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Hierarchies of Isomonodromic Deformations and Hitchin Systems

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Abstract

We investigate the classical limit of the Knizhnik-Zamolodchikov-Bernard equations, considered as a system of non-stationar Schröodinger equations on singular curves, where times are the moduli of curves. It has a form of reduced non-autonomous hamiltonian systems which include as particular examples the Schlesinger equations, Painlevé VI equation and their generalizations. In general case, they are defined as hierarchies of isomonodromic deformations (HID) with respect to changing the moduli of underling curves. HID are accompanying with the Whitham hierarchies. The phase space of HID is the space of flat connections of G bundles with some additional data in the marked points. HID can be derived from some free field theory by the hamiltonian reduction under the action of the gauge symmetries and subsequent factorization with respect to diffeomorphisms of curve. This approach allows to define the Lax equations associated with HID and the linear system whose isomonodromic deformations are provided by HID. In addition, it leads to description of solutions of HID by the projection method. In some special limit HID convert into the Hitchin systems. In particular, for $\mathrm{SL}(N,\mathbf{C})$ bundles over elliptic curves with a marked point we obtain in this limit the elliptic Calogero N-body system.

1 Introduction

1.1 It is known for a long time that equations provided isomonodromic deformations posses structures typical for integrable systems, such as the Lax representation [1, 2]. This paper concerns the studies of isomonodromic deformations in the spirit of Hitchin approach to integrable systems [3]. Hitchin discovered a wide class of hamiltonian integrable systems on the cotangent bundles to the moduli space of holomorphic vector bundles over Riemann curves. Some important facts about the Hitchin systems became clear later.

First of all, the Hitchin systems are related to the Knizhnik-Zamolodchikov-Bernard (KZB) equations [4, 5] on the critical level. KZB equations on the critical level take the form of second order differential equations, and their solutions are partition functions of the Wess-Zumino-Witten (WZW) theory on the corresponding Riemann curves. It turns out that these operators coincide with quantum second order Hitchin Hamiltonians [6, 7, 8, 9, 10, 11]. KZB eqs. for correlators of vertex operators need to include in the construction curves with marked points, where vertex operators are located. The KZB Hamiltonians for correlators define very important classes of quantum integrable equations such as the Gaudin equations (for genus zero curves), elliptic Calogero equations (for genus one curves), and their generalizations [12, 13, 14]. It means that their classical counterparts are particular cases of Hitchin systems [7, 8].

Hitchin approach, based on the hamiltonian reduction of some free field hamiltonian theory, claims to be universal in description of integrable systems. Essentially, it allows to present almost exhaustive information - integrals of motion, Lax pairs, action-angle variables, explicit solutions via the projective method.

The main goal of this paper is to go beyond the critical level in the classical limit of KZB eqs. and repeat the Hitchin program as far as possible. For generic values of level the KZB equations have the form of non-stationer Schrödinger equations, where the role of times is played by the coordinates of tangent vectors to the moduli space of curves [9, 10, 15]. On the classical level they correspond to the non-autonomous hamiltonian systems. We will call them the hierarchies of isomonodromic deformations (HID). In this situation the analog of the Hitchin phase space is the moduli space of flat connections $\mathcal{A}$ over Riemann curves $\Sigma_{g,n}$ of genus g with n marked points. While the flatness is the topological property of bundles, the polarization of connections $\mathcal{A} = (A, \bar{A})$ depends on the choice of complex structure on $\Sigma_{g,n}$. We consider a bundle $\mathcal{P}$ over the moduli space $\mathcal{M}_{g,n}$ of curves with flat connections $(A, \bar{A})$ as fibers. The fibers is supplemented by elements of coadjoint orbits $\mathcal{O}_a$ in the marked points x_a. The points of fibers $(A, \bar{A})$ are analogs of momenta and coordinates, while the base $\mathcal{M}_{g,n}$ serves as a set of "times". There exists a closed degenerated two-form ω on $\mathcal{P}$, which is non degenerated on the fibers. The connection $\bar{A}$ play the same role as in the Hitchin construction, while A replaces the Higgs field. Essentially, our construction is local - we work over a vicinity of some fixed curve $\Sigma_{g,n}$ in $\mathcal{M}_{g,n}$. As we already have mentioned, the coordinates of tangent vector to $\mathcal{M}_{g,n}$ at $\Sigma_{g,n}$ play role of times, while in the Hitchin times have nothing to do with the moduli space. The Hamiltonians of HID are the same quadratic Hitchin Hamiltonians, but now they are time depending.

