On average lower independence and domination numbers in graphs

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Abstract

The average lower independence number $i_{av}(G)$ of a graph $G=(V, E)$ is defined as $\frac{1}{|V|} \sum_{v \in V} i_v(G)$, and the average lower domination number $\gamma_{av}(G)$ is defined as $\frac{1}{|V|} \sum_{v \in V} \gamma_v(G)$, where $i_v(G)$ (resp. $\gamma_v(G)$) is the minimum cardinality of a maximal independent set (resp. dominating set) that contains $v$. We give an upper bound of $i_{av}(G)$ and $\gamma_{av}(G)$ for arbitrary graphs. Then we characterize the graphs achieving this upper bound for $i_{av}$ and for $\gamma_{av}$ respectively.

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1. Introduction

In a graph $G=(V(G), E(G))$, a subset $S \subseteq V$ of vertices is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex of $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. The independent domination number $i(G)$ is the minimum cardinality of a set that is both independent and dominating. The independence number $\alpha(G)$ is the maximum cardinality of an independent set. It is easy
to see that $\gamma(G) \leq i(G) \leq \alpha(G)$ holds for every graph $G$. For a comprehensive treatment of domination in graphs, see [6,7].

Henning [8] introduced the concept of average independence and average domination. For a vertex $v$ of a graph $G$, the lower independence number, denoted by $i_v(G)$, is the minimum cardinality of a maximal independent set of $G$ that contains $v$, and the lower domination number, denoted by $\gamma_v(G)$, is the minimum cardinality of a dominating set of $G$ that contains $v$. It is easy to see that every maximal independent set is a dominating set, and so $\gamma_v(G) \leq i_v(G)$ holds for every vertex $v$. The average lower independence number of $G$, denoted by $i_{av}(G)$, is the value $\frac{1}{|V(G)|} \sum_{v \in V(G)} i_v(G)$, and the average lower domination number of $G$, denoted by $\gamma_{av}(G)$, is the value $\frac{1}{|V(G)|} \sum_{v \in V(G)} \gamma_v(G)$. Since $\gamma_v(G) \leq i_v(G)$ holds for every vertex $v$, we have $\gamma_{av}(G) \leq i_{av}(G)$ for any graph $G$. Also, it is clear that $i(G) = \min \{i_v(G) \mid v \in V(G)\}$, $\gamma(G) = \min \{\gamma_v(G) \mid v \in V(G)\}$ and so $\gamma(G) \leq \gamma_{av}(G)$, and $i(G) \leq i_{av}(G)$.

Henning [8] established an upper bound for the average lower independence number of a tree and characterized the trees that achieve equality for this bound.

**Theorem 1 (Henning [8]).** If $T$ is a tree of order $n \geq 2$, then

$$i_{av}(T) \leq n - 2 + \frac{2}{n},$$

with equality if and only if $T$ is a star $K_{1,n-1}$.

In this paper, we give an upper bound for the average lower independence and domination numbers for any graph, improving Henning’s bound for trees. Then we characterize the graphs attaining this upper bound for the average lower independence and domination numbers respectively.

We finish this section by recalling some terminology and notation. Let $G=(V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v$ of $G$, the open neighbourhood of $v$ is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the closed neighbourhood of $v$ is $N[v] = N(v) \cup \{v\}$. Also we write $\overline{N}(v) = V(G) - N[v]$. If $S \subseteq V(G)$ then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The degree of a vertex $v$ of $G$, denoted by $d(v)$, is the size of its open neighbourhood. A vertex $v$ of degree 1 (resp. degree 0) is called a pendant vertex (resp. an isolated vertex). We denote by $n$ the order of $G$, which is the size of $V(G)$. For a subset $A$ of $V(G)$, $G[A]$ will denote the subgraph induced by the vertices of $A$.

**2. Upper bound**

A matching in a graph $G$ is a subset of pairwise non-incident edges. The matching number $\beta(G)$ is the size of a largest matching in $G$. A matching is said to be perfect if $\beta(G) = n/2$. For any vertex $v \in V(G)$, let $\beta_v(G)$ be the maximum cardinality of a matching in the graph induced by the vertices of $V(G) - N[v]$, that is, $\beta_v(G) = \beta(G[V(G) - N[v])]$. Recall that $\beta_v(G)$ can be computed for any graph $G$ in polynomial time (see [4]). Blidia et al. [2] gave an upper bound for the lower domination parameters $i(G)$ and $\gamma(G)$ for any graph $G$. 

