Nash Strategy for Markov Jump Stochastic Delay Systems

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Abstract—Nash games for a class of linear time-delay system with Markovian jumping parameters are investigated. By using a classical Lyapunov-Krasovskii method and a non-convex optimization approach as a sufficient condition, a strategy set in terms of matrix inequality is established. In order to obtain a strategy set numerically, new cross-coupled stochastic algebraic equations (CSAEs) are given based on Karush-Kuhn-Tucker (KKT) conditions. Furthermore, it is shown that the state feedback strategies can be obtained by solving linear matrix inequalities (LMIs) iteratively. Finally, a numerical example is detailed that shows the effectiveness of the proposed methods.

I. INTRODUCTION

With the maturity of Nash games, many works have been devoted to various control applications [1], [2]. For the practical control system with uncertainties, robust Nash strategy has been developed by means of the linear matrix inequalities (LMIs) defined by polytopes [3]. The $H_2/H_\infty$ control problems have also been investigated in [4], [5] by using Nash equilibrium.

In control engineering, there exists a wide class of dynamic systems subject to random abrupt variations of the operating point in the systems parameters. In order to capture these variations, Markovian jump linear systems (MJLSSs) have been proved to be useful to model a large number of them and these systems are adequate mathematical models for numerous processes and phenomena studied in control engineering. For example, in [24], the Markov process has been employed to model a vertical take-off and landing (VTOL) aircraft. It is well-known that MJLSSs belong to the category of hybrid systems with finite operation models. The stability analysis, stabilization, optimal control and $H_\infty$ control of MJLSSs have been widely investigated (see e.g., [7], [8], [9], [10], [11] and references therein). Recently, the control problem of time-delayed MJLSSs has been tackled in [12], [13], [14].

Nash games against time-delay and uncertainty have been one of the most challenging problems because these factors are frequently a source of instability and performance degradation of strategy. As an important result for eliminating time-delay, the $H_2/H_\infty$ control based on Nash equilibrium of time-delay system has paid too little early attention [15]. In [16], [17], [21], the state feedback Nash strategies for stochastic delay systems have been considered. On the other hand, Nash games of Markov jump linear stochastic systems (MJLSSs) governed by Itô’s differential equation have been solved successfully [18], [19], [20]. Up to now, to the best of the authors’ knowledge, pure Nash equilibria of strategic games of MJLSSs with time-delay have not yet been investigated. Thus, the combination of the theory of the MJLSSs with time-delay and the question of how control strategies overcome those harmful effects remain open.

In this paper, Nash games for a class of linear stochastic delay system governed by Itô’s differential equation with Markov jumping parameters are discussed. The novelty of this paper is that the abrupt variation of the system parameters and delay are both considered. It should be noted that although the results obtained in this paper extend those developed on the MJLSSs with time-delay in [16], [17], [18], [19], [20], [21], the used technique is more complicated. However, it is shown that such difficulties can be avoided via the SDP algorithm. First, by constructing an appropriate Lyapunov-Krasovskii functional, the sufficient condition is given for the existence of the delay-independent Nash strategy set via the cross-coupled matrix inequalities. Second, it is shown that new cross-coupled stochastic algebraic equations (CSAEs) are established using the Karush-Kuhn-Tucker (KKT) conditions instead of matrix inequalities. Moreover, in order to solve CSAEs, an iterative method based on LMIs is presented to reduce the complexity and large dimensions of the matrix calculus. Finally, a numerical example is given to illustrate the effectiveness and the usefulness of the proposed method.

Notation: The notations used in this paper are fairly standard. block diag denotes a block diagonal matrix. $\chi_A$ denotes indicator function. $E[\cdot| r_t = i]$ stands for the conditional expectation operator with respect to the event $\{r_t = i\}$. $M_{n,m}$ denotes space of all $S = (S(1), \ldots, S(s))$ with $S(i)$ being $n \times m$ matrix, $i \in D$, $D = \{1, 2, \ldots, s\}$.

II. DEFINITIONS AND PRELIMINARY

Let $r_t$, $t \geq 0$ with a finite state space $D = \{1, 2, \ldots, s\}$ be a right continuous homogeneous Markov chain on the filtered probability space $(\Omega, F, \{F_t\}_{t \geq 0}, P)$. It should be noted that a filtered probability space is a probability space equipped with the filtration $\{F_t\}_{t \geq 0}$ of its $\sigma$-algebra $F$. Throughout this paper we assume that $\{w(t)\}_{t \geq 0}$ and $\{r_t\}_{t \geq 0}$ are independent stochastic process. Furthermore, we also assume that the Markov process $r_t$ has the transition
probabilities given by
\[ P\{r_{t+\Delta t}=j \mid r_t=i\} = \begin{cases} 
\pi_{ij}\Delta t + o(\Delta t), & \text{if } i \neq j \\
1 + \pi_{ii}\Delta t + o(\Delta t), & \text{else}
\end{cases} \] (1)
where \( \pi_{ij} \geq 0 \) for \( i \neq j \) and \( \pi_{ii} = -\sum_{j=1, j \neq i}^s \pi_{ij} \).

