonian path between any pair of elements) under a weaker assumption, since if the mesh is connected, then the connectedness of its dual graph is equivalent to no cut vertices or cut edges, which is weaker than the assumption of no local cut vertices or local cut edges required in [6] (see discussions at the end of Section 2). The proof is constructive from which an efficient algorithm with optimal computational complexity can be derived. A nice feature of the algorithm is that it is incremental, i.e. new tetrahedra can be easily inserted into an existing path, which is useful when repartitioning an adaptively refined mesh.

The layout of the paper is as follows. In Section 2, we introduce the notations and definitions used in the paper. In Section 3, we state and prove our main theorem and give some corollaries. In Section 4, we present algorithms for computing Hamiltonian paths and cycles whose computational complexity is $O(n)$, where n is the number of elements in the mesh, and test it with some sample meshes. Finally in Section 5, we conclude with some discussions.

2. Notations and definitions

For the sake of convenience, we give all notations and definitions here, some of which are borrowed from [6].

A *tetrahedron* T is regarded as a closed domain in \mathbb{R}^3 . A tetrahedron contains four *vertices*, six *edges* and four triangular *faces*.

Let Ω be a connected open domain in \mathbb{R}^3 . A *tetrahedral mesh* \mathcal{M} on Ω is a set of tetrahedra $\{T_i\}$, such that $\bar{\Omega} = \cup T_i$, and $\overset{\circ}{T}_i \cap \overset{\circ}{T}_j = \emptyset$ for $i \neq j$, where $\overset{\circ}{T}$ denotes the interior of tetrahedron T , and \emptyset is the empty set. When no confusion may arise, we will also use the same term \mathcal{M} to represent the set of vertices, the set of edges, and the set of faces of the mesh. A vertex of \mathcal{M} is a *boundary vertex* if it lies on the boundary of Ω , and an *interior vertex* otherwise. The *size* of \mathcal{M} , $|\mathcal{M}|$, denotes the number of tetrahedra in \mathcal{M} . Two tetrahedra T_i and T_j in \mathcal{M} , $i \neq j$, are called *neighbours* if they have a common face.

A mesh is *conforming* if $\forall i \neq j$, $T_i \cap T_j$ is either a vertex, an edge, a face, or empty.

The *dual graph* of a conforming tetrahedral mesh \mathcal{M} , is an undirected graph which has a node for each tetrahedron and an edge for two tetrahedra that are neighbours.

A *path of length n* in a mesh \mathcal{M} is a sequence of tetrahedra, $T_1 T_2 \cdots T_n$, $T_i \in \mathcal{M}$, $i = 1, \dots, n$, with $T_i \cap T_{i+1} \neq \emptyset$, and $T_i \neq T_{i+1}$, $i = 1, n - 1$. A *cycle of length n* is a path of length $n + 1$ in

which $T_1 = T_{n+1}$. A *Hamiltonian path on \mathcal{M}* is a path in which each tetrahedron of \mathcal{M} appears exactly once. A *partial Hamiltonian path on \mathcal{M}* is a path in which each tetrahedron appears at most once. A *Hamiltonian cycle on \mathcal{M}* is a cycle in which every tetrahedron in \mathcal{M} appears exactly once except for $T_1 = T_{n+1}$.

A *through-vertex path* is a path in which the passage from one tetrahedron to the next is through a common vertex of the two tetrahedra, and the path does not pass through the same vertex when entering and exiting a tetrahedron. A (partial) through-vertex Hamiltonian path on \mathcal{M} is a through-vertex path that is also a (partial) Hamiltonian path on \mathcal{M} . A through-vertex path can be denoted by $H = T_1 v_1 T_2 \cdots v_{n-1} T_n$, where $v_i (1 \leq i < n)$ denotes the vertex in the path which connects T_i and T_{i+1} .