There is some free parameter κ (the level) in our construction. On the critical level ($\kappa = 0$), after rescaling the times, HID convert into the Hitchin systems. As the later, they can be derived by the symplectic reduction from the infinite affine space of the connections and the Beltrami differentials with respect of gauge action on the connection. In addition, to come to the moduli space we need the subsequent factorization under the action of the diffeomorphisms of $\Sigma_{g,n}$, which effectively acts on the Beltrami differentials only. Apart from the last step, our approach is closed to [16], where the KZB systems is derived as a quantization of the very similar symplectic quotient[1]. Due to this derivation, we find immediately the Lax pairs, prove that the equations of motion are consistency conditions of the isomonodromic deformations of the auxiliary linear problem, and, therefore, justify the notion HID. Moreover, we describe solutions via linear procedures (the projection method).

For genus zero our procedure leads to the Schlesinger equations. This case was discussed earlier [2]. We restrict ourselves to simplest cases in which we consider only simple poles of connections. Therefore, we don't include in the phase space the Stokes parameters. This phenomenon was investigated in the rational case in detail in [2]. Isomonodromic deformations on genus one curves were considered in [20] and on higher genus curves in [21]. Here we consider genus one curves with one marked point and obtain for $SL(2,\mathbf{C})$ bundles a particular family of the Painlevé VI equations. Generalization of this case on arbitrary simple groups leads to the multicomponent Painlevé VI related to these groups. If we introduce a few marked points we come to the elliptic generalization of the Schlesinger equations. In fact, the concrete systems we consider here are the deformation from the critical level of those Hitchin systems that described in [7].

1.2. Painlevé VI and Calogero equations The very instructive (but not generic) example of our systems is the Painlevé VI equation. It depends on four free parameters ($PVI_{\alpha,\beta,\gamma,\delta}$) and has the form

$$\frac{d^2X}{dt^2}=\frac{1}{2}\Big(\frac{1}{X}+\frac{1}{X-1}+\frac{1}{X-t}\Big)\Big(\frac{dX}{dt}\Big)^2-\Big(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{X-t}\Big)\frac{dX}{dt}+$$
$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\Big(\alpha+\beta\frac{t}{X^2}+\gamma\frac{t-1}{(X-1)^2}+\delta\frac{t(t-1)}{(X-t)^2}\Big). \tag{1.1}$$

$PVI_{\alpha,\beta,\gamma,\delta}$ has a lot of different applications (see, for example [22]). It is a hamiltonian systems [23]. We will write the symplectic form and the Hamiltonian below in another variables. This equation was derived firstly by Gambier [24], as an equations which has not solutions with moveable singularities. Among distinguish features of this equation we are interesting in its relation to the isomonodromic deformations of linear differential equations. This approach was investigated by Fuchs [25].

There exsits elliptic form of $PVI_{\alpha,\beta,\gamma,\delta}$ derived by Painlevé itself [26]. It was investigated recently by Manin [27] in connection with Frobenius varieties. It sheds light on connection of the Painlevé equations with the Hichin systems, and, thereby, with the KZB equations. We present shortly this approach.