Next, we first introduce the definitions of stochastic stabilizability, which are essential assumptions in the paper.

**Definition 1**: [6], [9], [10] Consider the following linear stochastically controlled system with Markovian jumps
\[ dx(t) = [A(r_t)x(t) + B(r_t)u(t)]dt + C(r_t)x(t)dw(t), \] (2)
where \( x(t) \in \mathbb{R}^n \) represents the state vector, \( u(t) \in \mathbb{R}^m \) represents the control input. The coefficients \( A, C \in \mathbb{M}_{n\times n}^s \) and \( B \in \mathbb{M}_{n\times m}^s \) with \( A(i), B(i), C(i), i \in D \), being constant matrices of compatible dimensions.

First, the stochastic system (2) or \( (A, C | D) \) is called stochastically stabilizable, if there exists a feedback control
\[ u(t) = \sum_{i=1}^s K(i)x(t)x_{r_{t-1}=i}, \] (3)
being constant matrices, such that for any initial state \( x(0) = x^0, r_0 = i \), the closed-loop system
\[ dx(t) = [A(r_t)x(t) + B(r_t)K(r_t)]x(t)dt + C(r_t)x(t)dw(t), \] (4a)
\[ y(t) = D(r_t)x(t) \] (4b)
is exponentially mean-square stable (EMSS), i.e.
\[ E|x(t)|^2 \leq \rho e^{-\psi(t-t_0)}E|x(t_0)|^2, \] \( \forall \rho, \psi > 0 \). (5)

Second, under the condition that \( B(i) \equiv 0, r_t = i \), \( (A, C) \) is called stable, if the stochastic system (2) is EMSS. Furthermore, when \( D \) is of full row-rank, \( (A, C | D) \) is stochastically detectable [6].

III. PROBLEM FORMULATION AND PRELIMINARY

Consider a linear stochastic time-delay system with Markovian switching under the multiple decision makers of the form.
\[ dx(t) = \left[ A(r_t)x(t) + A_h(r_t)x(t-h) \right. \]
\[ + \sum_{k=1}^N B_k(r_t)u_k(t) \] \[ \left. + A_p(r_t)x(t)dw(t), \right] \] (6)
\[ x(t) = \phi(t), t \in [-h, 0], \]
where \( x(t) \in \mathbb{R}^n \) represents the state vector. \( u_k(t) \in \mathbb{R}^{m_k}, k = 1, \ldots, N \) represents the \( i \)-th control vector. \( w(t) \in \mathbb{R} \) is a one-dimensional standard Wiener process defined in the filtered probability space [5], [22]. \( h > 0 \) is the time-delay of the MLSSs. \( \phi(t) \) is a real-valued initial function. It should be noted that since the multiplicative noise can easily be extended, one dimensional Wiener process is only considered.

The cost function for each decision maker is defined by
\[ J_k(u_1, \ldots, u_N, x(0), r_0) = E \left[ \int_0^\infty \left[ x^T(t)Q_k(r_t)x(t) + u_k^T(t)R_k(r_t)u_k(t) \right] dt \right| r_0 = i \] (7)
where \( k = 1, \ldots, N, r_t = 1, \ldots, s \), \( Q_k(r_t) = Q_k^T(r_t) > 0, R_k(r_t) = R_k^T(r_t) > 0 \).

It should be noted that \( u_j(t), j \neq l \) does not appear in the cost function. However, since they are included in the stochastic systems (6), they must have impacts on the cost functions (7).

It may be noted that in this study, the strategies \( u_k^* \) are restricted as the following linear feedback strategies.
\[ u_k(t) := \sum_{r_t=1}^s K_k(r_t)x(t)x_{r_{t-1}=i}, \] (8)

Let \( K_N \) denote the set of all \( (u_1(t), \ldots, u_N(t)) = (K_1(r_t)x(t), \ldots, K_N(r_t)x(t)) \) such that the following closed-loop stochastic system
\[ dx(t) = \left[ A(r_t) + \sum_{k=1}^N B_k(r_t)K_k(r_t) \right] x(t)dt \]
\[ + A_h(r_t)x(t-h)dt + A_p(r_t)x(t)dw(t) \] (9)
is EMSS.

