

# Locality and information transfer in quantum operations

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## Abstract

We investigate the situation in which no information can be transferred from a quantum system  $B$  to a quantum system  $A$ , even though both interact with a common system  $C$ .

## 1 Introduction

The universe can be divided up into subsystems that interact with one another. All parts of the universe are connected, directly or indirectly, by this web of interactions. Nevertheless, to predict the future state of a small subsystem  $A$ , it is not necessary to specify the past state of the whole universe. This is what we mean by “locality” of the dynamical evolution of  $A$  within the global system.

Beckman et al. [1] have investigated a related notion of locality in the context of quantum operations. Suppose we have a bipartite system  $AB$  whose quantum state evolves according to the map  $\mathcal{E}^{AB}$ . We say that this map is *semicausal* if it cannot be used to transfer information from  $B$  to  $A$ . That is, if we begin with a joint state  $\rho^{AB}$ , perform an operation  $\mathcal{B}$  on subsystem  $B$ , and finally apply the map  $\mathcal{E}^{AB}$  to the joint system then the

final state of  $A$  alone is independent of the choice of  $\mathcal{B}$ . A *causal* map is semicausal in both directions. In [1], these notions are related to other more constructive properties of the map  $\mathcal{E}^{AB}$ . Roughly speaking, we say that the map is *semilocalizable* if it can be represented as successive interactions with a common ancilla system  $R$ : first  $A$  interacts with  $R$  and then  $B$  interacts with  $R$ . The map is *localizable* if it is semilocalizable in both directions. Because of the order of these interactions, it can be seen that a semilocalizable map is also semicausal. Beckman et al. give an example of a map that is fully causal but not localizable. In [2] it is further shown that all semicausal maps are semilocalizable.

However, the framework of [1] and [2] does not seem sufficiently general to capture the notion of locality. From the outset, it is assumed that the joint system  $AB$  is effectively isolated. (While it is true that the map  $\mathcal{E}^{AB}$  may include interaction with an external environment, a knowledge only of the past state of  $AB$  itself is sufficient to predict the future state of  $AB$ .) Furthermore, if  $\mathcal{E}^{AB}$  is semicausal, then  $A$  itself is also effectively isolated—that is, there exists a map  $\mathcal{E}^A$  that yields future  $A$  states given only past  $A$  states as input. In other words, the future state of  $AB$  is determined by the past state of  $AB$ , and no influence can propagate from  $B$  to  $A$  during the time interval. But there are many situations in which these things are not true, but for which we would say that the dynamics is local.

For example, suppose we are considering the dynamics of a classical relativistic field  $\phi$  in spacetime. “Moments of time” are spacelike hypersurfaces in our spacetime. The state of  $\phi$  in a region  $A$  of one hypersurface is completely determined by the state of  $\phi$  in a somewhat larger region  $N(A)$  of an earlier hypersurface.[3] The dynamics of this field is local, inasmuch as we can ignore the rest of the universe outside of  $N(A)$  when predicting the future field configuration on  $A$ . Yet we cannot find two nonempty spatial regions  $A$  and  $B$  so that (1) the future joint field state on  $AB$  is determined only by the past field state on  $AB$ , and (2) no influence can propagate from  $B$  to  $A$  during the time interval. See Figure 1. The definition of semicausality cannot capture the notion of locality for the evolution of this kind of system.

We need an idea of locality based on a division of the universe into *three* subsystems. See Figure 2, in which these subsystems are represented by concentric planar regions. Subsystem  $A$  is surrounded by subsystem  $C$ , which includes the rest of the dynamical “neighborhood” of  $A$ . We call  $C$  the *context* of  $A$ . To predict the final state of  $A$ , we only need to know the initial state of the composite system  $AC$ . Beyond  $A$  and its context is subsystem  $B$ , which

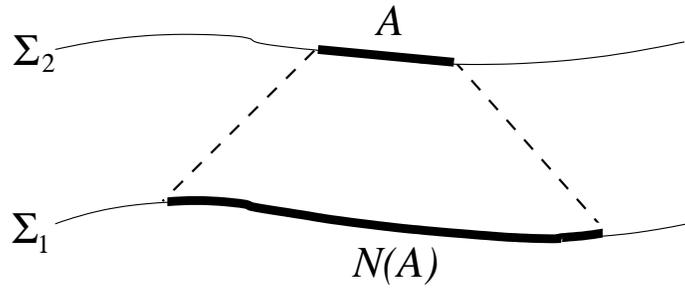


Figure 1: In spacetime, the state of a field in the region  $A$  on the later hypersurface  $\Sigma_2$  only depends on the state of the field in the region  $N(A)$  on the earlier hypersurface  $\Sigma_1$ .

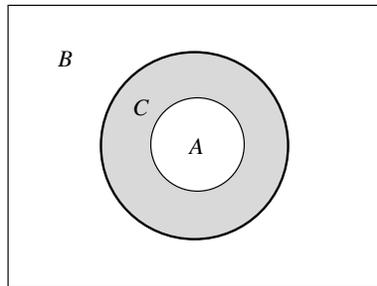


Figure 2: Three “concentric” systems.

contains the rest of our universe, and whose state is irrelevant to the final state of  $A$ .

Because the initial state of  $B$  does not affect the final state of  $A$ , no *information transfer* is possible from  $B$  to  $A$  under the dynamical evolution. We write this condition as  $B \not\leftrightarrow A$ .

In this paper we aim, first, to make precise the dynamical notion of locality in quantum mechanics and to clarify its relation to information transfer. Second, we will use these ideas to explore what sort of local dynamics is possible if the global quantum evolution is unitary.

## 2 Heuristics for quantum dynamical maps

We begin by reviewing some results about the dynamics of closed and open quantum systems. In a closed system, the evolution of the quantum state is described by a unitary operator  $U$ . An initial pure state vector  $|\psi\rangle$  evolves to a final pure state vector  $|\psi'\rangle$  according to

$$|\psi\rangle \longrightarrow |\psi'\rangle = U |\psi\rangle. \quad (1)$$

If instead we describe the initial state by a density operator  $\sigma$ , the final state is described by

$$\sigma \longrightarrow \rho = U\sigma U^\dagger. \quad (2)$$

An open quantum system interacts with its surroundings, and this interaction can lead to noise and decoherence in its time evolution. A more general description of this evolution would be a map  $\mathcal{E}$  from initial to final density operators—that is,

$$\sigma \longrightarrow \rho = \mathcal{E}(\sigma). \quad (3)$$

What properties must the map  $\mathcal{E}$  possess? It clearly must be trace-preserving, since  $\text{Tr } \sigma = \text{Tr } \rho = 1$ . (We will assume without further comment that all of our maps are trace-preserving.) Also,  $\mathcal{E}$  must be a positive map, always taking a positive operator  $\sigma$  to a positive operator  $\rho$ . Furthermore, it must be *completely positive* (CP), which means that when we extend the map to the map  $\mathbf{I} \otimes \mathcal{E}$  on a larger system, it remains positive. Physically, this means that we can append to our quantum system a second “ancilla” system that has trivial dynamics (described by the identity map  $\mathbf{I}$ ), and the overall

evolution of the composite system still takes positive density operators to positive density operators.

Every CP map  $\mathcal{E}$  has a *unitary representation*. That is, we can introduce an external “environment” system  $E$  that is initially in a standard state  $|0\rangle$  and find a unitary operator  $U$  on the composite system such that

$$\mathcal{E}(\sigma) = \text{Tr}_E U (\sigma \otimes |0\rangle\langle 0|) U^\dagger \quad (4)$$

for all  $\sigma$ . This not only gives a convenient representation for any CP map, it also makes a crucial physical point about when such maps are appropriate descriptions. The evolution can be described by a CP map only when the quantum system interacts with an external system with which it is not initially correlated. In more general situations where initial correlations may exist, we cannot treat the external system as an “environment” and derive a local CP map for the system of interest.