Let $\wp(u|\tau)$ be the Weierstrass function on the elliptic curve $T^2_\tau=\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$, and

$$e_i=\wp\Big(\frac{T_i}{2}|\tau\Big),\quad (T_0,\dots,T_3)=(0,1,\tau,1+\tau).$$

Consider instead of (X,t) in (1.1) the new variables

$$(u,\tau)\to\Big(X=\frac{\wp(u|\tau)-e_1}{e_2-e_1},t=\frac{e_3-e_1}{e_2-e_1}\Big). \tag{1.2}$$

[1]Quantization of isomonodromic deformations on rational and elliptic curves and their relations to KZB was considered in [17, 18, 19]

Then $PVI_{\alpha,\beta,\gamma,\delta}$ takes the form

$$\frac{d^2u}{d\tau^2} = -\partial_u U(u|\tau),$$

$$U(u|\tau) = \frac{1}{(2\pi i)^2}\sum_{j=0}^{3}\alpha_j\wp(u+\frac{T_j}{2}|\tau), \qquad (1.3)$$

$$(\alpha_0,\ldots,\alpha_3) = (\alpha,-\beta,\gamma,\frac{1}{2}-\delta).$$

The hamiltonian form of (1.3) is defined by the standard symplectic form

$$\omega_0 = \delta v\delta u, \qquad (1.4)$$

and the Hamiltonian

$$H = \frac{v^2}{2} + U(u|\tau). \qquad (1.5)$$

The equation of motion (1.3) can be derived from the action S

$$\delta S = v\delta u - H\delta\tau. \qquad (1.6)$$

We can consider (1.4),(1.5) as non-autonomous hamiltonian system with the time-dependent potential.

To find symmetries we consider two-form

$$\omega = \omega_0 - \delta H\delta\tau = \delta v\delta u - \delta H\delta\tau. \qquad (1.7)$$

The semidirect product of $\mathbf{Z}+\mathbf{Z}\tau$ and the modular group acting on the dynamical variables (v,u,τ) are the symmetry of (1.7). We consider them in detail in Sect.7.

Let us introduce the new parameter κ and instead of (1.6) consider

$$\omega = \omega_0 - \frac{1}{\kappa}\delta H\delta\tau. \qquad (1.8)$$

Then, (5.1) takes the form

$$\kappa^2\frac{d^2u}{d\tau^2} = -\partial_u U(u|\tau). \qquad (1.9)$$

It corresponds to the overall rescaling of constants $\alpha_j \to \frac{\alpha_j}{\kappa^2}$. Put $\tau = \tau_0 + \kappa t^H$ and consider the system in the limit $\kappa \to 0$. We come to the equation

$$\frac{d^2u}{(dt^H)^2} = -\partial_u U(u|\tau_0), \qquad (1.10)$$

It is just the rank one elliptic Calogero-Inozemtsev equation [28, 29, 30], which we denote $CI_{\alpha,\beta,\gamma,\delta}$. Thus, we have in this limit

$$PVI_{\alpha,\beta,\gamma,\delta} \xrightarrow{\kappa\to 0} CI_{\alpha,\beta,\gamma,\delta}. \qquad (1.11)$$

So far we don't now how to manage with the general forms of both types of equations for arbitrary values of constants. Here we consider only the one-parametric family in (1.9) $PVI_{\frac{\nu^2}{4},-\frac{\nu^2}{4},\frac{\nu^2}{4},\frac{1}{2}-\frac{\nu^2}{4}}$

$$\alpha_j = \frac{\nu^2}{4}, \quad (\alpha = \frac{\nu^2}{4}, \beta = -\frac{\nu^2}{4}, \gamma = \frac{\nu^2}{4}, \delta = \frac{1}{2} - \frac{\nu^2}{4}).$$