A graph is *connected* means that there exists a path between any pair of nodes. A vertex v is called a *cut vertex* if $\overline{\Omega} \setminus v$ is disconnected. A vertex v is called a *local cut vertex* if $\exists R > 0$ such that for any $r, 0 < r \leq R$, $(B(v, r) \cap \overline{\Omega}) \setminus v$ is disconnected [6], here $B(v, r)$ denotes the ball of radius r centred at v . Cut vertex and local cut vertex are illustrated in Figures 1 and 2, in

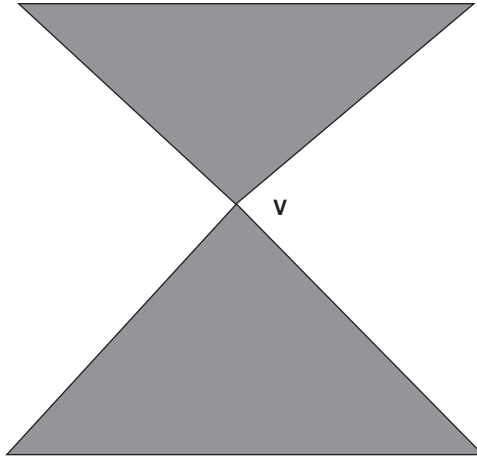


Figure 1. A mesh with a cut vertex v .

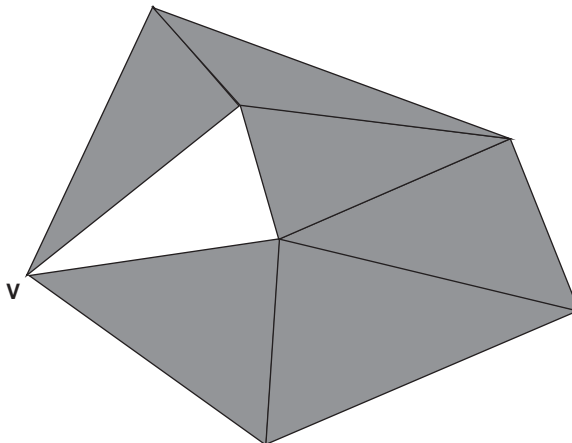


Figure 2. A mesh with a local cut vertex v .

which two-dimensional triangular meshes are used for clarity. *Cut edge* and *local cut edge* can be defined similarly.

It is easy to see that for any conforming tetrahedral mesh, its dual graph is connected if and only if it has no cut vertices or cut edges. On the other side, a conforming tetrahedral mesh with the connected dual graph may have local cut vertices or local cut edges.

3. Existence of through-vertex Hamiltonian paths and cycles

In this section, we prove the existence of through-vertex Hamiltonian paths and cycles on conforming tetrahedral meshes.

We first give three lemmas which will be used in the proof of the main theorem. The first lemma says that, we can always find two tetrahedra that share a common face, and it also tells us that neighbour tetrahedra have common vertices. These common vertices are essential when we extend a Hamiltonian path by inserting a new tetrahedron.

LEMMA 3.1 *Let \mathcal{M} be a conforming tetrahedral mesh whose dual graph is connected and $|\mathcal{M}| \geq 2$. Let \mathcal{M}_1 and \mathcal{M}_2 be non-empty subsets of \mathcal{M} , $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$, $\mathcal{M}_1 \cup \mathcal{M}_2 = \mathcal{M}$, then:*

- (1) *there exist $T_1 \in \mathcal{M}_1$ and $T_2 \in \mathcal{M}_2$ such that T_1 and T_2 have a common face.*
- (2) *if T , T_1 and T_2 are distinct tetrahedra of \mathcal{M} and T_1 and T_2 are neighbours of T , then T , T_1 and T_2 have exactly two common vertices.*
- (3) *if T , T_1 , T_2 and T_3 are distinct tetrahedra of \mathcal{M} and T_1 , T_2 and T_3 are neighbours of T , then T , T_1 , T_2 and T_3 have exactly one common vertex.*

Proof Statement (1) is a direct consequence of the connectedness of the dual graph of the mesh.

Statement (2): since both T_1 and T_2 are neighbours of T , each of them shares a common face with T , and it is easy to verify that the two common faces share a common edge, thus T_1 , T_2 and T share a common edge and the statement follows.