Any CP map  $\mathcal{E}$  also has an *operator-sum representation*, which means that there are operators  $A_\mu$  such that

$$\mathcal{E}(\sigma) = \sum_{\mu} A_{\mu} \sigma A_{\mu}^{\dagger} \quad (5)$$

for all  $\sigma$ . The operators  $A_{\mu}$  satisfy  $\sum_{\mu} A_{\mu}^{\dagger} A_{\mu} = 1$ . A given CP map has many different operator-sum representations.

When there is any chance of confusion, we indicate the particular system to which a state, operator or map applies by a superscript. Thus,  $|\psi^C\rangle$  is a pure state vector for  $C$ ,  $X^{AB}$  is an operator for the composite system  $AB$ , and  $\mathcal{E}^Q$  is a map on  $Q$  states. We will also need to consider maps *between* two distinct systems—in other words, maps that take states of a system  $X$  as input and yield states of a system  $Y$  as output. We will indicate this using both superscripts and subscripts, like so:

$$\rho^Y = \mathcal{E}_X^Y(\sigma^X). \quad (6)$$

(If the CP map is written with no subscript, the input and output spaces are the same.) The partial trace operation is a simple example of this type of map.

To specify a CP map, we would in general need to say how it acts on many different input states. However, there is a way to specify the map by

describing the action of its extension on a single input state. Let  $\mathcal{E}^Q$  be a CP map on  $Q$  states, and let us append an ancilla system  $R$  whose Hilbert space is at least as large as  $\mathcal{H}^Q$ . The composite system evolves according to  $\mathbf{I}^R \otimes \mathcal{E}^Q$ . Let  $|\Psi\rangle$  be a maximally entangled state of  $RQ$ . Then specifying the output state

$$\rho^{RQ} = \mathbf{I}^R \otimes \mathcal{E}^Q (|\Psi\rangle\langle\Psi|) \quad (7)$$

completely specifies the CP map  $\mathcal{E}^Q$ . This is a handy characterization. If we can show that two CP maps lead to the same output from a given maximally entangled input, then we can conclude that the two maps are the same.

We end this section with an observation about the states of composite systems. We call a pure state  $|\Psi^{RQ}\rangle$  a *purification* of the state  $\rho^Q$  if

$$\rho^Q = \text{Tr}_R |\Psi^{RQ}\rangle\langle\Psi^{RQ}|. \quad (8)$$

A given density operator  $\rho^Q$  will admit many possible purifications by  $R$ , provided  $\dim \mathcal{H}^R$  is at least as large as the rank of  $\rho^Q$ . If  $|\Psi_1^{RQ}\rangle$  and  $|\Psi_2^{RQ}\rangle$  are two purifications of the same state  $\rho^Q$ , then there exists a unitary operator  $V^R$  on  $\mathcal{H}^R$  such that

$$|\Psi_2^{RQ}\rangle = (V^R \otimes 1^Q) |\Psi_1^{RQ}\rangle. \quad (9)$$

In other words, any purification of a given state of  $Q$  can be turned into any other by the application of a unitary transformation that only affects the purifying system  $R$ .

### 3 Locality

How can we express the condition  $B \not\rightsquigarrow A$  more precisely? Let us imagine that  $A$ ,  $B$  and  $C$  are quantum systems, and that we have the task of predicting the future state (or the outcomes of future measurements) of  $A$ . The global evolution of the composite system  $ABC$  is described by a CP map  $\mathcal{E}^{ABC}$ .

First of all, we can say that  $B \not\rightsquigarrow A$  if the future state of  $A$  is a function of the initial quantum state of the subsystem  $AC$  only. The global initial state is described by the density operator  $\sigma^{ABC}$ , but for making  $A$  predictions we only need  $\sigma^{AC} = \text{Tr}_B \sigma^{ABC}$ . Call this condition ‘‘Locality (I)’’:

**Locality (I).** There exists a CP map  $\mathcal{E}_{AC}^A$  such that,

$$\mathcal{E}_{AC}^A \circ \text{Tr}_B = \mathcal{E}_{ABC}^A = \text{Tr}_{BC} \circ \mathcal{E}^{ABC}. \quad (10)$$

That is, for all initial  $ABC$ -states  $\sigma^{ABC}$ ,

$$\mathcal{E}_{AC}^A(\sigma^{AC}) = \text{Tr}_{BC} \mathcal{E}^{ABC}(\sigma^{ABC}). \quad (11)$$

To find the final state of subsystem  $A$ , therefore, it suffices to know only the initial state  $\sigma^{AC} = \text{Tr}_B \sigma^{ABC}$  of the subsystem  $AC$ , rather than the global state  $\sigma^{ABC}$ .

Alternately, we may focus on the special case when subsystems  $A$ ,  $B$  and  $C$  all have definite states to begin with. In this case,  $B \not\rightsquigarrow A$  means that ignorance of the initial  $B$  state will have no adverse effect on our ability to make predictions about  $A$ . This is ‘‘Locality (II)’’:

**Locality (II).** Given pure states  $|\gamma\rangle$  of  $C$  and  $|\alpha\rangle$  of  $A$ , suppose that  $|0\rangle$  and  $|1\rangle$  are two pure states (not necessarily orthogonal) of  $B$ . For  $k \in \{0, 1\}$ , let

$$\rho_k^A = \text{Tr}_{BC} \mathcal{E}^{ABC} (|\alpha\rangle\langle\alpha| \otimes |k\rangle\langle k| \otimes |\gamma\rangle\langle\gamma|). \quad (12)$$

$B \not\rightsquigarrow A$  means that, for all choices of  $|\alpha\rangle$ ,  $|\gamma\rangle$  and the  $B$ -states  $|k\rangle$ ,  $\rho_0^A = \rho_1^A$ .

Finally,  $B \rightsquigarrow A$  means that no prior intervention in the  $B$  system will affect any prediction that we make about the future state of  $A$  alone. This is ‘‘Locality (III)’’:

**Locality (III)** Suppose  $ABC$  starts in some arbitrary state  $\sigma^{ABC}$ , and suppose that  $\mathcal{F}_0$  and  $\mathcal{F}_1$  are two CP maps on  $B$  states. Given  $k \in 0, 1$ , define

$$\rho_k^A = \text{Tr}_{BC} \mathcal{E}^{ABC} (\mathcal{F}_k \otimes \mathbf{I}^{AC} (\sigma^{ABC})). \quad (13)$$

$B \rightsquigarrow A$  means that, for all choices of  $\sigma^{ABC}$  and the  $B$ -maps  $\mathcal{F}_k$ ,  $\rho_0^A = \rho_1^A$ .

We can give a heuristic summary of the three conditions as follows. Locality (I) says that ignorance (about  $B$ ) doesn’t hurt. Locality (II) says that

knowledge (of the state of  $B$ ) doesn't help. Locality (III) says that nothing we can do (to  $B$ ) will make any difference. In fact, as we will now show, these three conditions are completely equivalent, so any of them may be used as the definition for the locality of the dynamical evolution of  $C$  with context  $B$ .