Theorem 2 (Blidia et al. [2]). Let $G$ be a graph of order $n$. Then for every vertex $v$ of $G$, 
$$
\gamma(G) \leq i(G) \leq i_v(G) \leq n - d(v) - \beta_v(G).
$$

Proof. Let $v$ be any vertex of $G$, and let $S$ be a smallest maximal independent set of the graph $G[N(v)]$. Then $S \cup \{v\}$ is a maximal independent set of $G$, so $i(G) \leq |S| + 1 \leq \gamma(G[N(v)])$. However, it is known [13] that $\gamma(G) + \beta(G) \leq |V(G)|$ holds for every graph $G$. So we have $\gamma(G[N(v)]) \leq n - (d(v) + 1) - \beta(G[N(v)])$ and therefore $i(G) \leq n - d(v) - \beta_v(G)$. $\square$

Our next result is an upper bound for the average lower independence number of a graph $G$.

Proposition 3. For any graph $G$ with $n$ vertices and $m$ edges,
$$
i_{av}(G) \leq n - 2m/n - 1/n \sum_{v \in V(G)} \beta_v(G). \quad (2)
$$

Proof. By Theorem 2, $i_{av}(G) = 1/n \sum_{v \in V(G)} i_v(G) \leq n - 1/n \sum_{v \in V(G)} (d(v) + \beta_v(G))$. Since $\sum_{v \in V(G)} d(v) = 2m$, the result follows. $\square$

Corollary 4. For any graph $G$ with $n$ vertices and $m$ edges,
$$
\gamma_{av}(G) \leq n - 2m/n - 1/n \sum_{v \in V(G)} \beta_v(G). \quad (3)
$$

Since every tree $T$ of order $n$ contains $n - 1$ edges, Proposition 3 leads immediately to the following corollary for any tree, which improves the upper bound in (1).

Corollary 5 (Henning [8]). For every tree $T$,
$$
\gamma_{av}(T) \leq i_{av}(T) \leq n - 2 + 2/n - 1/n \sum_{v \in V(T)} \beta_v(T). \quad (4)
$$

We can remark that the second part of Theorem 1 follows easily from Corollary 5.

For more simplicity, let us write $\beta_{av}^*(G) = 1/n \sum_{v \in V(T)} \beta_v(T)$. We are interested in characterizing graphs attaining the upper bounds in (2) and (3).

3. Graphs with equality in (2)

The corona of a graph $H$, denoted by $H \circ K_1$, is a graph containing $2|V(H)|$ vertices and constructed from a copy of the graph $H$ where each vertex of $V(H)$ is adjacent to exactly one vertex of degree one. Note that if $G$ is a corona of a graph $H$ then $i(G) = \gamma(G) = \beta(G) = |V(H)|$.

A graph $G$ is called well covered if every maximal independent set is maximum, that is $i(G) = \gamma(G)$. In [11], Ravindra characterize the well covered trees.
Theorem 6 (Ravindra [11]). A tree $T$ is well covered if and only if $T$ is a single vertex or $T$ is a corona of a tree.

A graph is called very well covered if $G$ is well covered and $i(G) = z(G) = n/2$. Rautenbach and Volkmann [10] characterized the graphs $G$ such that $G$ is very well covered and $i(G) + \beta(G) = n$, and pointed out that this characterization gives a polynomial-time algorithm for the recognition of such graphs. We will call such graphs extremely well covered.

Let $G$ be a graph and $v$ a vertex of $G$. Let $I(v)$ denote the set of isolated vertices in $G[N(v)]$, $N^0(v) = \{x \in N(v) \mid x \text{ has no neighbour in } N(v)\}$ and $N^I(v) = \{x \in N(v) \mid x \text{ has a neighbour in } I(v)\}$. We next give a necessary and sufficient condition for sharpness of the inequality $i_{av}(G) \leq n - 2m/n - \beta_{av}^*(G)$.

Theorem 7. Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. Then $i_{av}(G) = n - 2m/n - \beta_{av}^*(G)$ if and only if for every vertex $v$ of $G$ the subgraph $G[N(v)]$ is extremely well covered.

Proof. The if part of the theorem is easy to check. Let us now prove the only if part. Let $G = (V, E)$ be a graph such that $i_{av}(G) = n - 2m/n - \beta_{av}^*(G)$. Then, by the same argument as in the proof of Proposition 3, every vertex $v$ satisfies

$$i_u(G) = n - d(u) - \beta_u(G) \leq n$$

and so,

$$\beta_u(G) = |N(u)|.$$ 

Consequently, $G[N(u)]$ is extremely well covered.