The potential (1.9) takes the form

$$U(u|\tau) = \frac{1}{(2\pi i)^2}\nu^2 \wp(2u|\tau), \tag{1.12}$$

We will prove that (1.4)(1.5) with the potential (1.12) describe the dynamic of flat connections of SL$(2, \mathbf{C})$ bundles over elliptic curves T_τ with one marked point $\Sigma_{1,1}$. In fact, u lies on the Jacobian of T_τ, (v, u) are related to the flat connections, and τ defines a point in $\mathcal{M}_{1,1}$. Roughly speaking, the triple (v, u, τ) are the coordinates in the total space of the bundle $\mathcal{P}$ over $\mathcal{M}_{1,1}$, ω (1.8) is the two-form on $\mathcal{P}$, and ω_0 is the symplectic form on its fibers. This finite-dimensional Hamiltonian system is derived as a quotient of infinite-dimensional phase space of $(A, \bar{A})$ connections and the Beltrami differentials as times under the action of gauge transforms and diffeomorphisms of $\Sigma_{1,1}$. This approach directly leads to the Lax linear system and allows to define solutions of the Cauchy problem via the projection procedure. Simultaneously, we describe the auxiliary linear problem whose isomonodromic deformations are governed by this particular family of the Painlevé VI. The discrete symmetries of (1.6) are nothing else as the remnant gauge symmetries. On the critical level it is just two-body elliptic Calogero system. The corresponding quantum system is identified with the KZB equation for the one-vertex correlator. In the similar way $PVI_{\frac{\nu^2}{4},-\frac{\nu^2}{4},\frac{\nu^2}{4},\frac{1}{2}-\frac{\nu^2}{4}}$ is the classical limit of the KZB for $\kappa \neq 0$. This example will be analyzed in detailed in Sect.7.

Solutions of $PVI_{\frac{\nu^2}{4},-\frac{\nu^2}{4},\frac{\nu^2}{4},\frac{1}{2}-\frac{\nu^2}{4}}$ in particular case ($\nu^2 = \frac{1}{2}$) were found by Hitchin [31] in connection with his investigations of anti-self-dual Einstein metrics. They are written in terms of theta functions. But for generic values of ν solutions are genuine Painlevé transcendents [32].

1.3. Whitham equations The previous example is not entirely exhaustive. A special phenomena occurs if $\dim \mathcal{M}_{g,n} > 1$. We have as many Hamiltonians as $\dim \mathcal{M}_{g,n}$ - each Hamiltonian H_s is attached to the tangent vector t_s to $\mathcal{M}_{g,n}$ at some fixed point $\Sigma_{g,n}$. There are consistency conditions of the equations of motion for the non-autonomous multi-time hamiltonian systems. They take a form of the classical zero-curvature conditions for the connections

$$\partial_{t_s} + H_s, \tag{1.13}$$

where the commutator is replaced by the Poisson brackets. The later is the inverse tensor to the symplectic form, defined on the fibers. The flatness conditions is the so-called the Whitham hierarchy, (WH) defined in the same terms by Krichever [33], though the starting point of his construction is different and based on the averaging procedure. Let S be the action for HID ($\delta S = \delta^{-1}\omega$ as in(1.6)). The τ-function of WH is

$$\log \tau = S.$$

It allows to find the Hamiltonians. For HID in the rational case the τ-functions were investigated in [2]. For the Painlevé I-VI they were considered in [34].

The quantum analog of the Whitham equations was exploited in [15, 9] to construct the KZB connections. The quantum version of (1.13) is an operator, acting in the space V of sections of holomorphic line bundle over moduli of flat connections. The quantum WH is the flatness of the bundle of projective spaces $\mathbf{P}V$ over the Teichmüller space $\mathcal{T}_{g,n}$.

1.4. Outline We start in sect. 2 with a general setup about flat bundles over singular curves. In sect. 3 we present the basic facts about the abstract nonautonomous hamiltonian equations. They defined by a degenerate closed two-form, on the extended phase space, which is a bundle over the space of times. The Whitham equations occur on this stage. We remind the symplectic reductions technique, which is applicable in degenerate case as well. Sect. 4 contains our main result - the derivation of the HID, corresponding to the flat bundles over Riemann curves. In sect. 5 we discuss two limits - the level zero limit of HID to the Hitchin systems and the classical limit of general KZB equations to HID. Then we analyze in detail two feasible examples - flat bundles over rational curves, leading to the Schlesinger equations (sect. 6), and flat bundles over elliptic curves, responsible for the Painlevé VI type equations, and the elliptic Schlesinger equations (sect. 7). In particular, we apply the projection method to construct perturbatively solutions of the Schlesinger equations. In the Appendix we summarize some results about the elliptic functions, which are used in sect. 7.