Locality (III) clearly implies Locality (II), since the input state  $\sigma^{ABC}$  could possibly be a product pure state, and the operations  $\mathcal{F}_k$  could simply reset the state of  $B$  to given fixed states  $|k\rangle$ . Locality (I) also implies Locality (III). Given trace-preserving maps  $\mathcal{F}_k$  on  $B$  states, we can define

$$\sigma_k^{ABC} = \mathcal{F}_k \otimes \mathbf{I}^{AC} (\sigma^{ABC}). \quad (14)$$

From this we can see that  $\text{Tr}_B \sigma_k^{ABC} = \sigma^{AC}$ , the same state for every choice of  $k$ . By Locality (I),

$$\text{Tr}_{BC} \mathcal{E}^{ABC} (\sigma_k^{ABC}) = \mathcal{E}_{AC}^A (\text{Tr}_B \sigma_k^{ABC}) = \mathcal{E}_{AC}^A (\sigma^{AC}), \quad (15)$$

which is manifestly independent of  $k$ , and so Locality (III) holds. To show that all three conditions are independent, therefore, we need to prove that Locality (II) implies Locality (I).

For a given system, we can find an operator basis of pure states, so that any operator  $X$  can be written as a linear combination of projections:

$$X = \sum_n c_n |n\rangle\langle n|. \quad (16)$$

If the underlying Hilbert space has dimension  $d$ , then the set of pure states  $\{|n\rangle\}$  will have  $d^2$  elements. (It follows that the vectors  $|n\rangle$  cannot form an orthogonal set.) Suppose we choose states  $|\alpha\rangle$  to yield an operator basis for  $A$ ,  $|\beta\rangle$  to yield an operator basis for  $B$ , and  $|\gamma\rangle$  to yield an operator basis for  $C$ . Then the product states  $|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle$  will yield an operator basis for the composite system  $ABC$ . This means that any density operator  $\sigma^{ABC}$  can be written

$$\sigma^{ABC} = \sum_{\alpha\beta\gamma} s_{\alpha\beta\gamma} |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \otimes |\gamma\rangle\langle\gamma|. \quad (17)$$

If we take a partial trace over  $B$ , then the subsystem state  $\sigma^{AC}$  is written

$$\sigma^{AC} = \sum_{\alpha\gamma} s_{\alpha\gamma} |\alpha\rangle\langle\alpha| \otimes |\gamma\rangle\langle\gamma| \quad (18)$$

where  $s_{\alpha\gamma} = \sum_{\beta} s_{\alpha\beta\gamma}$ .

Suppose Locality (II) holds for the evolution  $\mathcal{E}^{ABC}$ . We wish to construct the map  $\mathcal{E}_{AC}^A$  that takes initial  $AC$ -states to final  $A$ -states. Fix a particular  $B$ -state  $|0\rangle$  (which should be one of the states  $|\beta\rangle$  that give the operator basis), and define

$$\mathcal{E}_{AC}^A(\sigma^{AC}) = \text{Tr}_{BC} \mathcal{E}^{ABC} (|0\rangle\langle 0| \otimes \sigma^{AC}). \quad (19)$$

This is by construction a trace-preserving CP map. Now, for any states  $|\alpha\rangle$ ,  $|\beta\rangle$  and  $|\gamma\rangle$ , Locality (II) implies that

$$\text{Tr}_{BC} \mathcal{E}^{ABC} (|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \otimes |\gamma\rangle\langle\gamma|) = \text{Tr}_{BC} \mathcal{E}^{ABC} (|\alpha\rangle\langle\alpha| \otimes |0\rangle\langle 0| \otimes |\gamma\rangle\langle\gamma|). \quad (20)$$

Therefore, for any initial  $ABC$ -state  $\sigma^{ABC}$ ,

$$\begin{aligned} \text{Tr}_{BC} \mathcal{E}^{ABC} (\sigma^{ABC}) &= \sum_{\alpha\beta\gamma} s_{\alpha\beta\gamma} \text{Tr}_{BC} \mathcal{E}^{ABC} (|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \otimes |\gamma\rangle\langle\gamma|) \\ &= \sum_{\alpha\beta\gamma} s_{\alpha\beta\gamma} \text{Tr}_{BC} \mathcal{E}^{ABC} (|\alpha\rangle\langle\alpha| \otimes |0\rangle\langle 0| \otimes |\gamma\rangle\langle\gamma|) \\ &= \sum_{\alpha\gamma} s_{\alpha\gamma} \text{Tr}_{BC} \mathcal{E}^{ABC} (|\alpha\rangle\langle\alpha| \otimes |0\rangle\langle 0| \otimes |\gamma\rangle\langle\gamma|) \\ &= \text{Tr}_{BC} \mathcal{E}^{ABC} (|0\rangle\langle 0| \otimes \sigma^{AC}) \\ &= \mathcal{E}_{AC}^A(\sigma^{AC}). \end{aligned} \quad (21)$$

The map  $\mathcal{E}_{AC}^A$  thus satisfies the requirement of Locality (I). The three conditions are all equivalent, as promised. Each of them captures the notion of the locality of the dynamical evolution of  $A$  with context  $C$ .

Suppose that an independent system  $R$  is appended to  $ABC$ , so that the overall system evolves according to  $\mathcal{E}^{ABC} \otimes \mathbf{I}^R$ , where  $\mathbf{I}^R$  is the identity map. Then if  $B \not\rightsquigarrow A$ , a straightforward derivation using Locality (I) shows that  $B \not\rightsquigarrow AR$  and  $BR \not\rightsquigarrow A$ . Note that this is a statement about the CP maps and remains true even if the initial quantum state has entanglement between  $R$  and  $ABC$ .

## 4 Precursor subspaces

Our next task is to explore some of the implications of locality in the evolution of quantum systems. To do this, we will find it convenient (as we will see in

the next section) to introduce the idea of a *precursor subspace*.

Let  $\mathcal{E}$  be a trace-preserving CP map on density operators. (We make no assumption about the input and output states of  $\mathcal{E}$ ; these may be states of the same system, or of different systems.) It may happen that  $\mathcal{E}$  takes a pure input state to a pure output state:

$$\mathcal{E}(|\phi\rangle\langle\phi|) = |\psi\rangle\langle\psi|. \quad (22)$$

In this case, we say that  $|\phi\rangle$  is a *precursor* of  $|\psi\rangle$  under  $\mathcal{E}$ . In this section we make some observations about pure states and their precursors.

Suppose the operators  $A_\mu$  give an operator sum representation for  $\mathcal{E}$ . If  $|\phi\rangle$  is a precursor of  $|\psi\rangle$  under  $\mathcal{E}$ , then for all  $\mu$ ,

$$A_\mu |\phi\rangle = a_\mu |\psi\rangle \quad (23)$$

where the  $a_\mu$ 's are scalars. To see this, let  $|\hat{\psi}_\mu\rangle = A_\mu |\phi\rangle$ . (The “hat” reminds us that this vector will not in general be normalized, even if  $|\phi\rangle$  is.) Then

$$|\psi\rangle\langle\psi| = \mathcal{E}(|\phi\rangle\langle\phi|) \quad (24)$$

$$= \sum_{\mu} A_\mu |\phi\rangle\langle\phi| A_\mu^\dagger \quad (25)$$

$$= \sum_{\mu} |\hat{\psi}_\mu\rangle\langle\hat{\psi}_\mu|. \quad (26)$$

The only way that the positive operators  $|\hat{\psi}_\mu\rangle\langle\hat{\psi}_\mu|$  could sum to the rank-1 projection  $|\psi\rangle\langle\psi|$  would be if each of them were multiples of  $|\psi\rangle\langle\psi|$ . This means that  $|\hat{\psi}_\mu\rangle = a_\mu |\psi\rangle$  for every  $\mu$ .