As a consequence we have a characterization of trees $T$ for which $i_{av}(T) = n - 2 + 2/n - \beta_{av}^*(T)$.

Corollary 8. Let $T$ be a tree of order $n \geq 2$. Then the following statements are equivalent:

(a) $i_{av}(T) = n - 2 + 2/n - \beta_{av}^*(T)$,

(b) $T[N(v)]$ is a well covered tree for every vertex $v$ of $T$,

(c) $T$ is a star $K_{1,n-1}$ or $T$ is a corona of tree.

Proof. Let $T$ be a tree of order $n \geq 2$.

(a) $\Rightarrow$ (b) follows from Theorem 7 and the fact that for a tree, being well covered tree is equivalent to being extremely well covered.

Let us prove (b) $\Rightarrow$ (c). Let $v$ be a vertex of $T$ and assume that $\beta_v(T) = 0$ for every vertex $v$ of $T$. Thus $\beta_{av}^*(T) = 0$ and $T[N(v)]$ contains no edge. Clearly now if $I(v) = \emptyset$ then $T$
is a star $K_{1,n-1}$. So assume that $I(v) \neq \emptyset$. Then $d(v) = 1$ for otherwise $\beta_u^*(T) \neq 0$ for some $w \in I(v)$, a contradiction. Thus $T = K_{1,n-1}$. We now suppose that $\beta_{av}^*(T) \neq 0$. Let us consider the following two cases:

**Case 1: $v$ is a pendant vertex.** Let $s$ be the neighbour of $v$. Then $T[\overline{N}(v)]$ must contain at least one edge for otherwise $T$ is a star. We also have $I(v) = \emptyset$. Suppose to the contrary that $I(v) \neq \emptyset$. Let $w$ be a pendant vertex of $T$ in $T[\overline{N}(v) - I(v)]$. A such pendant vertex exists since $T[\overline{N}(v) - I(v)]$ contains at least one component that is a corona of tree. In this case, $s$ will be adjacent to at least two pendant vertices in $T[\overline{N}(w)]$ and so $T[\overline{N}(w)]$ is not well covered, a contradiction. Thus each component of $T[\overline{N}(v)]$ is nontrivial.

If all the components of $T[\overline{N}(v)]$ are of order two or every pendant vertex of $T[\overline{N}(v)]$ is a pendant vertex in $T$, then $T$ is a corona of a tree and we are finished. So assume there is a vertex $z$ in a component $C = H \circ K_1$ of $T[\overline{N}(v)]$ with at least 4 vertices, such that $z$ is a pendant vertex in $C$ but not in $T$. Then $d(z) = 2$ since $C$ is a corona. Let $u \in V(H)$ be the neighbour of $z$ and $y \in V(H)$ be a neighbour of $u$. Then $s$ will have $v$ and $z$ as two pendant neighbours in $T[\overline{N}(y)]$, a contradiction.

**Case 2: $v$ is not a pendant vertex in $T$.** We first see that every vertex of $N^I(v)$ is adjacent to exactly one vertex of $I(v)$. Suppose there is a vertex $z \in N^I(v)$ such that $|N(z) \cap I(v)| \geq 2$. Since $d(v) \geq 2$, for a vertex $u \in N(v)$ different from $z$, $(N(z) \cap I(v)) \cup \{z\}$ will induce a star of order at least three in $T[\overline{N}(u)]$, a contradiction. Likewise, we can see that $|N^0(v)| \leq 1$. It suffices to consider the non-neighbourhood of a vertex in $\overline{N}(v)$, a such vertex exists since $T$ is not a star. Suppose now that $N^0(v) = \emptyset$. If $\beta_v(T) = 0$ then $T$ is a subdivided star and so for every pendant vertex $w$ of $T$, $T[\overline{N}(w)]$ is not a corona of tree, a contradiction. Assume that $\beta_v(T) \neq 0$ and let $z \in N(v)$ be a vertex that has a neighbour in a nontrivial component of $T[\overline{N}(v)]$ and pick $y \in N(v) - \{z\}$. Clearly $N(y) \cap \overline{N}(v) \neq \emptyset$ since $N^0(v) = \emptyset$. Depending on whether $z$ has a neighbour in $I(v)$ or not, we see that the component containing $z$ is not a corona in $T[\overline{N}(x)]$, where $x \in N(y) \cap \overline{N}(v)$ or $x = y$ respectively, and we get a contradiction. Thus $|N^0(v)| = 1$, and so $v$ has a pendant neighbour.