**Lemma 1**: [11], [12], [13] The trivial solution of a stochastic differential equation is defined as follows:
\[ dx(t) = f(t, x, r_t)dt + g(t, x, r_t)dw(t) \] (10)
where \( f(t, x, r_t) \) and \( g(t, x, r_t) \) are sufficiently differentiable maps, is EMSS if there exists a function \( V(t, x, r_t) \) such that the following inequalities are satisfied:
\[ a_1|x(t)|^2 \leq V(t, x, i) \leq a_2|x(t)|^2, \] \( a_1, a_2 > 0 \). (11a)
\[ \nabla V(t, x, i) := \frac{\partial V(t, x, i)}{\partial t} + \frac{\partial V(t, x, i)}{\partial x}f(t, x, i) \]
\[ + \sum_{j=1}^s \pi_{ij}[V(t, x, j) - V(t, x, i)] \]
\[ + \frac{1}{2}Tr \left[ g^T(t, x, i)\frac{\partial^2 V(t, x, i)}{\partial x^2}g(t, x, i) \right] \]
\[ \leq -c||x(t)||^2, \] \( c > 0 \) (11b)
for \( x(t) \neq 0 \).

In this paper, it is assumed that matrix inequality (11b) have a solution satisfying (11a). Nash equilibrium inequality as a formal definition is given.

**Definition 2**: [1] The strategy set
\[ \mathcal{G}^* = (u_1^*, \ldots, u_N^*), \]
\[ u_k^*(t) := \sum_{r_t=1}^s K_k^*(r_t)x(t)x_{r_{t-1}=i}, \] (12)
\[ k = 1, \ldots, N \] is a stochastic Nash equilibrium strategy set if for each \( k = 1, \ldots, N \), the following inequality holds.
\[ J_k(u_1^*, \ldots, u_N^*, x(0), r_0) \]
\[ \leq J_k(u_1^*, \ldots, u_{k-1}^*, u_k, u_{k+1}^*, \ldots, u_N^*, x(0), r_0) \] (13)
for all \( x(0) \) and \( (K_1(i)x(t), \ldots, K_N(i)x(t)) \in K_N \).

For the stochastic delay systems (6), the solution of the stochastic Nash games are given below.
Theorem 1: Assume that the set \( K_N \) is not empty. Suppose that \( N \) real symmetric matrices \( P_k(i) > 0 \) and \( N \) real symmetric matrices \( W_k > 0 \) exist such that
\[
\mathbf{F}_k(P_k(1), \ldots, P_k(N), W_k, i) := \begin{bmatrix}
\Xi_k(i) & P_k(i)A_k(i) \\
A_k^T(i)P_k(i) & -W_k
\end{bmatrix} \leq 0,
\]
where \( k = 1, \ldots, N, i = 1, \ldots, s, \), \( \Xi_k(i) := P_k(i)A_k(i) + A_k^T(i)P_k(i) + A_k^T(i)P_k(i)A_k(i) - P_k(i)A_k(i)P_k(i) + Q_k(i) + \sum_{j \neq k} \pi_{jk}P_k(j) + W_k, \)
\( A_k(i) := A_k(i) - \sum_{k = 1, \ k \neq i}^N S_k(i)P_k(i), S_k(i) := B_k(i)R_k^{-1}(i)B_k^T(i). \)
Define the set \( (K_1^*, \ldots, K_N^*)(x(t), \ldots, K_N^*(x(t)) \in K_N. \)
Furthermore,
\[
J_k(K_1^*(x), \ldots, K_N^*(x)) \in K_N. \]
Then, \( G^* = (K_1^*(x(t), \ldots, K_N^*(x(t)) \in K_N. \)
Furthermore,
\[
J_k(K_1^*(x), \ldots, K_N^*(x)) \leq E[x(t)^0 P_k(0)x(0) | v_r = 0] + E \left[ \int_{-h}^{t} \phi^T(\tau)W_k \phi^T(\tau) d\tau \right] \]
Proof: Its proof can be demonstrated by using completion of squares. First, define the following Lyapunov-Krasovskii functional with \( W_k = W_k^T > 0. \)
\[
V_k(t, i) := x^T(t)P_k(i)x(t) + \int_{-h}^{t} x^T(\tau)W_k x(\tau) d\tau.
\]
Let us consider the following stochastic system with \( u_k(t) := u_k^*(t), i \neq j. \)
\[
dx(t) = \begin{bmatrix}
A_{-k}(r_i)x(t) + A_h(r_i)x(t - h) + B_k(r_i)u_k(t) \\
A_p(r_i)x(t) + d\tau(t), x(t) = \phi(t), t \in [-h, 0]
\end{bmatrix}
\]
By using the Itô formula, the weak infinitesimal generator, along with the stochastic system (18), can be obtained.
\[
D[V_k(t, i)] + x^T(t)(P_k(i)x(t) + u_k^T(t)(R_k(i)u_k(t)) := x^T(t)(\Xi_k(i)x(t) + 2x^T(t)P_k(i)A_k(i)x(t - h) + [u_k(t) + R_k^{-1}(i)B_k^T(i)P_k(i)x(t)]^T R_k(i)
\]
\[
+ [u_k(t) + R_k^{-1}(i)B_k^T(i)P_k(i)x(t)]^T R_k(i)
\]
\[-x^T(t - h)W_k x(t - h).
\]
Hence, if \( u_k^*(t) := -R_k^{-1}(i)B_k^T(i)P_k(i)x(t) \) and the matrix inequality (14) holds, the closed-loop system is EMSS and we have \( E[V_k(\infty, i) | r_0 = i] = 0. \) Thus, integrating both sides of the above equation and using \( E[V_k(\infty, i) | r_0 = i] = 0 \) results in
\[
J_k(u_1^*, \ldots, u_N^*, \ x, u_i^*(0), r_0) - E[V_k(0, r_0) | r_0 = i]
\geq E \left[ \int_{0}^{\infty} \eta^T(t)F_k(P_1(i), \ldots, x, P_N(i), W_k, i) \eta(t) d\tau \right] | r_0 = i
\]
\[
= J_k(u_1^*, \ldots, u_N^*, x(0), r_0) - E[V_k(0, r_0) | r_0 = i] \geq 0.
\]
Thus, the strategy set (15) satisfies the stochastic Nash equilibrium (13). On the other hand,
\[
J_k(u_1^*, \ldots, u_N^*, x(0), r_0) - E[V_k(0, r_0) | r_0 = i]
= E \left[ \int_{0}^{\infty} \eta^T(t)F_k(P_1(i), \ldots, P_N(i), W_k, i) \eta(t) d\tau \right] \leq 0.
\]
Thus, if (14) holds, then the desired result is obtained. \( \blacksquare \)