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2 Flat bundles over singular curves

We will describe the general setup more or less naively. We don't consider in this section the mechanism of symplectic reduction in detail and postpone it on Sect.4 We will consider three cases: 1) smooth proper (compact) algebraic curves; 2) smooth proper algebraic curves with punctures; 3) proper algebraic curves with nodal singularities (double points).

Let G be a semisimple group and V be its exact representation; f.e. $G = \mathrm{SL}(N, \mathbf{C})$ and V is a N-dimensional vector space with a volume form.

2.1. Smooth curves.

Let S be a smooth oriented compact surface of genus g. Let us consider the moduli space $FBun_{S,G}$ of flat V-bundles on S. This space can be considered as the quotient of the space $FConn$ of flat C^∞ connections on trivial V-bundle by action of the gauge group $\mathcal{G}$ of G-valued C^∞-functions on S, or as a result of hamiltonian reduction of space $Conn$ of all connections by action of the gauge group. The symplectic form on the space $Conn$ is the form

$$\omega = \int_S < \delta\mathcal{A}, \delta\mathcal{A} >, \tag{2.1}$$

where $< \cdot, \cdot >$ denotes the Killing form, $\delta\mathcal{A}$ is a Lie(G)-valued one-form on S. So, $< \delta\mathcal{A}, \delta\mathcal{A} >$ is a two-form, and the integral is well defined. The dual space to the gauge algebra Lie is the space of Lie(G)-valued two-forms on S, and the momentum map corresponds to any connection $\mathcal{A}$ its curvature $F_{\mathcal{A}} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$. Hence, the preimage of 0 under momentum map is the space of flat connections.

A complex structure Σ on S is a differential operator $\overline{\partial} : \Omega^0_{C^\infty} \to \Omega^1_{C^\infty}$; its kernel is the space of holomorphic functions. For any connection we can consider $\overline{\partial}$-part of connection; it defines a holomorphic bundle. A section is holomorphic, if it belongs to the kernel of this operator. Let us recall that:
1. Connection on the holomorphic bundle $\mathcal{E}$ is the operator of type $\partial + A$.
2. Connection is holomorphic if this operator commutes with operator of complex structure on the bundle.
3. Connection is flat if its square vanishes; for curves this condition is empty.
4. The space of all holomorphic connections on $\mathcal{E}$ is an infinite dimensional affine space and the gauge group $\mathcal{G}$ acts on it by affine transformations.
5. The quotient of stable holomorphic connections with respect to the gauge transformations is the moduli space $Bun_{\Sigma,G}$ of holomorphic vector bundles on Σ.

Let b be a point in $Bun_{\Sigma,G}$ and $\mathcal{V}_b$ is the corresponding V-bundle on Σ. Denote by ad_b the bundle of endomorphisms of $\mathcal{V}_b$ as bundle of Lie algebras and by $Flat_b$ the space of flat holomorphic connections on holomorphic bundle $\mathcal{V}_b$.

Consider the map from $FBun_{S,G}$ onto the moduli space $Bun_{\Sigma,G}$. The fiber of the projection $FBun_{S,G} \to Bun_{\Sigma,G}$ over a point b is naturally isomorphic to $Flat_b$. These fibers are Lagrangian with respect to ω (2.1) - any complex structure on S defines a polarization on $FBun_{S,G}$.

The tangent space to $Bun_{\Sigma,G}$ at the point b is canonically isomorphic to the first cohomology group $H^1(\Sigma, ad_b)$. The space $Flat_b$ of holomorphic connections on $\mathcal{V}_b$ is an affine space over the vector space $H^0(\Sigma, ad_b \otimes \Omega^1)$ of holomorphic ad_b-valued one-forms, since a difference between any connections is a ad_b-valued differential form. The vector spaces $H^1(\Sigma, ad_b)$ and $H^0(\Sigma, ad_b \otimes \Omega^1)$ are dual.