Now, if every pendant vertex of a component of $T[\overline{N}(v) - I(v)]$ of size at least 4 is also pendant in $T$ then $T$ is a corona of tree. Thus assume there is a component $C = H \circ K_1$ of $T[\overline{N}(v)]$ with at least 4 vertices and with a pendant vertex $z$ that is not be a pendant vertex in $T$. Then $d(z) = 2$ in $T$. Let $y$ be the unique neighbour of $z$ in $N(v)$ and $x$ be the neighbour of $z$ in $H$. If $y \in N^I(v)$, then $C \cup (N[y] \cap I(v))$ is not a corona of a tree in $T[\overline{N}(w)]$ where $w$ is the pendant vertex adjacent to $v$, a contradiction. If $y \notin N^I(v)$, then $v$ will be adjacent to two pendant vertices in $T[\overline{N}(x)]$, again a contradiction.

The implication (c) $\rightarrow$ (a) is easy to show, which completes the proof. □

4. Graphs with equality in (3)

As an immediate consequence of Corollary 8, the following corollary provides us a characterization of trees $T$ for which the average lower domination number achieves equality in (4).

**Corollary 9.** Let $T$ be a tree of order $n \geq 2$. Then $\gamma_{av}(T) = n - 2 + 2/n - \beta_{av}^*(T)$ if and only if $T$ is a star $K_{1,n-1}$ or $T$ is a corona of tree.
A graph $G$ is said to be a strong crowned graph if for every vertex $v$ of $G$ each component of $G[N(v)]$ is either an isolated vertex or a cycle $C_4$ or a corona. Crowned graphs were defined similarly in [2]. Clearly, strong crowned graphs are extremely well covered.

The graphs $G$ of even order and without isolated vertices with $\gamma(G) = n/2$ have been characterized independently by Payan and Xuong and Fink, Jacobson, Kinch and Roberts.

**Theorem 10** (Fink et al. [5], Payan and Xuong [9]). Let $G$ be a graph of even order $n$ and without isolated vertices. Then $\gamma(G) = n/2$ if and only if each component of $G$ is either a cycle $C_4$ or the corona of a connected graph.

**Proposition 11.** Let $G$ be a graph with $\gamma_{av}(G) = n - 2m/n - \beta_{av}(G)$. Then $G$ is a strong crowned graph.

**Proof.** Let $G$ be a graph such that $\gamma_{av}(G) = n - 2m/n - \beta_{av}(G)$. Then for every vertex $v$ of $G$, we have $\gamma_{v}(G) = n - d(v) - \beta_{v}(G)$. Let $v$ be a vertex of $G$ and $Y$ a minimum dominating set of the subgraph $G[N(v) - I(v)]$. Then $\{v\} \cup I(v) \cup Y$ is a dominating set of $G$ that contains $v$, so $\gamma_{v}(G) \leq 1 + |I(v)| + |Y|$. Thus

$$\gamma_{v}(G) + d(v) + \beta_{v}(G) \leq 1 + |I(v)| + |Y| + d(v) + \beta_{v}(G) \leq n.$$ 

Since $\gamma_{v}(G) + d(v) + \beta_{v}(G) = n$, we have $n \leq 1 + |I(v)| + |Y| + d(v) + \beta_{v}(G) \leq n$ and so

$$|Y| + \beta_{v}(G) = n - (d(v) + 1) - |I(v)| = |N(v) - I(v)|.$$ 

Consequently, $|Y| = \beta_{v}(G) = |N(v) - I(v)|/2$. Since $G[N(v) - I(v)]$ contains no isolated vertices, by Theorem 10 every component of $G[N(v) - I(v)]$ is a cycle $C_4$ or a corona, so the result follows. □

Note that the converse of Proposition 11 is not true for every strong crowned graph. It can be seen by the graph $G$ formed from a cycle $C_4$ by adding a vertex attached to the two non-adjacent vertices of $C_4$. Then $G$ is a strong crowned graph but $\gamma_{av}(G) = 2$ and $n - 2m/n - \beta_{av}(G) = 13/5$.