IV. MAIN RESULTS

It should be noted that the upper bound of the cost function in Theorem 1 depends on the initial condition. To remove this dependence, we suppose that the initial state is arbitrary but belongs to the following set.
\[
E[x(0)x^T(0)] = I_n, \quad E \left[ \int_{-h}^{0} \phi^T(\tau)W_k \phi^T(\tau) d\tau \right] = M, \quad M = M^T > 0.
\]

Theorem 2: Consider the stochastic system and the cost function.
\[
dx(t) = [(A(r_t) + B(r_t)K(r_t))x(t) + A_h(r_t)x(t - h)]dt + A_p(r_t)x(t)d\tau(t), x(t) = \phi(t), t \in [-h, 0],
\]
\[
J = E \left[ \int_{0}^{\infty} x^T(t)(Q(r_t) + K^T(r_t)R(r_t)K(r_t))x(t) dt \right] | r_0 = i,
\]
\[
Q(r_t) = Q^T(r_t) \geq 0, \quad R(r_t) = R^T(r_t) > 0.
\]

For the above-mentioned stochastic system, assume that real symmetric matrices \( \tilde{P}(i) > 0 \) and \( \tilde{W} > 0 \) exist such that
\[
\tilde{F}_k(\tilde{P}(i), \tilde{W}, \tilde{K}(i), i) \leq 0,
\]
where \( \tilde{F}_k(\tilde{P}(i), \tilde{W}, \tilde{K}(i), i) := \tilde{P}(i)[A(i) + B(i)\tilde{K}(i)] + [A(i) + B(i)\tilde{K}(i)]^T \tilde{P}(i) + A_k^T(i)\tilde{P}(i)A_p(i) + Q(i) + \sum_{j = 1}^{N} \pi_{ij}P(j) + K^T(i)R(i)K(i) + \tilde{W} + \tilde{P}(i)A_h(i)\tilde{W}^{-1}A_k^T(i)\tilde{P}(i).
\]

Then, we have
\[
J \leq E[x^T(0)\tilde{P}(r_0)x(0) | r_0 = i] + E \left[ \int_{-h}^{0} \phi^T(\tau)\tilde{W} \phi^T(\tau) d\tau \right] | r_0 = i
\]
\[
= \text{Tr} [\tilde{P}(r_0)] + \text{Tr} [M\tilde{W}].
\]
Second, let us now consider the minimization problem of the cost bound of (25). If the following two conditions hold
i) \( [A(i) + B(i)\tilde{K}(i) + A_h(i)\tilde{W}^{-1}A_k^T(i)\tilde{P}(i), A_p(i) | I_n] \)
\]
\[
is \text{stochastically detectable.}
\]
ii) \( [A(i) + B(i)\tilde{K}(i) + A_h(i)\tilde{W}^{-1}A_k^T(i)\tilde{P}(i), A_p(i) | I_n] \)
\]
\[
is \text{stable.}
\]
and \( \nu^* \) is a local minimum that satisfies the constraint qualification, then there exists a unique positive definite
solution $\tilde{G}^* > 0$ such that