The converse of this is also true. If  $A_\mu |\phi\rangle = a_\mu |\psi\rangle$  for all  $\mu$ , then  $\mathcal{E}(|\phi\rangle\langle\phi|) = |\psi\rangle\langle\psi|$ . (The only issue here is normalization, which follows from the fact that  $\mathcal{E}$  is trace-preserving.)

For a state vector  $|\phi\rangle$  in the output space, we define

$$\mathcal{S}_\psi = \left\{ |\hat{\phi}\rangle : \mathcal{E}(|\hat{\phi}\rangle\langle\hat{\phi}|) = \lambda |\psi\rangle\langle\psi|, \lambda \geq 0 \right\}. \quad (27)$$

This is the set of input vectors which are (up to normalization) precursors of  $|\psi\rangle$ . This set  $\mathcal{S}_\psi$  is a subspace, as can be seen from the previous fact. Pick

an operator sum representation for  $\mathcal{E}$  given by operators  $A_\mu$ . If  $|\phi\rangle$  and  $|\phi'\rangle$  are in  $\mathcal{S}_\psi$ , then

$$A_\mu (\alpha |\phi\rangle + \alpha' |\phi'\rangle) = a_\mu \alpha |\psi\rangle + a'_\mu \alpha' |\psi\rangle = b_\mu |\psi\rangle. \quad (28)$$

Therefore,  $\mathcal{E}$  will take the superposition of  $|\phi\rangle$  and  $|\phi'\rangle$  to a multiple of  $|\psi\rangle$ , and so the superposition lies in  $\mathcal{S}_\psi$ . The set  $\mathcal{S}_\psi$  is therefore a subspace. We call this the *precursor subspace* of  $|\psi\rangle$ . Notice that, even though the map  $\mathcal{E}$  acts on operators, the precursor subspace exists in the underlying Hilbert space.

Given a map  $\mathcal{E}$  and any  $|\psi\rangle$ , there is a precursor subspace  $\mathcal{S}_\psi$ . However, it may be the case that this subspace is null. For example, suppose we have a qubit whose pure states are spanned by computational basis states  $|0\rangle$  and  $|1\rangle$ . Consider the map  $\mathcal{E}$  which takes every input state  $\sigma$  to  $\mathcal{E}(\sigma) = |0\rangle\langle 0|$ . Then the precursor space of  $|0\rangle$  is the whole Hilbert space for the qubit, but the precursor space of any other state will be null.

How are the precursor subspaces for two distinct pure states related to each other? It is easy to see that the two precursor subspaces can only intersect in the null space. Now suppose that  $|\phi\rangle$  and  $|\phi'\rangle$  are precursors for  $|\psi\rangle$  and  $|\psi'\rangle$ , respectively. Since fidelity is monotonic under CP maps [4],

$$|\langle\phi|\phi'\rangle|^2 \leq |\langle\psi|\psi'\rangle|^2. \quad (29)$$

As a corollary, if  $|\psi\rangle$  and  $|\psi'\rangle$  are orthogonal, their precursors must also be orthogonal. The precursor subspaces for orthogonal states are orthogonal subspaces.

## 5 Autonomy

Suppose quantum system  $A$  is described by a Hilbert space  $\mathcal{H}^A$  of dimension  $d_A$ . Every trace-preserving CP map  $\mathcal{E}$  on  $A$  has a unitary representation—in fact, many different unitary representations, employing environment systems of various sizes. However, any CP map on  $A$  states can be represented using an environment system  $E$  whose Hilbert space dimension is no larger than  $d_A^2$ . We can classify the maps by their *rank*, the Hilbert space dimension of the smallest environment needed to give a unitary representation. For any  $\mathcal{E}$ ,

$$1 \leq \text{rank } \mathcal{E} \leq d_A^2. \quad (30)$$

(The rank of  $\mathcal{E}$  is also the minimum number of operators required for an operator-sum representation of  $\mathcal{E}$ .) The minimal-rank operations are those which require no environment system at all—that is, the maps that are already unitary.

What are the minimal-rank operations in the case where the input and output states belong to different systems? Consider a CP map  $\mathcal{E}_{AC}^A$  that takes states of a composite system  $AC$  to states of its subsystem  $A$ . Systems  $A$  and  $C$  are described by Hilbert spaces of dimension  $d_A$  and  $d_C$ , respectively, and the Hilbert space for  $AC$  has dimension  $d_A d_C$ . The minimal-rank operations of this type are those that do not require an external environment for their unitary representation. We call this property *autonomy*:

**Autonomy.** The map  $\mathcal{E}_{AC}^A$  is *autonomous* if there exists a unitary operator  $U$  on  $AC$  such that

$$\mathcal{E}_{AC}^A(\sigma) = \text{Tr}_C U \sigma U^\dagger. \quad (31)$$

for any  $AC$ -state  $\sigma$ . In other words, a unitary representation for an autonomous CP map does not require the introduction of any additional environment system.

It will turn out that autonomy is equivalent to two other technical conditions on the map  $\mathcal{E}_{AC}^A$ , which are:

**Uniform dimension condition (UDC).** The map  $\mathcal{E}_{AC}^A$  satisfies the *uniform dimension condition* if, for any  $|\psi\rangle \in \mathcal{H}^A$ ,  $\dim \mathcal{S}_\psi = d_C$ .

**Output rank condition (ORC).** Suppose we add an ancilla system  $R$  to  $AC$  and prepare the overall system in an initial state  $|\Psi\rangle$  in which  $R$  is maximally entangled with  $AC$ . The entire system evolves according to the map  $\mathbf{I}^R \otimes \mathcal{E}_{AC}^A$ , leading to a final state  $\rho$  of  $RA$ . We say that  $\mathcal{E}_{AC}^A$  satisfies the *output rank condition* if  $\rho$  has rank  $d_C$ .

To show that these three are equivalent, we will prove that autonomy implies the UDC, the ORC implies autonomy, and the UDC implies the ORC.

**Autonomy  $\Rightarrow$  UDC.** Suppose  $\mathcal{E}_{AC}^A$  is autonomous, with  $U$  being the implied unitary operator on  $AC$ . Let  $|\psi\rangle \in \mathcal{H}^A$ . Then

$$\mathcal{S}_\psi = \{U^\dagger (|\gamma\rangle \otimes |\psi\rangle) : |\gamma\rangle \in \mathcal{H}^C\}. \quad (32)$$

This clearly has dimension  $d_C$ .

**ORC  $\Rightarrow$  Autonomy.** Suppose we add the ancilla system  $R$ , and start with the maximally entangled state  $|\Psi\rangle$  of  $RAC$ , which maps under  $\mathbf{I}^R \otimes \mathcal{E}_{AC}^A$  to the density operator  $\rho^{RA}$ . Also suppose that  $\text{rank } \rho = d_C$ . Then we can purify the final state  $\rho^{RA}$  by appending a system of dimension  $d_C$ —in particular, by appending  $C$  itself. This yields a pure state  $|\Psi'\rangle$  such that

$$\rho^{RA} = \text{Tr}_C |\Psi'\rangle\langle\Psi'|. \quad (33)$$

The state of  $R$  alone has not changed under the evolution by  $\mathbf{I} \otimes \mathcal{E}_{AC}^A$ . Both  $|\Psi\rangle$  and  $|\Psi'\rangle$  are purifications of the same state of  $R$ , and hence are related by some unitary operator  $U^{AC}$  on  $\mathcal{H}^{AC}$ . Thus,

$$|\Psi'\rangle = (1^R \otimes U^{AC}) |\Psi\rangle. \quad (34)$$

The unitary operator  $U^{AC}$ , together with the partial trace over  $C$ , defines an autonomous CP map from  $AC$  states to  $A$  states. But such a map is completely specified by its action on a single maximally entangled input state of  $RAC$ , namely  $|\Psi\rangle$ . Thus, this map must be the same as  $\mathcal{E}_{AC}^A$  itself, and so  $\mathcal{E}_{AC}^A$  is autonomous.