**Observation 12.** In a strong crowned graph $G$, let $v$ be a vertex of $G$, $C$ be a component of $N(v)$ and $w$ be any vertex of $N(v)$ that has a neighbour in $C$. Then exactly one of the following holds:

1. $|C| = 1$;
2. $|C| = 2$ and $|N(w) \cap C| = 1$;
3. $|C| = 2$ and $|N(w) \cap C| = 2$;
4. $C$ is a 4-vertex cycle or path and $|N(w) \cap C| = 4$;
5. $C$ is a 4-vertex cycle or path and $|N(w) \cap C| \leq 3$ and, if $C$ is a $P_4$, then $w$ is adjacent to at least one pendant vertex of $C$;
6. $C$ is the corona of a graph $H$ with $|H| \geq 2$ and $w$ is not adjacent to any pendant vertex of $C$;
7. \( C \) is the corona of a graph \( H \) with \( |H| \geq 3 \) and \( w \) is adjacent to at least one pendant vertex of \( C \).

(In the last two cases, \( w \) may be adjacent arbitrarily to vertices of \( H \).

In cases 1, 3, 5 above we say that \( w \) is a \( C \)-candidate and that \( C \) is a candidate-generating component.

**Lemma 13.** In a strong crowned graph \( G \), let \( v \) be a vertex of \( G \), \( C \) be a component of \( \overline{N}(v) \) and \( w \) be any vertex of \( N(v) \) that has a neighbour in \( C \). If \( v \) satisfies \( \gamma_v(G) = n - d(v) - \beta_v(G) \) then:

- Cases 4 and 7 cannot occur for \( C \) and \( w \);
- No vertex of \( N(v) \) can be a \( C_1 \)-candidate and a \( C_2 \)-candidate for two different components \( C_1, C_2 \) of \( \overline{N}(v) \);
- If there is a candidate-generating component of size \( \geq 2 \) in \( \overline{N}(v) \), then there is no other component in \( \overline{N}(v) \).

**Proof.** Let \( D \) be a set defined as follows: put \( v \) and all the isolated vertices of \( \overline{N}(v) \) in \( D \); in addition, for every component \( C \) of \( \overline{N}(v) \) that is a 4-cycle, add in \( D \) two non-adjacent vertices of \( C \), and for every component \( C \) of \( \overline{N}(v) \) that is the corona of a graph \( H \), put in \( D \) all the vertices of \( H \). It is easy to see that \( D \) is a dominating set containing \( v \), with \( |D| = n - d(v) - \beta_v(G) = \gamma_v(G) \).

1. Assume that Case 4 occurs for \( C \) and \( w \). Define a set \( D' \) obtained from \( D \) by removing the 2 vertices of \( C \cap D \) and adding \( w \). It is easy to see that \( D' \) is a dominating set containing \( v \), with \( |D'| < |D| \), a contradiction.

Now assume that Case 7 occurs for \( C \) and \( w \), where \( C \) is the corona of a graph \( H \) with \( |H| \geq 3 \).

First suppose that \( w \) is adjacent to at least three pendant vertices \( x^*, y^*, z^* \) of \( C \), and call \( x, y, z \) their respective neighbours in \( H \). Define a set \( D' \) obtained from \( D \cup \{w\} \) by removing vertices as follows: if there are at least two edges among \( x, y, z \), say the edges \( xy, xz \), then remove \( y, z \) from \( D \cup \{w\} \); if there is exactly one edge among \( x, y, z \), say the edge \( xy \), then remove \( y, z \) (note that in that case \( z \) has a neighbour in \( H - \{x, y, z\} \)); if there is no edge among \( x, y, z \) then remove \( x, y, z \) (note that in that case each of \( x, y, z \) has a neighbour in \( H - \{x, y, z\} \)). It is easy to see that in either case \( D' \) is a dominating set containing \( v \) with \( |D'| < |D| \), a contradiction.

Now suppose that \( w \) is adjacent to exactly two pendant vertices \( x^*, y^* \) of \( C \), and call \( x, y \) their respective neighbours in \( H \). Let \( z \) be a vertex of \( H - \{x, y\} \) and \( z^* \) be the pendant vertex of \( C \) adjacent to \( z \). Note that \( v, w, x, y, x^*, y^* \) are non-neighbours of \( z^* \) and lie in one component \( F \) of \( \overline{N}(z^*) \). Since \( G \) is strong crowned, \( F \) must be a corona. If \( x \) is not pendant in \( F \), it should have a pendant neighbour \( x' \) in \( F \), and \( x' \) can only be in \( N(v) \); but this is impossible since then \( x'v \) is another edge of \( F \). So \( x \) and similarly \( y \) are pendant vertices of \( F \). This implies that their only neighbour in \( H \) is \( z \). But then \( D \cup \{w\} - \{x, y\} \) is a dominating set containing \( v \) with \( |D'| < |D| \), a contradiction.