$$F(\tilde{P}^*(i), \tilde{W}^*, i) = 0,$$  \hspace{1cm} (26a)

$$I(\tilde{G}^*(i), \tilde{P}^*(i), \tilde{W}^*, i) = 0,$$  \hspace{1cm} (26b)

$$\tilde{H}(\tilde{G}^*(i), \tilde{P}^*(i), \tilde{W}^*, i) = 0,$$  \hspace{1cm} (26c)

$$\tilde{K}^*(i) = -R^{-1}(i)B^T(i)\tilde{P}^*(i),$$  \hspace{1cm} (26d)

where $S(i) := B(i)R^{-1}_{-i}B^T(i), \tilde{F}(\tilde{P}(i), \tilde{W}, i) := \tilde{P}(i)A(i) + A^T(i)\tilde{P}(i) + A^T(i)\tilde{P}(i)A(i) - \tilde{P}(i)S(i)\tilde{P}(i) + Q(i) + \sum_{j=1}^N\pi_{ij}P(j) + W + \tilde{P}(i)\tilde{A}_{pi}W^{-1}\tilde{A}_{pi}^T\tilde{P}(i), I(\tilde{G}(i), \tilde{P}(i), \tilde{W}, i) := (A(i) - S(i)\tilde{P}(i) + A_{hi}W^{-1}A_{hi}^T\tilde{P}(i))\tilde{G}(i) + \tilde{G}(i)(A(i) - S(i)\tilde{P}(i) + A_{hi}W^{-1}A_{hi}^T\tilde{P}(i))^T + A_p(i)\tilde{G}(i)A_p^T(i) + \pi_{ii}\tilde{G}(i) + I_n, H(\tilde{G}(i), \tilde{P}(i), \tilde{W}, i) := \tilde{W}(G(i) + M)\tilde{W} - A_{hi}^T(\tilde{P}(i)\tilde{G}(i)\tilde{P}(i)A_{hi}(i), v^* = [\text{vec} \tilde{P}^*(i)]^T, [\text{vec} \tilde{W}^*]^T$.

In other words, let $v^*$ be the solution set that gives a local minimum. Then, the constrained minimization of the cost bound is attained such that CSAEs (26) hold. That is,

$$\min_{(P(r_0), W)} \left[ \text{Tr} \left[ \tilde{P}(r_0) \right] + \text{Tr} \left[ MW \right] \right] = \text{Tr} \left[ \tilde{P}(r_0) \right] + \text{Tr} \left[ MW^* \right].$$  \hspace{1cm} (27)

**Proof:** Since the proof of the first part of Theorem 2 can be obtained by using the same technique used in the previous section, we only prove the second part. This proof can be obtained by using the optimization technique proposed in [16], [17], [21]. According to the KKT conditions [23], the problem of determining a strategy that minimizes the cost bound (25) subject to the constraint in (24) can be converted into the following optimization problem. Let us consider the Lagrangian $L$.

$$L(\tilde{P}(i), \tilde{W}, \tilde{G}(i), \tilde{K}(i)) = \text{Tr} \left[ \tilde{P}(i) \right] + \text{Tr} \left[ MW \right] + \text{Tr} \left[ \tilde{G}(i)\tilde{K}(i) \tilde{P}(i) + \tilde{W} \tilde{K}(i), i \right],$$  \hspace{1cm} (28)

where $\tilde{G}$ is a symmetric matrix of Lagrange multipliers.

It is clear that $\text{Tr} \left[ \tilde{P} \right] + \text{Tr} \left[ MW \right]$ and $\tilde{F}_K$ are continuously differentiable at the point $v^*$. Using the KKT conditions, we have

$$\tilde{F}_K(\tilde{P}^*(i), \tilde{W}^*, \tilde{K}^*(i), i) \leq 0, \tilde{G}^*(i) \geq 0,$$  \hspace{1cm} (29a)

$$\tilde{G}^*(i)\tilde{F}_K(\tilde{P}^*(i), \tilde{W}^*, \tilde{K}^*(i), i) = 0,$$  \hspace{1cm} (29b)

$$\frac{\partial L}{\partial \tilde{P}(i)} = I_K(\tilde{G}(i), \tilde{P}^*(i), \tilde{W}^*, \tilde{K}^*(i), i) = 0,$$  \hspace{1cm} (29c)

$$\frac{\partial L}{\partial \tilde{W}} = \tilde{G}^*(i) + M - [\tilde{W}^*]^{-1} \times A_{hi}^T(\tilde{P}^*(i)\tilde{G}(i)\tilde{P}^*(i)A_{hi}^T(i)[\tilde{W}]^{-1} = 0,$$  \hspace{1cm} (29d)

$$\frac{\partial L}{\partial \tilde{K}(i)} = 2(B(i)\tilde{P}^*(i) + R(i)\tilde{K}^*(i))\tilde{G}^*(i) = 0,$$  \hspace{1cm} (29e)

where $I_K(\tilde{G}(i), \tilde{P}(i), \tilde{W}, \tilde{K}(i), i) := [A(i) + B(i)\tilde{K}(i) + A_{hi}W^{-1}A_{hi}^T(i)\tilde{P}(i)]\tilde{G}(i) + \tilde{G}(i)[A(i) + B(i)\tilde{K}(i) + A_{hi}W^{-1}A_{hi}^T(i)\tilde{P}(i)]^T + A_p(i)\tilde{G}(i)\tilde{P}(i)A_{hi}(i)$.