**UDC  $\Rightarrow$  ORC.** It remains to show that the uniform dimension condition implies the output rank condition. The ORC states that, for an input state  $|\Psi\rangle$  that is maximally entangled between  $R$  and  $AC$ , the output state  $\rho^{RA} = \mathbf{I}^R \otimes \mathcal{E}_{AC}^A (|\Psi\rangle\langle\Psi|)$  has rank  $d_C$ .

In fact, without any assumptions about  $\mathcal{E}_{AC}^A$ , we can show that this output state has rank at least  $d_C$ . We can write

$$\rho^{RA} = \sum_{\alpha} P_{\alpha} |\Phi_{\alpha}\rangle\langle\Phi_{\alpha}|, \quad (35)$$

where  $\alpha$  runs from 1 to  $\text{rank } \rho^{RA}$ , and the states  $|\Phi_{\alpha}\rangle$  are pure entangled states of  $RA$ . If we let

$$\begin{aligned} \rho_{\alpha}^R &= \text{Tr}_A |\Phi_{\alpha}\rangle\langle\Phi_{\alpha}| \\ \rho_{\alpha}^A &= \text{Tr}_R |\Phi_{\alpha}\rangle\langle\Phi_{\alpha}| \end{aligned} \quad (36)$$

then  $\text{rank } \rho_{\alpha}^R = \text{rank } \rho_{\alpha}^A \leq d_A$ . Since

$$\rho^R = \sum_{\alpha} P_{\alpha} \rho_{\alpha}^R, \quad (37)$$

it follows that  $\text{rank } \rho^R \leq (\text{rank } \rho^{RA})d_A$ . In our case, the input state  $|\Psi\rangle$  is maximally entangled between  $R$  and  $AC$  and the system  $R$  evolves according to the identity map  $\mathbf{I}^R$ , so that  $\text{rank } \rho^R = d_A d_C$ . Therefore,  $\text{rank } \rho^{RA} \geq d_C$ .

Now we show that if  $\mathcal{E}_{AC}^A$  satisfies the uniform dimension condition, then  $\text{rank } \rho^{RA} \leq d_C$  as well. This will require a much lengthier proof. Our argument is based on the following general fact. Suppose  $\mathcal{E}$  is a CP map with operator-sum representation  $\{A_\mu\}$ , and suppose we have vectors  $|\alpha_1\rangle, \dots, |\alpha_n\rangle$  which are precursor states to pure states:

$$\mathcal{E}(|\alpha_k\rangle\langle\alpha_k|) = |\phi_k\rangle\langle\phi_k| \quad (38)$$

for  $k = 1, \dots, n$ . Let  $|\phi\rangle = \sum_k c_k |\alpha_k\rangle$ . Then the density operator

$$\rho = \mathcal{E}(|\phi\rangle\langle\phi|) \quad (39)$$

has rank no larger than  $n$ . To see this, note first that

$$A_\mu |\alpha_k\rangle = \beta_{k\mu} |\phi_k\rangle. \quad (40)$$

Then

$$\rho = \sum_{k,k'} \left( \sum_{\mu} \beta_{k\mu} \beta_{k'\mu}^* c_k c_{k'}^* \right) |\phi_k\rangle\langle\phi_{k'}|. \quad (41)$$

This operator obviously has support contained in the subspace spanned by the image states  $|\phi_k\rangle$ , which has dimension no larger than  $n$ . Thus  $\text{rank } \rho \leq n$ .

Our plan is to write a maximally entangled input state of  $RAC$  as a superposition of  $d_C$  states that are precursors of pure states under  $\mathbf{I}^R \otimes \mathcal{E}_{AC}^A$ . It will follow that  $\text{rank } \rho^{RA} \leq d_C$ .

Now for the details. Suppose that  $\mathcal{E}_{AC}^A$  satisfies the UDC. Pick an orthonormal basis  $\{|k\rangle\}$  for  $\mathcal{H}^A$ . For each  $|k\rangle$ , we have a precursor subspace  $\mathcal{S}_k$ . There are  $d_A$  such subspaces, and they must be orthogonal to each other (since otherwise two non-orthogonal input states could map to orthogonal output states).

Assuming the uniform dimension condition, each of the precursor subspaces has dimension  $d_C$ . We will now construct another subspace  $\mathcal{T}_1$  of dimension  $d_A$  that ‘‘cuts across’’ the precursor subspaces in a special way. To begin with, we note that any vector  $|\phi\rangle \in \mathcal{H}^{AC}$  can be written  $|\phi\rangle =$

$\sum_k a_k |\phi_k\rangle$ , where  $|\phi_k\rangle \in \mathcal{S}_k$ . Furthermore, for a given  $|\phi\rangle$ , the (normalized) vectors  $|\phi_k\rangle$  are unique up to phase.

Now pick a particular  $|\psi\rangle \in \mathcal{H}^A$  so that  $|\psi\rangle = \sum_k c_k |k\rangle$  with  $c_k \neq 0$ .

Let  $|\phi\rangle$  be some precursor of  $|\psi\rangle$ . We can write the precursor state  $|\phi\rangle$  as a superposition of states in the subspaces  $\mathcal{S}_k$ :

$$|\phi\rangle = \sum_k a_k |\phi_k\rangle. \quad (42)$$

Since the magnitudes of inner products cannot decrease under  $\mathcal{E}$ , we know that  $|a_k|^2 \geq |c_k|^2$ . But since both  $|\psi\rangle$  and its precursor  $|\phi\rangle$  are normalized,

$$\sum_k |a_k|^2 = \sum_k |c_k|^2 = 1. \quad (43)$$

Therefore,  $|a_k|^2 = |c_k|^2$  for all values of  $k$ . By adjusting the phases of the  $|\phi_k\rangle$  basis precursor states, we can arrange for  $a_k = c_k$ . Once we have done this, the precursor of  $|\psi\rangle = \sum_k c_k |k\rangle$  will be

$$|\phi\rangle = \sum_k c_k |\phi_k\rangle. \quad (44)$$

Our subspace  $\mathcal{T}_1$  is the subspace spanned by the basis precursor states  $|\phi_k\rangle$  that we have chosen. There are  $d_A$  of these, so that is the dimension of  $\mathcal{T}_1$ . Our next step is to show that *any* pure state in  $\mathcal{T}_1$  is a precursor of some pure state in  $\mathcal{H}^A$ . Introduce an operator-sum representation for the map  $\mathcal{E}_{AC}^A$ , given by operators  $A_\mu$ . (The operators  $A_\mu$  act on vectors in  $\mathcal{H}^{AC}$  and map them to vectors in  $\mathcal{H}^A$ .) From Equation 23, we see that

$$\begin{aligned} A_\mu |\phi\rangle &= \alpha_\mu |\psi\rangle \\ A_\mu |\phi_k\rangle &= \beta_{\mu k} |k\rangle \end{aligned} \quad (45)$$

for some scalars  $\alpha_\mu$  and  $\beta_{\mu k}$ . Writing  $|\phi\rangle$  in terms of the basis precursors  $|\phi_k\rangle$  and  $|\psi\rangle$  in terms of the basis states  $|k\rangle$ , we obtain

$$\sum_k \alpha_\mu c_k |k\rangle = \sum_k \beta_{\mu k} c_k |k\rangle. \quad (46)$$

This implies that  $\alpha_\mu = \beta_{\mu k}$  for all values of  $k$  and  $\mu$ .