The Dolbeault representation of cohomology classes corresponds to the description based on the hamiltonian reduction. Indeed, let us decompose $\mathcal{A}$ into $(1,0)$ and $(0,1)$ parts: $\mathcal{A} = A + \overline{A}$. From the zero curvature condition we get that $(1,0)$-part δA must be $\overline{A}$-holomorphic: $\overline{\partial}_A \delta A \equiv (\overline{\partial} + \overline{A})\delta A = 0$, and $\delta\overline{A}$ is defined up to the infinitesimal gauge transformations $\delta\overline{A} \to \delta\overline{A} + \overline{\partial}_A h$.

2.2. Curves with punctures

For any two isomorphic representations V and V' of G denote by $Isom(V, V')$ the space of G-isomorphisms between them; this space is a principal homogeneous space over G. For a curve Σ_n with n marked points x_j we replace the moduli space $Bun_{\Sigma,G}$ by moduli space $Bun_{\Sigma,\mathbf{x},G}$, ($\mathbf{x} = (x_1, \dots, x_n)$) of holomorphic V-bundles $\mathcal{V}$ with the trivializations $g_j : \mathcal{V}_{x_j} \to V$ of fibers at the marked points. We have natural "forgetting" projection $\pi : Bun_{\Sigma,\mathbf{x},G} \to Bun_{\Sigma,G}$. The fiber of this projection is the product $\prod Isom(\mathcal{V}_{x_j}, V)$ of the spaces of isomorphisms between fibers of bundle at marked points and V. The projection π can be treated as a reduction by action of n copies of the group G which acts transitively on the fibers

$$Bun_{\Sigma,G} = Bun_{\Sigma,\mathbf{x},G}/\prod G.$$

The bundle of endomorphisms of this data is the bundle $ad_b(-\mathbf{x})$ of endomorphisms, vanishing at the marked points, since non vanishing endomorphisms change trivializations. Hence, the tangent space to $Bun_{\Sigma,\mathbf{x},G}$ is isomorphic to $H^1(\Sigma, ad_b(-\mathbf{x}))$. The dual space to this space is $H^0(\Sigma, ad_b(\mathbf{x}) \otimes \Omega^1)$. Consequently, in order to get the symplectic variety we must replace the affine space $Flat_b$ of flat holomorphic connection by the affine space $Flat_b(\log)$ of flat connections with logarithmic singularities. As a result, we get the moduli space $FBun_{\Sigma,\mathbf{x},G}$ of triples: (holomorphic bundle, trivializations at marked points, holomorphic connection with logarithmic singularities). Again, we have the Lagrangian projection $FBun_{\Sigma,\mathbf{x},G} \to Bun_{\Sigma,\mathbf{x},G}$. The fiber of this projection is an affine space over $H^0(\Sigma, ad_b(\mathbf{x}) \otimes \Omega^1)$.

The product of n copies of G acts on $FBun_{X,\mathbf{x},G}$ by changing trivialization of fibers, this action is hamiltonian. We can consider the hamiltonian reduction with respect to this action. As a result, for any collection of G-orbits of adjoint action we get the moduli space $FBun_{\Sigma,\mathbf{x},G,\{\mathcal{O}_j\}}$ of pairs (holomorphic bundle, holomorphic connection with logarithmic singularities with residues in orbits $\mathcal{O}_j$).

According to the Dolbeault theorem, the tangent space $H^1(\Sigma, ad_b(-\mathbf{x}))$ can be realized as the space of $(0,1)$ ad_b-valued forms *holomorphically* vanishing at the marked points modulo $\overline{\partial}_A$-coboundaries of *holomorphically* vanishing at marked points ad_b-valued functions. A function (or one-form) is referred to be holomorphically vanishing at the point x, if it has asymptotics $(z-x)O(1)$ at this point.