On applying the conditions i) and ii) to (29c), it follows immediately that (29c) has a unique positive definite solution $\tilde{G}^*(i) > 0$. Hence, from equation (29e), (26d) can be derived, and from equations (29c) and (26d), (26b) can be derived. Moreover, pre- and post-multiplying both sides of the equation $\partial L/\partial \tilde{W} = 0$ of (29d) by $\tilde{W}^*$, (29c) can also be obtained. From the remaining equations (29b), (26d) and $\tilde{F}_K(i) = 0$ because of $\tilde{G}^*(i) > 0$, (26a) holds.

Taking into consideration of Theorem 2, we arrive at the following result.

**Theorem 3:** Consider the stochastic systems (6) and the cost function (7). Suppose that the real symmetric positive definite matrices $P_k(i) > 0, G_k(i) > 0$ and $W_k > 0$ exist such that

$$F(P_k(i), W_k, i) = 0,$$  \hspace{1cm} (30a)

$$I(G_k(i), P_k(i), W_k, i) = 0,$$  \hspace{1cm} (30b)

$$H(G_k(i), P_k(i), W_k, i) = 0,$$  \hspace{1cm} (30c)

where $k = 1, \ldots, N,$ $\tilde{A}(i) := A(i) - \sum_{k=1}^N S_k(i)P_k(i), P_k(i), W_k, i) := P_k(i)A(i) + A^T(i)P_k(i) + A^T(i)P_k(i)A_p(i) + \sum_{j=1}^N\pi_{ij}P(j) + W_k + P_k(i)A_{hi}(i)W^{-1}A_{hi}(i)P_k(i), I(G_k(i), P_k(i), W_k, i) := (\tilde{A}(i) + A_{hi}W^{-1}A_{hi}^T(i)P_k(i))G_k(i) + G_k(i)\tilde{A}(i) + A_{hi}W^{-1}A_{hi}^T(i)P_k(i))A_p(i)G_k(i)A_p^T(i) + \pi_{ii}G_k(i) + I_n, H(G_k(i), P_k(i), W_k, i) := W_k(G_k(i) + M)W_k - A_{hi}^T(i)P_k(i)G_k(i)P_k(i)A_{hi}(i)$.

Furthermore, assume that the following two conditions hold.

i) $\tilde{A}(i) + A_{hi}W^{-1}A_{hi}^T(i)P_k(i), A_p(i) | I_n$ is stochastically detectable.

ii) $\tilde{A}(i) + A_{hi}W^{-1}A_{hi}^T(i)P_k(i), A_p(i)$ is stable.

Then, the constrained minimization of the cost bound is attained such that CSAEs (30) hold. That is,

$$\min_{(P_k(r_0), W_k)} \left[ \text{Tr} \left[ P_k(r_0) \right] + \text{Tr} \left[ MW_k \right] \right] = \text{Tr} \left[ [P_k(r_0)] + \text{Tr} \left[ MW_k \right] \right].$$  \hspace{1cm} (31)

The state feedback Nash strategy set is given below.

$$u_k(t) = \sum_{i=1}^N K_k(i)x(t)\chi_{r_i=i},$$  \hspace{1cm} (32)

$$= -\sum_{i=1}^N [R_k(i)]^{-1}B_k^T(i)P_k(i)x(t)\chi_{r_i=i},$$  \hspace{1cm} (32)

**Proof:** Now, let us consider the following problem, in which cost function (33) is minimal at $K_k(i) = K_k^*(i)$.

$$\phi(K_k(i)) := E \left[ \int_0^\infty x(t)^T(Q_k(i) + K_k^*(i)R_k(i)K_k(i)x(t)dt \right] \bigg| r_0 = i, i \right),$$  \hspace{1cm} (33)

where $x(t)$ follows from

$$dx(t) = \left[ A_{-k}(i) + B_k(i)K_k(i)x(t) + A_{hi}(i)x(t-h) \right]dt + A_p(i)x(t)dw(t), x(t) = \phi(t), t \in [-h, 0].$$  \hspace{1cm} (34)
It should be noted that function $\phi$ coincides with function $J$ in Theorem 2. Applying Theorem 2 to this minimization problem as $P_k(i) \Rightarrow P(i), W_k \Rightarrow W, G_k(i) \Rightarrow G(i), A_{-k}(i) \Rightarrow A(i), S_k(i) \Rightarrow S(i)$ and $Q_k(i) \Rightarrow Q(i)$ yields the fact that function $\phi$ is minimal at $K_k(i) = K_k^*(i)$.