Now consider another vector  $|\phi'\rangle$  in  $\mathcal{T}_1$ , which is a superposition of our basis precursor states:

$$|\phi'\rangle = \sum_k c'_k |\phi_k\rangle. \quad (47)$$

The operators of the operator-sum representation act on this vector to yield

$$A_\mu |\phi'\rangle = \alpha_\mu \sum_k c'_k |k\rangle = \alpha_\mu |\psi'\rangle, \quad (48)$$

where  $|\psi'\rangle = \sum_k c'_k |k\rangle$ . This in turn implies that

$$\mathcal{E}_{AC}^A(|\phi'\rangle\langle\phi'|) = |\psi'\rangle\langle\psi'|. \quad (49)$$

To sum up, we have found a subspace  $\mathcal{T}_1$  such that every vector in  $\mathcal{H}^A$  has a precursor in  $\mathcal{T}_1$ , every vector in  $\mathcal{T}_1$  is the precursor of some vector in  $\mathcal{H}^A$ , and the relation between precursor and image is linear. Furthermore, the intersection of  $\mathcal{T}_1$  with any precursor subspace  $\mathcal{S}_\psi$  is one-dimensional. We let  $|\Upsilon_{1k}\rangle = |\phi_k\rangle$ ; the vectors  $|\Upsilon_{1k}\rangle$  form a basis for  $\mathcal{T}_1$ .

Now we turn our attention to  $\mathcal{T}_1^\perp$ , which is a subspace of  $\mathcal{H}^{AC}$  of dimension  $(d_C - 1)d_A$ . Every state  $|\psi\rangle$  in  $\mathcal{H}^A$  will have a precursor subspace within  $\mathcal{T}_1^\perp$  of dimension  $d_C - 1$ . Therefore,  $\mathcal{T}_1^\perp$  satisfies a uniform dimension condition with a reduced precursor subspace dimension  $d_C - 1$ . This in turn means that we can repeat our process to arrive at a new subspace  $\mathcal{T}_2$  orthogonal to  $\mathcal{T}_1$  such that every vector in  $\mathcal{T}_2$  is a precursor of some pure state and the relation between precursor and image is linear. Also, the intersection of  $\mathcal{T}_2$  and any precursor subspace  $\mathcal{S}_\psi$  will be one-dimensional. We let the vectors  $|\Upsilon_{2k}\rangle$  in  $\mathcal{T}_2$  be the precursors of the basis vectors  $|k\rangle$ .

We can generalize this process. At the  $n$ th stage, we find the subspace  $\mathcal{T}_n$  that is perpendicular to the linear span of  $\mathcal{T}_1$  through  $\mathcal{T}_{n-1}$ . The new subspace  $\mathcal{T}_n$  has dimension  $d_A$ , and each of its elements is a precursor of some state in  $\mathcal{H}^A$ . The relation between precursor in  $\mathcal{T}_n$  and image in  $\mathcal{H}^A$  is linear. Every precursor subspace  $\mathcal{S}_\psi$  in  $\mathcal{H}^{AC}$  has a one-dimensional intersection with  $\mathcal{T}_n$ . Finally, we identify basis vectors  $|\Upsilon_{nk}\rangle$  that are precursors of basis vectors  $|k\rangle$ .

Now introduce the ancilla system  $R$  and let the whole system evolve according to the map  $\mathbf{I}^R \otimes \mathcal{E}_{AC}^A$ . Imagine that we have an input pure state

$$|\Psi_n\rangle = \sum_k c_k |\alpha_{nk}^R\rangle \otimes |\Upsilon_{nk}\rangle. \quad (50)$$

In other words, the input state  $|\Psi_n\rangle$  is an entangled state whose support in  $\mathcal{H}^{AC}$  is entirely contained in  $\mathcal{T}_n$ . By our construction of the subspace  $\mathcal{T}_n$ ,

$$\mathbf{I}^R \otimes \mathcal{E}_{AC}^A (|\Psi_n\rangle\langle\Psi_n|) = |\Phi_n\rangle\langle\Phi_n| \quad (51)$$

where

$$|\Phi_n\rangle = \sum_k c_k |\alpha_{nk}^R\rangle \otimes |k\rangle. \quad (52)$$

A pure entangled state that is supported within  $\mathcal{T}_n$  maps to a pure entangled output state.

Now, any entangled input state  $|\Psi\rangle$  of  $RAC$  can be written as

$$|\Psi\rangle = \sum_n a_n |\Psi_n\rangle, \quad (53)$$

where the  $|\Psi_n\rangle$  states have  $AC$  support in  $\mathcal{T}_n$ . That is,  $|\Psi\rangle$  is a superposition of  $d_B$  states each of which is a precursor of some pure state under  $\mathbf{I}^R \otimes \mathcal{E}_{AC}^A$ . The output state

$$\rho^{RA} = \mathbf{I}^R \otimes \mathcal{E}_{AC}^A (|\Psi\rangle\langle\Psi|) \quad (54)$$

therefore has a rank no larger than  $d_C$ , as we wished to prove.

We have now shown that  $\text{rank } \rho^{RA} \geq d_C$  and  $\text{rank } \rho^{RA} \leq d_C$ , so  $\text{rank } \rho^{RA} = d_C$ , and the output rank condition (ORC) holds for  $\mathcal{E}_{AC}^A$ . Autonomy, the UDC and the ORC are all equivalent conditions on  $\mathcal{E}_{AC}^A$ .

Autonomy will prove to be a useful idea when considering locality in systems which have unitary global evolution. This is the subject of the next section.

## 6 Global unitarity

Return to the situation in which the joint system  $ABC$  evolves according to  $\mathcal{E}^{ABC}$  such that  $B \not\leftrightarrow A$ . This implies the existence of a local CP map  $\mathcal{E}_{AC}^A$ . What can we say about the local evolution  $\mathcal{E}_{AC}^A$  if we know that the global evolution  $\mathcal{E}^{ABC}$  is in fact unitary?

Let  $|\psi\rangle$  be a pure output state of  $A$ . We wish to consider  $|\psi\rangle$  as an output of each of the maps  $\mathcal{E}_{AC}^A$  and  $\mathcal{E}_{ABC}^A$ . Let  $\mathcal{S}_\psi^{AC}$  and  $\mathcal{S}_\psi^{ABC}$  be the precursor subspaces of  $|\psi\rangle$  for these two maps. If  $|\phi\rangle \in \mathcal{S}_\psi^{AC}$ , then every vector of the form  $|\beta\rangle \otimes |\phi\rangle$  (where  $|\beta\rangle$  is a  $B$  state) must be in  $\mathcal{S}_\psi^{ABC}$ . This implies that  $\mathcal{H}^B \otimes \mathcal{S}_\psi^{AC} \subseteq \mathcal{S}_\psi^{ABC}$ .