Now suppose that \( w \) is adjacent to exactly one pendant vertex \( x^* \) of \( C \), and call \( x \) the neighbour of \( x^* \) in \( H \). Since \( H \) is connected, there is a vertex \( z \) of \( H - x \) such that \( H - z \).
is connected. Let \( z^* \) be the pendant vertex of \( C \) adjacent to \( z \). Note that all the vertices of \((C - \{z, z^*\}) \cup \{v, w\}\) are non-neighbours of \( z^* \) and lie in one component \( F \) of \( \overline{N}(z^*) \). Since \( G \) is strong crowned, \( F \) must be a corona. We see that \( x \) is not pendant in \( F \), so there should be a pendant vertex \( x' \) in \( F \) adjacent to \( x \), and \( x' \) can only be in \( N(v) \); but this is impossible since then \( x'v \) is another edge of \( F \). This proves the first point of the lemma.

2. Assume that some vertex \( w \) of \( N(v) \) is a \( C_1 \)-candidate and a \( C_2 \)-candidate for two different components \( C_1, C_2 \) of \( N(v) \). We build a set \( D' \) by modifying \( D \cup \{w\} \) as follows: if \( C_1 \) is in Case 1 or 3, remove the vertex of \( C_1 \cap D \); if \( C_1 \) is a 4-vertex cycle (Case 5), remove the two vertices of \( C_1 \cap D \) and add a center vertex of \( C_1 - N(w) \); if \( C_1 \) is a 4-vertex path (Case 5), remove the vertex \( x \) of \( H \) such that \( x^* \) is adjacent to \( w \). At this point, it is easy to see that \( D' \) is a dominating containing \( v \) and \( w \) with \( |D'| = |D| \). Now we do the same modifications with respect to \( C_2 \) (instead of \( C_1 \)), starting from \( D' \). Then it is easy to see that the resulting set \( D'' \) is a dominating set containing \( v \) and \( w \) with \( |D''| < |D| \), a contradiction. This proves the second point of the lemma.

3. Let us prove the third point. Let \( C \) be a candidate-generating component of size \( \geq 2 \) in \( \overline{N}(v) \), and suppose that there is another component \( C' \) in \( \overline{N}(v) \). Let \( w \) be a \( C \)-candidate. Let \( z \) be a vertex of \( C' \) that is not adjacent to \( w \). Such a \( z \) exists: if \( C' \) is a candidate-generating component this is by the second point of the lemma; if \( C' \) is not a candidate-generating component we are in Case 2 or 6 for \( C' \) and we can let \( z \) be any pendant vertex of \( C' \). Note that all the vertices of \( C \cup \{v, w\} \) are non-neighbours of \( z \) and lie in one component \( F \) of \( \overline{N}(z) \). Since \( G \) is strong crowned, this component must be a corona. Let \( x \) be a neighbour of \( w \) in \( C \), such that, if \( C \) is not a 4-cycle, \( x \) is a pendant vertex of \( C \) (such an \( x \) exists by the definition of candidates). We see that \( x \) is not pendant in \( F \), so there should be a pendant vertex \( x' \) in \( F \) adjacent to \( x \), and \( x' \) can only be in \( N(v) \); but this is impossible since then \( x'v \) is another edge of \( F \). This completes the proof of the lemma. \( \square \)

Let \( \mathcal{F} \) be the family of graphs in Fig. 1. Let \( \mathcal{G} \) be the family of graphs \( G \) such that the components of \( \overline{G} \) are chordless cycles or paths.

**Theorem 14.** A connected graph \( G \) satisfies \( \gamma_{av}(G) = n - 2m/n - \beta_{av}(G) \) if and only if either \( G \) is a corona or \( G \) is in family \( \mathcal{F} \) or \( \mathcal{G} \).
Proof. The ‘if’ part is simple to check and we omit it. Now assume that \( G \) is a connected graph that satisfies \( \gamma_{av}(G) = n - 2m/n - \beta_{av}(G) \). By Proposition 11, \( G \) is a strong crowned graph. Note that this equality and the definition of \( \gamma_{av} \) implies that for every vertex \( v \) of \( G \) we have \( \gamma_v(G) = n - d(v) - \beta_v(G) \).