The second description: the tangent space $H^1(\Sigma, ad_b(-\mathbf{x}))$ is isomorphic to the space of all $(0,1)$ ad_b-valued forms modulo $\overline{\partial}_A$-coboundaries of vanishing at marked points ad_b-valued functions. Indeed, we have a natural embedding of the space of holomorphically vanishing forms to the space of all forms and the space of holomorphically vanishing functions to the space of vanishing functions. It defines the map ϕ from the space from first description to space from second description. We can assume that $\overline{A}$ vanishes at small vicinities of marked points. Then $\overline{\partial}_{\overline{A}}$ equals $\overline{\partial}$ at these vicinities. Let h be a function, vanishing at the marked points which is not holomorphically vanishing,

$$h \equiv \sum_{i=1}^{\infty} a_i \overline{(z-x_j)}^i + (z-x_j)o(1).$$

Then

$$\overline{\partial} h \sim (\sum_{i=1}^{\infty} i a_i \overline{(z-x_j)}^{(i-1)} + (z-x_j)o(1)) d\,\overline{z}$$

is not holomorphically vanishes. So, the map ϕ is injective. At the other hand, for any form ν with asymptotics

$$\nu \sim (\sum_{i=0}^{\infty} a_i \overline{(z-x_j)}^i + (z-x_j)o(1))d\,\overline{z}$$

at the points x_j we can choose some function ϕ such that $\nu - \overline{\partial}_A\phi$ has asymptotics $(z-x_j)o(1)$, since for any collection of asymptotics $(\sum_{i=o}^{\infty} \frac{1}{i+1} a_i \overline{(z-x_j)}^{i+1} + (z-x_j)o(1))$ a function with such asymptotic exists. So, the map ϕ is surjective.

The third equivalent description of this space is defined as the cokernel of the map:

$$\Omega^{0,0}_{C^\infty}(ad_b) \to \Omega^{0,1}_{C^\infty}(ad_b) + \sum ad_b|_{x_i};\ h \to (\overline{\partial}_A h, h(x_i)).$$

Indeed, for any element $(\delta\overline{A}, \{a_i\})$ in the cokernel we can choose its representative with vanishing second part $\{a_i\}$. This choice is unique up to $\overline{\partial}_A h$ for vanishing at the marked points functions h. This description is adapted to the forgetting map π. The local part $\{a_i\}$ corresponds to the tangent space of fiber and the global part $\delta\overline{A}$ corresponds to the tangent space of base.

By "integration" of the action of the gauge Lie algebra ad_b to the action of the gauge Lie group $\mathcal{G}$ we get two description of the moduli space.

A. The moduli space $Bun_{\Sigma,\mathbf{x},G}$ is the quotient of the space of $\overline{\partial}$-connections $\overline{A}$ on the trivial V-bundle by action of the reduced gauge group of G-valued functions, whose values at the marked points are equal to the neutral element 1 of the group G. The isomorphisms of fibers with V are identical maps.

B. The moduli space $Bun_{\Sigma,\mathbf{x},G}$ is the quotient of the space of pairs (a $\overline{\partial}$-connection $\overline{A}$ on the trivial V-bundle, a collection g_j of elements of G) by action of the gauge group $\mathcal{G}$. The isomorphisms of fibers with V are g_i. The isomorphism of these descriptions can be proved by the following consideration. For any collection $(\overline{A}, g_j)$ we can choose the gauge equivalent collection with $g_j = 1$, and such collections are equivalent up to action of the reduced gauge group.

In what follows we assume the second description of $Bun_{X\Sigma,\mathbf{x},G}$

Let us consider the space of data ($\overline{\partial}$-connections $\overline{A}$ on the trivial V-bundle, a collection g_j of elements of G, ∂-connection A with logarithmic singularities at marked points). This space is symplectic with the form $FBun_{X,\{x_j\},G}$ equals to Hamiltonian reduction of this space by action of gauge group $\mathcal{G}$.