Conversely, suppose that  $|\Phi\rangle \in \mathcal{S}_\psi^{ABC}$ . This state may have entanglement between  $B$  and  $AC$ , but in any case we can write it as

$$|\Phi\rangle = \sum_k \alpha_k |k\rangle \otimes |\phi_k\rangle \quad (55)$$

where the  $|k\rangle$  states are an orthonormal basis for  $\mathcal{H}^B$ . The partial trace over  $B$  of this state yields the mixed  $AC$  state

$$\sigma^{AC} = \sum_k |\alpha_k|^2 |\phi_k\rangle\langle\phi_k|. \quad (56)$$

We know that  $\mathcal{E}_{AC}^A(\sigma^{BC}) = |\psi\rangle\langle\psi|$ . Therefore, it must be that  $|\phi_k\rangle \in \mathcal{S}_\psi^{AC}$  for every  $k$  with  $\alpha_k \neq 0$ , and

$$|\Phi\rangle \in \mathcal{H}^B \otimes \mathcal{S}_\psi^{AC}. \quad (57)$$

Thus  $\mathcal{S}_\psi^{ABC} = \mathcal{H}^B \otimes \mathcal{S}_\psi^{AC}$ .

The map  $\mathcal{E}_{ABC}^A$  is clearly autonomous, and so satisfies the uniform dimension condition. The precursor subspace  $\mathcal{S}_\psi^{ABC}$  has dimension  $d_{BC} = d_B d_C$ . Since this subspace is a tensor product of  $\mathcal{H}^B$  and  $\mathcal{S}_\psi^{AC}$ , we find that  $\dim \mathcal{S}_\psi^{AC} = d_C$ . Because this is independent of the choice of  $|\psi\rangle$ , we see that  $\mathcal{E}_{AC}^A$  satisfies the uniform dimension condition. In short, if  $\mathcal{E}^{ABC}$  is unitary and  $B \not\leftrightarrow A$ , then  $\mathcal{E}_{AC}^A$  must be autonomous.

Furthermore, we can show that such a unitary map decomposes in an elegant way. Let  $U$  be the unitary operator acting on  $\mathcal{H}^{ABC}$  that gives rise to the unitary map  $\mathcal{E}^{ABC}$ . We append ancilla systems  $R_B$  and  $R_{AC}$  and prepare the initial state so that  $B$  and  $R_B$  are maximally entangled, as are  $AC$  and  $R_{AC}$ . (This of course means that  $R_B R_{AC}$  is maximally entangled with  $ABC$ .) Call this input state  $|\Psi\rangle$ .

Now we construct two scenarios. In the first, our composite system evolves according to the unitary operator  $U$ . (Technically, for the entire system  $ABC R_B R_{AC}$  this would be the operator  $U^{ABC} \otimes 1^{R_B} \otimes 1^{R_{AC}}$ .) At the end, the joint state is  $|\Phi_U\rangle$ .

In the second scenario, we consider a unitary operator  $V$  that gives a unitary representation of  $\mathcal{E}_{AC}^A$ . Since we have shown that  $\mathcal{E}_{AC}^A$  is autonomous,  $V$  is chosen only to act on  $\mathcal{H}^{AC}$ . That is, for any  $AC$  state  $\sigma$ ,

$$\mathcal{E}_{AC}^A(\sigma) = \text{Tr}_B V \sigma V^\dagger. \quad (58)$$

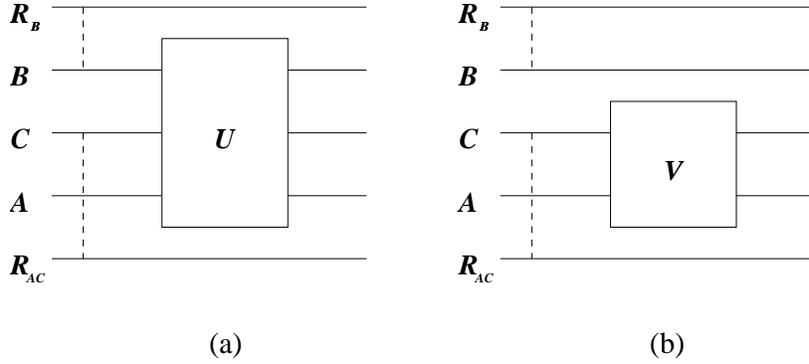


Figure 3: Two scenarios for the evolution of the five-part system  $ABCR_B R_{AC}$ . In (a),  $ABC$  evolves by  $U$ , while in (b) only  $AC$  evolves according to  $V$ . Dotted lines indicate the initial entanglement of the system.

The final state in this case is  $|\Phi_V\rangle$ . The two scenarios are illustrated as circuit diagrams in Figure 3.

How do the two final global states  $|\Phi_U\rangle$  and  $|\Phi_V\rangle$  differ from one another? By the definition of the local map  $\mathcal{E}_{AC}^A$ ,

$$\mathcal{E}_{AC}^A \circ \text{Tr}_B = \mathcal{E}_{ABC}^A = \text{Tr}_{BC} \circ \mathcal{E}^{ABC} \quad (59)$$

(Equation 10 above). Thus for our extended system,

$$\text{Tr}_{BC} \circ (\mathbf{I}^{R_B} \otimes \mathcal{E}^{ABC} \otimes \mathbf{I}^{R_{AC}}) = (\mathcal{E}_{AC}^A \otimes \mathbf{I}^{R_B} \otimes \mathbf{I}^{R_{AC}}) \circ \text{Tr}_B. \quad (60)$$

It follows that the final state of the subsystem  $AR_B R_{AC}$  is

$$\rho^{AR_B R_{AC}} = \text{Tr}_{BC} |\Phi_U\rangle\langle\Phi_U| = \text{Tr}_{BC} |\Phi_V\rangle\langle\Phi_V|. \quad (61)$$

The states  $|\Phi_U\rangle$  and  $|\Phi_V\rangle$  are thus purifications of the same marginal state on the subsystem  $AR_B R_{AC}$ . This implies that there is a unitary operator  $W$  that acts only on the complementary system  $BC$  such that

$$\begin{aligned} |\Phi_U\rangle &= (W^{BC} \otimes 1^{R_B} \otimes 1^{AR_{AC}}) |\Phi_V\rangle \\ (U^{ABC} \otimes 1^{R_B} \otimes 1^{R_{AC}}) |\Psi\rangle &= (W^{BC} \otimes 1^{R_B} \otimes 1^{AR_{AC}}) \\ &\quad \times (1^{R_B B} \otimes V^{AC} \otimes 1^{R_{AC}}) |\Psi\rangle \end{aligned} \quad (62)$$

Since this is true for the input state  $|\Psi\rangle$ , which is maximally entangled between  $R_B R_{AC}$  and  $ABC$ , it follows that

$$U^{ABC} = (W^{BC} \otimes 1^A) (1^B \otimes V^{AC}). \quad (63)$$

We have shown that any unitary map  $\mathcal{E}^{ABC}$  for which  $B \not\leftrightarrow A$  can be decomposed as shown in Figure 4. In this decomposition, systems  $A$  and  $C$  interact

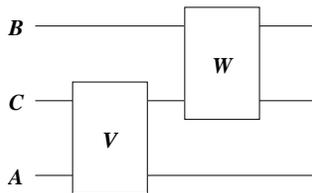


Figure 4: Decomposition of a unitary operator for which  $B \not\leftrightarrow A$ .

first, and then systems  $B$  and  $C$  interact. This causal structure clearly guarantees that no information can be transferred from  $B$  to  $A$ ; we have now shown that this sort of structure is the *only* way to guarantee  $B \not\leftrightarrow A$  in a unitary map.

The causal structure illustrated in Figure 4 could apply to more general CP maps as well. If an overall map  $\mathcal{E}^{ABC}$  could be decomposed as shown in Figure 5, then it would be clearly true that  $B \not\leftrightarrow A$ . But does the converse hold? If  $B \not\leftrightarrow A$  in this more general context, can we always decompose  $\mathcal{E}^{ABC}$  as shown in Figure 5?