First suppose that \( \beta_{av}(G) = 0 \). This means that for every vertex \( v \) the set \( \overline{N}(v) \) is independent, and every vertex \( x \) of \( \overline{N}(v) \) is adjacent to every vertex of \( N(v) \) (else \( \overline{N}(x) \) contains an edge). So every component of \( \overline{G} \) is a clique, so \( G \) is the join of several independent sets \( S_1, \ldots, S_k \). Note that for a vertex \( v \) we have \( \gamma_v = 1 \) if \( v \in S_i \) with \( |S_i| = 1 \) and \( \gamma_v = 2 \) if \( v \in S_i \) with \( |S_i| \geq 2 \). Putting these constraints in the equation \( \gamma_{av}(G) = n - 2m/n - \beta_{av}(G) \) implies that each \( S_i \) has size at most 2, i.e., \( G \) is the complement of a graph whose component are edges or isolated vertices, so \( G \) is in class \( \mathcal{G} \).

Now suppose that \( \beta_{av}(G) > 0 \). This means that there exists a vertex \( v \) such that \( \beta_v(G) > 0 \). Note that Observation 12 and Lemma 13 hold for \( v \). So every component \( C \) of \( \overline{N}(v) \) is either candidate-generating (Cases 1, 3, 5 of Observation 12) or not (Cases 2, 6).

Suppose that there is a candidate-generating component \( C \) of size 4 in \( \overline{N}(v) \). By Lemma 13, there is no other component in \( \overline{N}(v) \) and \( C \) is either a cycle or path on 4 vertices. Let \( a, b, c, d \) be the vertices of \( C \), its edges being \( ab, bc, cd \) and (if \( C \) is a 4-cycle) \( ad \). Let \( x \) be any vertex of \( C \). We claim that there are at most two non-neighbours of \( x \) in \( N(v) \). For suppose there are at least three vertices in \( N(v) \cap \overline{N}(x) \). Then \( v \) and these vertices lie in a component \( F \) of \( \overline{N}(x) \), and \( F \) must be a corona, so at least two vertices of \( N(v) \cap \overline{N}(x) \) must have a pendant neighbour in \( F \); but the corresponding pendant vertices can only be in \( \overline{N}(x) \cap C \), and that set does not contain two non-adjacent vertices. So the claim holds, and since no vertex of \( N(v) \) is adjacent to all of \( C \), it follows that \( N(v) \) has size \( \leq 8 \). So \( G \) has size at most 13. Now exhaustive search is one way to determine all the graphs \( G \) such that: \( G \) has at most 13 vertices, \( G \) is strong crowned, for some vertex \( v \) the non-neighbours of \( v \) induce a 4-vertex cycle or path, and for every vertex \( v \) we have \( \gamma_v(G) = n - d(v) - \beta_v(G) \).

These graphs can also be found by a more refined case analysis; the details can be found in [3]. These graphs are shown in Fig. 1.

From now on we may assume that for every vertex \( x \) of \( G \) there is no candidate-generating component \( C \) of size 4 in \( \overline{N}(x) \).