Statement (3): since T_1 , T_2 and T_3 are neighbours of T , each of them shares a common face with T , and the three common faces share a common vertex. ■

LEMMA 3.2 *Let \mathcal{M} be a conforming tetrahedral mesh whose dual graph is connected and $|\mathcal{M}| \geq 2$. Let $H = T_1v_1T_2v_2 \cdots T_{k-1}v_{k-1}T_k$ ($k \geq 2$) be a partial through-vertex Hamiltonian path on \mathcal{M} . Then:*

- (1) *if T_1 and T_2 are neighbours, then any neighbour of T_1 not belonging to H can be inserted into H between T_1 and T_2 .*
- (2) *if T_{k-1} and T_k are neighbours, then any neighbour of T_k not belonging to H can be inserted into H between T_{k-1} and T_k .*

Proof Since the proofs of the two statements are similar, we only give a proof here for statement (1).

Let T be a neighbour of T_1 not belonging to H . According to Lemma 3.1, T_1 , T_2 and T have two common vertices, denoted by u_1 and u_2 .

If H just contains T_1 and T_2 , i.e., $k = 2$, then T can be inserted into H as $T_1u_1Tu_2T_2$.

Now assume $k > 2$. If $v_1 \in \{u_1, u_2\}$, without loss of generality, assume $v_1 = u_2$, then T can be inserted into H as $T_1u_1Tv_1T_2v_2 \cdots T_k$. If $v_1 \notin \{u_1, u_2\}$, then one of u_1 and u_2 , say u_1 , is different from v_2 , in this case T can be inserted into H as $T_1v_1Tu_1T_2v_2 \cdots T_k$. ■

The third lemma states that, for a non-empty partial through-vertex Hamiltonian path, we can enlarge it by inserting new tetrahedra.

LEMMA 3.3 *Let \mathcal{M} be a conforming tetrahedral mesh whose dual graph is connected and $|\mathcal{M}| \geq 2$. Let $H = T_1 v_1 \cdots T_n$ be a non-empty partial through-vertex Hamiltonian path containing all neighbours of T_1 and T_n . Let $\mathcal{M}_1 = \{T_1, \dots, T_n\}$, then as long as $\mathcal{M}_1 \neq \mathcal{M}$, H can be enlarged by inserting a new tetrahedron after T_1 and before T_n .*

Proof Denote $\mathcal{M} \setminus \mathcal{M}_1$ by \mathcal{M}_2 . If \mathcal{M}_2 is not empty, by Lemma 3.1, there exist $T \in \mathcal{M}_2$ and $T_i \in \mathcal{M}_1$ such that T and T_i are neighbours, where $1 < i < n$.

Since T_i has four vertices, one of the three common vertices of T_i and T must be either T_i 's in-vertex v_{i-1} or out-vertex v_i , without loss of generality, suppose it is v_i (otherwise, we can reverse the path and after finishing the insertion of the new tetrahedron reverse it again). And one of the three common vertices of T_i and T , denoted by v , must not be v_{i-1} or v_i . Then a new partial through-vertex Hamiltonian path on \mathcal{M} can be constructed as $H = T_1 v_1 \cdots T_i v T v_i T_{i+1} \cdots T_n$. ■

Remark 1 The condition in Lemma 3.3 that the path includes all neighbours of T_1 and T_n is necessary. For example, assume $H = T_1 v T_n$, where T_1 and T_n share just one vertex v , if T_1 has a neighbour T that does not contain the vertex v , then we cannot insert T between T_1 and T_n .

Now we are ready to state the main theorem.

THEOREM 3.4 *Let \mathcal{M} be a conforming tetrahedral mesh whose dual graph is connected and $|\mathcal{M}| \geq 2$. For any pair of tetrahedra $T_1 \in \mathcal{M}$ and $T_n \in \mathcal{M}$, $T_1 \neq T_n$, there exists a through-vertex Hamiltonian path on \mathcal{M} from T_1 to T_n .*

Proof Since the dual graph of \mathcal{M} is connected, we can find a sequence of tetrahedra $T_1 T_2 \cdots T_n$ in \mathcal{M} such that T_i and T_{i+1} ($1 \leq i < n$) are neighbours. It is easy to construct a partial through-vertex Hamiltonian path $H = T_1 v_1 T_2 v_2 \cdots T_n$ as follows: we start with an initial through-vertex Hamiltonian path consisting of the single tetrahedron T_1 , assume we have constructed $T_1 v_1 \cdots T_{i-1} v_{i-1} T_i$, $1 \leq i < n$, since T_{i+1} has three common vertices with T_i , let v_i be any one of the common vertices if $i = 1$ or any one of the common vertices which is different from v_{i-1} if $i > 1$, then the partial through-vertex Hamiltonian path can be extended as $T_1 v_1 \cdots T_{i-1} v_{i-1} T_i v_i T_{i+1}$, and the process can be repeated for $i = 1, \dots, n - 1$. Using Lemma 3.2, H can be enlarged to include all neighbours of T_1 and T_n . Then using Lemma 3.3, all other tetrahedra of \mathcal{M} can be inserted into H . ■

The next theorem states existence of through-vertex Hamiltonian cycles.