V. MODE-INDEPENDENT STRATEGY SET

The results of the previous section assume the complete access for the jumping mode. However, it is generally difficult if not impossible to know the present mode accurately. Thus, a design of mode-independent strategy set is more challenging. In this section, the mode-independent strategy set is considered. Such strategies are assumed to be the following form

$$ u_k(t) = K_k x(t), \quad k = 1, \ldots, N. \tag{35} $$

In order to obtain the fixed controller gain $K_k$, the following constraints are assumed with respect to (14):

$$
\begin{align*}
K_k(1) &= K_k(2) = \cdots = K_k(s) = K_k, \\
P_k(1) &= P_k(2) = \cdots = P_k(s) = P_k.
\end{align*}
$$

It should be noted that since $\sum_{j=1}^{s} \pi_{ij} = 0$, $\sum_{j=1}^{s} \pi_{ij} P_k(j) = \sum_{j=1}^{s} \pi_{ij} P_k = 0$ under the assumption (36). Hence, we have the following corollary.

**Corollary 1:** Consider the stochastic delay systems described by (6) and cost function (7). Suppose that the following LMIs (37) have the solution set of a symmetric positive definite matrices $X_k \in \mathbb{R}^{n \times n}$, $Z_k \in \mathbb{R}^{n \times n}$ and matrix $Y_k \in \mathbb{R}^{m \times n}$,

$$
\begin{bmatrix}
\Psi_k(i) & X_k A_P^T(i) & X_k & X_k \\
A_p(i)X_k & -X_k & 0 & 0 \\
X_k & 0 & -Z_k & 0 \\
X_k & 0 & 0 & -(Q_k(i))^{-1}
\end{bmatrix} < 0, \tag{37}
$$

where $i = 1, \ldots, s, k = 1, \ldots, N$ with $\Psi_k(i) := A(i)X_k + X_k A_P^T(i) + \sum_{m=1}^{N} [B_m(i)Y_m + Y_m^T B_m^T(i)] + B_k(i)[R_k(i)]^{-1} B_k^T(i) + A_p(i) Z_k A_p^T(i), Z_k = W_k^{-1}$.

If these conditions are satisfied, then $K_k = Y_k X_k^{-1}$ is the gain matrix for the closed-loop stochastic delay systems. Furthermore, the value of (16) satisfies the following inequality.

$$
J_k(K_k^*(i)x, \ldots, K_k^N(i)x, x(0), r_0) < E[(x^T(0)X_k^{-1}x(0)]
+ E \left[ \int_0^T \phi^T(\tau) Z_k^{-1} \phi(\tau) d\tau \right]. \tag{38}
$$

**Proof:** Since this proof can be done by using the similar technique for the proof of Theorem 1, it is omitted.

The advantages of the mode-independent design method lie in the fact that it enables us to compute the proposed optimization problem without the knowledge of the instantaneous current mode. Obviously, the approach using mode-independent strategy seems to be conservative because for all modes, all LMI should be satisfied.

Nash strategy $K_k$ of (35) can be obtained by solving the matrix inequalities (37). It should be noted that the matrix inequalities (37) are cross-coupled equations and cannot be applied by using the LMI Control Toolbox with Matlab directly. Therefore, we now propose a numerical approach for solving the matrix inequalities (37).

Let us consider the following new algorithm which is based on optimization problems.

$$
\min_{X_k^{(r)}} \left( \text{Tr}(S_k^{(r+1)} + \text{Tr}(M_k^{(r+1)})) \right),
$$

$$
A_k^{(r)} \in (X_k^{(r+1)}, Z_k^{(r+1)}, S_k^{(r+1)}, T_k^{(r+1)}, Y_k^{(r+1)}) \tag{39}
$$

such that (40) and the following LMIs hold.

$$
\begin{bmatrix}
-s_k^{(r+1)} & I_n & I_n & I_n \\
I_n & -X_k^{(r+1)} & 0 & 0
\end{bmatrix} < 0,
\begin{bmatrix}
-T_k^{(r+1)} & I_n & I_n & I_n \\
I_n & -Z_k^{(r+1)} & 0 & 0
\end{bmatrix} < 0.
$$

The iterative procedure for solving the matrix inequalities (37) by means of the above-mentioned semidefinite programming (SDP) is given below:

**Step 1.** Initialization: Set $X_k^{(0)} = I_n$, $Z_k^{(0)} = I_n$ and $Y_k^{(0)} = -1/s \sum_{j=1}^{s} \sum_{m=1}^{N} [R_k(j)]^{-1} B_m(j)$. For all $i, i = 1, \ldots, N$.