Suppose that there is a candidate-generating component \( C \) of size 2 in \( \overline{N}(v) \). By Lemma 13, there is no other component in \( \overline{N}(v) \). Let \( a, b \) be the vertices of \( C \), let \( w \) be a \( C \)-candidate, so \( w \) is adjacent to both \( a, b \). We claim that \( a \) (and similarly \( b \)) has at most one non-neighbour in \( N(v) \); the proof is as above. Now if a vertex \( z \) of \( N(v) \) is not adjacent to any of \( w, a, b \) then the triangle \( wab \) lies in a component \( F \) of \( \overline{N}(z) \) and \( F \) must be a corona; call \( w^*, a^*, b^* \) the pendant vertices in \( F \) adjacent to \( w, a, b \) respectively; clearly \( w^*, a^*, b^* \) are in \( N(v) \); but then \( z, w^*, a^* \) are three non-neighbours of \( b \) in \( N(v) \), a contradiction. So every vertex of \( G \) has a neighbour in the triangle \( wab \), and the diameter of \( G \) is 2. Let \( z \) be a neighbour of \( v \). Suppose that \( \overline{N}(z) \) has at least two components \( C_1, C_2 \). If \( a \) (or similarly \( b \)) lies in one of \( C_1, C_2 \), say in \( C_1 \), then \( z \) and any vertex of \( C_2 \) are two non-neighbours of \( a \) in \( N(v) \), a contradiction. It follows that \( C_1 \cup C_2 \subseteq N(v) \), and so \( v \) is a \( C_1 \)-candidate and a \( C_2 \)-candidate with respect to \( z \), a contradiction. So \( \overline{N}(z) \) has at most one component. Moreover if \( \overline{N}(z) \) does not induce a clique, then \( \overline{N}(z) \) induces a corona or \( C_4 \); then \( \overline{N}(z) \notin N(v) \) (for otherwise the first point of Lemma 13 would be violated with respect to \( z \)), so one of \( a, b \) is in \( \overline{N}(z) \), but this implies that one of \( a, b \) has at least two non-neighbours in \( N(v) \), a contradiction.
So $\overline{N}(z)$ is a clique, of size 1 or 2 since $G$ is strong crowned. In summary, for every vertex $x$ the non-neighbours of $x$ induce a clique of size at most 2, so $G$ is in class $\mathcal{G}$.

From now on we may assume that for every vertex $x$ of $G$ there is no candidate-generating component $C$ of size $\geq 2$ in $\overline{N}(x)$.

If $v$ has degree 1, it is easy to see that $G$ is a corona. So we assume that $v$ has degree $\geq 2$.

Suppose that there is a component $C$ of size at least 2 in $\overline{N}(v)$ that contains a pendant vertex $x^*$ of $G$, and call $x$ the neighbour of $x^*$. Let $w$ be a vertex of $\overline{N}(v)$ adjacent to $C$; we may assume that $w$ is adjacent to $x$ (else consider another pendant vertex of $C$) and that $w$ has minimum degree among all such vertices. Suppose $w$ has degree $\geq 3$. Then, since $G$ is strong crowned, $v$ must have a pendant neighbour $v^*$ in $\overline{N}(x^*)$, and every vertex of $\overline{N}(v) - v^*$ must have a pendant neighbour in $\overline{N}(x^*)$ and such a vertex is in $\overline{N}(v)$; so, $G$ is a corona. Now we may assume that $w$ has degree 2. Note that $w$ is a pendant neighbour of $v$ in $\overline{N}(x^*)$. Thus there is no other vertex $w'$ of $\overline{N}(v)$ of degree 2 adjacent to $x$ (for otherwise, $w, w'$ would both be pendant neighbours of $v$ in $\overline{N}(x^*)$). Let $u$ be a vertex of $\overline{N}(v) - w$. Recall that in $\overline{N}(x^*)$ there is a pendant neighbour $u^*$ of $u$, and $u^* \in \overline{N}(v)$. Now consider $\overline{N}(u^*)$, which contains $v, w, x, x^*$. If $\overline{N}(u^*) = \{v, w, x, x^*\}$ then $vwxx^*$ is a $P_4$ which is a candidate-generating component for $u^*$ (and we have treated such a case before). If $\overline{N}(u^*) \neq \{v, w, x, x^*\}$, it must be that $\overline{N}(v) - \{u, w\} \neq \emptyset$ and so $\overline{N}(u^*)$ is connected and is a corona; but we see that in that component $w$ is not pendant and has no pendant vertex, a contradiction.

Finally, we may assume that every component of size at least 2 in $\overline{N}(v)$ contains no pendant vertex, which is possible only if any such component has size exactly 2 (there may also be components of size 1). We may in fact assume that for every vertex $x$ every component of $\overline{N}(x)$ has size 1 or 2. Since $\beta_v(G) > 0$, there actually exists such a component $C$. Let $a, b$ be the vertices of $C$, and $x$ (resp. $y$) be a neighbour of $a$ (resp. $b$) in $\overline{N}(v)$. Note that $xb$ and $ya$ are not edges or else $C$ would be a candidate-generating component. Suppose $\overline{N}(v) - \{x, y\}$ contains a vertex $z$. We may assume that $za$ is not an edge since $C$ is not candidate-generating. But now $v, y, z$ lie in one component of $\overline{N}(a)$, a contradiction. So $\overline{N}(v) = \{x, y\}$, and $G$ is the complement of $P_5$ or $C_5$ ($G$ is in class $\mathcal{G}$). This completes the proof of the Theorem. 

References

Further reading