THEOREM 3.5 *Let \mathcal{M} be a conforming tetrahedral mesh whose dual graph is connected and $|\mathcal{M}| \geq 2$, then there exists a through-vertex Hamiltonian cycle on \mathcal{M} .*

Proof We choose two tetrahedra T_1 and T_n where T_1 is a neighbour of T_n . By Theorem 3.4 there exists a through-vertex Hamiltonian path $T_1 v_1 T_2 v_2 \cdots v_{n-1} T_n$. Since T_1 and T_n are neighbours, they have three common vertices and one of the three vertices, denoted by v_n , must be different from v_1 and v_{n-1} . Then we can construct a through-vertex Hamiltonian cycle as $T_1 v_1 T_2 v_2 \cdots v_{n-1} T_n v_n T_1$. ■

Table 1. Timing results with some sample meshes.

	# tetrahedra	# faces	# vertices	time (s)
Mesh 1	155,456	323,264	32,238	0.127770
Mesh 2	239,616	480,704	40,781	0.240720
Mesh 3	453,736	923,336	85,338	0.515140
Mesh 4	1,243,648	2,536,704	232,288	1.265506
Mesh 5	3,211,264	6,498,304	573,628	3.736147

4. Algorithm and test results

The proofs in the last section can be easily turned into efficient algorithms for computing through-vertex Hamiltonian paths and cycles. Here, we present the algorithm for computing through-vertex Hamiltonian paths and briefly analyse its computational complexity. The algorithm for computing through-vertex Hamiltonian cycles can be similarly derived, with the same computational complexity.

ALGORITHM 4.1 *Let \mathcal{M} be a conforming tetrahedral mesh whose dual graph is connected and $n = |\mathcal{M}| \geq 2$. For any given pair of tetrahedra T_1 and T_n , a through-vertex Hamiltonian path $H = T_1 v_1 T_2 \cdots v_{n-1} T_n$ can be constructed by the following three steps.*

Step 1. Construct a partial through-vertex Hamiltonian path H from T_1 to T_n in which any two successive tetrahedra are neighbours. Details for constructing such a path are given in the proof of Theorem 3.4.

Step 2. Enlarge H by inserting all neighbours of T_1 and T_n into the path. This can be done by following the proof of Lemma 3.2.

Step 3. Enlarge H by inserting all the remaining tetrahedra into the path, one at a time. This can be done by following the proof of Lemma 3.3.

The first step can be implemented using the well-known *breadth first* search algorithm whose computational complexity is $O(n)$. It is clear that the complexity of the second step is $O(1)$ since each T_1 and T_n can have at most four neighbours. The third step consists of stepping along the path and inserting all the neighbours of each tetrahedron, which can obviously be accomplished with $O(n)$ complexity. Thus, the overall computational complexity of the algorithm is $O(n)$.

We have implemented a sequential version of this algorithm in our parallel adaptive finite-element toolbox PHG (<http://1sec.cc.ac.cn/phg>) [8]. Table 1 gives timing results with some sample meshes. The tests were run on a DELL PowerEdge 2950 server (2 quad-core 1.60 GHz Intel E5310 CPUs, 4 MB secondary cache). The sample meshes were generated using the mesh generator NETGEN by Schöberl [7]. Less than 4 s are needed to generate a through-vertex Hamiltonian path for a mesh consisting of more than 3 million tetrahedra, and the running time is proportional to the number of tetrahedra in the mesh.

5. Conclusion

We proved the existence of through-vertex Hamiltonian paths and cycles on conforming tetrahedral meshes and implemented an algorithm with $O(n)$ computational complexity for computing through-vertex Hamiltonian paths in the adaptive finite-element toolbox PHG, which provides a way, among other methods including graph partitioning and space filling curves, to partition the mesh into submeshes.

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