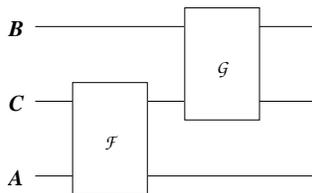


Figure 5: Decomposition of an arbitrary CP map implying that  $B \not\leftrightarrow A$ .

The answer is no. It is easy to come up with a CP map on  $ABC$  for which  $B \not\leftrightarrow A$ , but which cannot be written in this way. Consider for instance a map in which a measurement is performed on  $A$ , and its result is written in

the state of  $B$  (erasing any previous state). System  $C$  evolves via the identity map. This example cannot be decomposed in the way suggested by Figure 5, but clearly  $B \not\rightsquigarrow A$ .

On the other hand, we can find a unitary representation for any CP map. If  $B \not\rightsquigarrow A$ , what can we say about the structure of such a representation? First, let us consider the case of two systems  $A$  and  $B$  which evolve according to a global CP map  $\mathcal{E}^{AB}$ , and for which  $B \not\rightsquigarrow A$ . To construct a unitary representation for  $\mathcal{E}^{AB}$ , we introduce an environment system  $E$  in a standard initial state  $|0\rangle$ . Then there exists a unitary operator  $U$  on  $ABE$  such that

$$\mathcal{E}^{AB}(\sigma^{AB}) = \text{Tr}_E (U (\sigma^{AB} \otimes |0\rangle\langle 0|) U^\dagger) \quad (64)$$

for any  $\sigma^{AB}$ . Since  $B \not\rightsquigarrow A$ , there is a local map  $\mathcal{E}^A$ , and this local map itself has a unitary representation. Appending the environment  $E$  initially in  $|0\rangle$ , there is a unitary  $V$  acting on  $\mathcal{H}^{AE}$  so that

$$\mathcal{E}^A(\sigma^A) = \text{Tr}_E (V (\sigma^A \otimes |0\rangle\langle 0|) V^\dagger) \quad (65)$$

for all inputs  $\sigma^A$ .

Now append ancilla systems  $R_A$  and  $R_B$  which initially are maximally entangled with  $A$  and  $B$ , respectively. The global initial state is  $|\Psi\rangle$ . To this initial state we can apply either  $U$  (to  $ABE$ ) or  $V$  (to  $AE$  alone). In either case, we will arrive at a final state that has the same marginal state for  $AR_AR_B$ , and thus the two final states differ only by a unitary transformation  $W$  affecting only  $B$  and  $E$ . Pictorially, we have Figure 6. This is not exactly

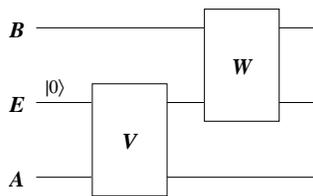


Figure 6: Unitary representation for  $\mathcal{E}^{AB}$  when  $B \not\rightsquigarrow A$ .

the same as our previous result, since the input state of the environment  $E$  is fixed to be  $|0\rangle$ , unlike the system  $C$  which can have any input state. We have nevertheless shown that  $B \not\rightsquigarrow A$  implies that the global map  $\mathcal{E}^{AB}$  has a unitary representation in which the environment interacts with  $A$  and with  $B$

sequentially. Any information transfer between the two systems is mediated by the system  $E$ , and this transfer can only occur in one direction.

A very similar argument can be applied to the three-system situation, in which the global map  $\mathcal{E}^{ABC}$  permits no information transfer from  $B$  to  $A$ . In this case, the map must have a unitary representation of the form shown in Figure 7. In general, the map  $\mathcal{E}^{ABC}$  can *not* be written as the compo-

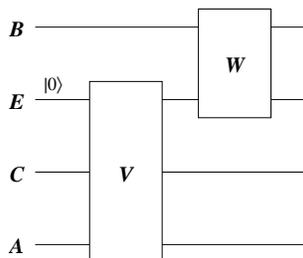


Figure 7: Unitary representation for  $\mathcal{E}^{ABC}$  with  $B \not\leftrightarrow A$ .

sition of two maps acting on  $AC$  and  $BC$  alone, because these subsystems each interact with the same environment  $E$ . On the other hand, if  $\mathcal{E}_{AC}^A$  is autonomous, then we can find a unitary representation for it that does not include any interaction with the external environment  $E$ . This means that the global map  $\mathcal{E}^{ABC}$  can be decomposed in this way

$$\mathcal{E}^{ABC} = (\mathbf{I}^A \otimes \mathcal{F}^{BC}) \circ (\mathbf{I}^B \otimes \mathcal{U}^{AC}) \quad (66)$$

where  $\mathcal{U}^{AC}$  is unitary. This is shown schematically in Figure 8.

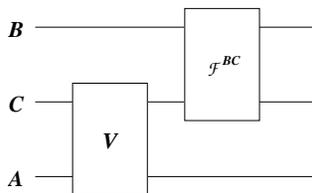


Figure 8: If  $\mathcal{E}_{AC}^A$  is autonomous and  $B \not\leftrightarrow A$ , then  $\mathcal{E}^{ABC}$  can be decomposed into the product of a unitary operation on  $AC$  followed by a general operation on  $BC$ .

## 7 Remarks

Our discussion of dynamical locality in quantum mechanics has so far been very general.

Classical cellular automata are idealized systems consisting of a spatial grid of cells. Each cell can only take on a finite number of internal states at a series of discrete time steps. During each time step, the internal state of each cell is updated according to a rule that is the same for every cell. This rule takes as input the states of the cell itself and a finite number of its immediate neighbors. Thus, during a given time step, each cell only receives information from the cells of its neighborhood.[5]

A quantum cellular automaton is a spatial grid of cells, each of which is a quantum system described by a finite Hilbert space. During each discrete time step, the state of each cell is updated according to a CP map which takes as input the joint state of the cell and its neighbors. In other words, the update rule for a cell  $A$  with neighbors  $C$  is of the form  $\mathcal{E}_{AC}^A$ . The system  $B$  containing the rest of the grid beyond the neighborhood does not influence the new  $A$  state, so that  $B \not\leftrightarrow A$ . [6]

There are complications in the quantum case that are not present in the classical case. For example, any local classical rule can be extended to a global update rule for the entire grid. However, this is not true for a quantum cellular automaton. It is possible to devise local CP maps  $\mathcal{E}_{AC}^A$  that cannot be “woven together” in overlapping neighborhoods to form a global CP map for the entire system. An important (and to our knowledge, open) question is what class of local CP maps can be linked together consistently.

Armed with our analysis of locality, we can draw some interesting conclusions about quantum cellular automata in general. For instance, if the global update map is unitary, then the local update maps  $\mathcal{E}_{AC}^A$  must be autonomous. This and other issues will be discussed in a later paper.

It is possible that our analysis could have application to the theory of quantum cryptography. The requirement that  $B \not\leftrightarrow A$  is a kind of “security condition”: no information about secret system  $B$  can find its way to system  $A$  (perhaps accessible to an eavesdropper), despite the fact that both have interacted with  $C$ . Our decomposition results tell under what circumstances this condition holds exactly for arbitrary initial states.

We also remark that our decomposition results are most intuitively represented as statements about the rearrangement of a quantum circuit. A complicated circuit can be replaced as shown in Figure 4 if and only if the

condition  $B \not\rightsquigarrow A$  holds. This may be a useful idea for the design of quantum algorithms.

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