**Step 2.** For $k = 1, \ldots, N$ repeat the following steps: Solve the SDP problem, with respect to $X_k^{(r)}$, subject to (40).

**Step 3.** If the algorithm converges, then $X_k^{(r+1)}$ is the solution of SDP STOP. Otherwise, increment $r \rightarrow r + 1$ and go to Step 2, until all LMIs (37) are simultaneously satisfied.

It should be noted that convergence of the above algorithm cannot be guaranteed. However, we found the proposed algorithm to work well in practice.

VI. NUMERICAL EXAMPLE

In order to demonstrate the effectiveness of the proposed method for time delay stochastic systems, we present a simple numerical result. The system matrices are given as follows.

$$
s = 2, \begin{bmatrix}
\pi_{11} & \pi_{12} \\
\pi_{21} & \pi_{22}
\end{bmatrix} = \begin{bmatrix}
-0.2 & 0.2 \\
0.8 & -0.8
\end{bmatrix},
$$

$$
A(1) = \begin{bmatrix}
-1 & 1 \\
1 & -2
\end{bmatrix}, A(2) = \begin{bmatrix}
-1 & 0 \\
1 & 1
\end{bmatrix},
$$

$$
A_h(1) = \text{block diag} \begin{bmatrix}
0.01 & 0.01
\end{bmatrix},
$$

$$
A_h(2) = \text{block diag} \begin{bmatrix}
0.02 & 0.01
\end{bmatrix},
$$

$$
A_p(1) = 0.01A(1), A_p(2) = 0.02A(2),
$$

$$
B_1(1) = \begin{bmatrix}
0 \\
1
\end{bmatrix}, B_1(2) = \begin{bmatrix}
1 \\
1
\end{bmatrix},
$$

$$
B_2(1) = \begin{bmatrix}
1 \\
1
\end{bmatrix}, B_2(2) = \begin{bmatrix}
1 \\
1
\end{bmatrix},
$$

$$
Q_k(1) = \text{block diag} \begin{bmatrix}
2 & 1
\end{bmatrix},
$$

$$
Q_k(2) = \text{block diag} \begin{bmatrix}
1 & 2
\end{bmatrix},
$$

$$
R_k(1) = 1, R_k(2) = 2, k = 1, 2, h = 1,
$$

$$
\phi(t) = \begin{bmatrix}
1 & 0.5
\end{bmatrix}, -1 \leq t \leq 0.
$$

It should be noted that the considered stochastic system cannot be treated by using the existing technique [16], [17], [18], [19], [20], [21] because the variation of each element...
of coefficient matrices are quite large and there exists a time-
delay. For example, it is verified that for (2, 2)-th element of
A(i) jumps from −2 to 1.

In this example there are two modes and it is supposed that
the state transition may occur governed by the probability
Π and the state transition may occur governed by the probability
A with an accuracy of 1.0e−7 after 57 iterations. Although the convergence of the algorithm is not
given due to adherence to the page limitation, it seems to be
linearly. The exact Nash strategies are given below.

\[
\begin{align*}
\Psi_k(i) = \begin{bmatrix}
A_k^T(i) & X_k^{(r+1)} & X_k^{(r+1)} & X_k^{(r+1)} \\
X_k^{(r+1)} & -X_k^{(r+1)} & 0 & 0 \\
X_k^{(r+1)} & 0 & -Z_k^{(r+1)} & 0 \\
X_k^{(r+1)} & 0 & 0 & -Q_k(i) \end{bmatrix}^{-1} < 0,
\end{align*}
\]

(40)

where \( r = 0, 1, \ldots, i = 1, \ldots, s, k = 1, \ldots, N \) with \( \Psi_k(i) := A(i)X_k^{(r+1)} + X_k^{(r+1)}A^T(i) + B_k(i)Y_k^{(r+1)} + \\
Y_k^{(r+1)T}B_k^T(i) + \sum_{m=1, m\neq k}^N Y_m^{(r+1)T}B_m^T(i) + B_k(i)R_k(i)^{-1}B_k(i) + A_k(i)Z_k^{(r)}A_k^T(i). \)

VII. Conclusion

The local state feedback Nash strategies for Markov
jump stochastic systems with time-delay have been studied.
In particular, the sufficient conditions for the existence of
the Nash strategies have been developed using the cross-
coupled matrix inequalities. The necessary conditions for the optimization of the cost bounds have been established on the
basis of the KKT conditions. Finally, the mode independent
Nash strategy set and the numerical algorithm were both
studied. It may be noted that the obtained solution Nash strategy
set can be attained the robust equilibrium point even if the
delay and random abrupt variations of the operating point
both exist.

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