It is worthwhile to note that (in contrast with Case 1) this construction is essentially based on the complex structure on the surface, since $\overline{A}$ is nonsingular and A has singularities at the marked points. We can consider the moduli space of flat bundles over noncompact surface $S \setminus \bigcup x_j = G$-representations of fundamental group of $S \setminus \bigcup x_j$. According to Deligne, for any complex structure on S any stable representation can be realized by connections with logarithmic singularities on holomorphic bundle. Hence, any connection $\mathcal{A}$ is gauge equivalent to a connection with regular $\overline{\partial}$-part, but corresponding gauge transformation has any singularities at marked points a priori. This fact makes such approach very complicated.

2.3. Curves with double points

A curve with a double point (the nodal singularity) can be treated as a "limit" of nonsingular curves under pinching of some circle. If this circle is homological to zero, then the resulting curve is the union of two intersecting smooth curves; and sum of genera of these curves is equal to the genus of nonsingular curves. If this circle is homologically nontrivial, then the singular curve is a smooth curve with "glued" two different points. The genus of this curve is less by one that genus of initial curves. The normalization of singular curve ("disglueing" of singularity) is a smooth curve (not connected, in general) with marked points.

Let us fix some notations. Denote by $\mathcal{M}_g$ the moduli space of smooth curves of genus g, and denote by $\mathcal{M}_{g,n}$ the moduli space of smooth curves of genus g with n different marked points ($\mathcal{M}_g = \mathcal{M}_{g,0}$) . Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$. Then the compactification divisor $D_\infty = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is the union of components $D_\infty^{g_1,g_2}$, $g_1+g_2 = g$ and D_∞^{g-1}. These components are covered by $\overline{\mathcal{M}}_{g_1,1}\times\overline{\mathcal{M}}_{g_2,1}$ and $\overline{\mathcal{M}}_{g-1,2}$ correspondingly.

Consider the moduli space $Bun_{g,\Sigma}$ of pairs (smooth curve of genus g, G-bundle on it). Evidently, this space is fibered over $\mathcal{M}_g$ with $Bun_{X,G}$ as fibers. Unfortunately, we don't now the canonical compactification $\overline{Bun_{g,X}}$, which is fibered over $\overline{\mathcal{M}}_g$. Let us assume that such compactification does exist. Then the open part of fiber over singular curve must be described as moduli space of the following data (G-bundle over normalized curve, isomorphisms of fibers over "glued" points).

Denote by Σ^0 a singular curve with nodal points $y_1, y_2 \cdots y_n$, and denote by Σ its normalization, x_a and x_{n+a} are preimages of y_a under normalization map. Then moduli space $Bun_{\Sigma^0,G}$, corresponding to Σ^0 is the moduli space of (holomorphic V-bundle on Σ, isomorphisms between fibers of this bundle over x_a and x_{n+a}). This space is quotient of $Bun_{\Sigma,\mathbf{x},G}$ by action of n-copies of group G, j-th G acts on $Isom(\mathcal{V}_{x_a}, V)$ and $Isom(\mathcal{V}_{x_{n+a}}, V)$ from the right:

$$\{h_a\}_{1\le a\le n} : \{g_a, g_{a+n}\}_{1\le a\le n} \to \{g_a h_a, g_{a+n} h_a\}_{1\le a\le n}.$$

Corresponding symplectic variety $FBun_{\Sigma^0,G}$ is the result of symplectic reduction of $FBun_{\Sigma,\mathbf{x},G}$ by the described above action with zero level of momentum map. This space is the moduli space of data (holomorphic V-bundle on X, isomorphisms between fibers of this bundle over x_a and x_{n+a}, connection with logarithmic singularities such that residue at points x_a and x_{n+a} are opposite). In the last part of data we use isomorphisms of fibers from the second part of data.

3 Hamiltonian formalism

We consider nonautonomous hamiltonian systems. For this type of systems it is the custom in the classical mechanics to deal with a degenerate symplectic form on the extended phase space, which includes beside the usual coordinates and momenta, the space of times and corresponding to them Hamiltonians as the conjugate variables. Then the hamiltonian equations of motion are defined as variations of dynamical variables along the null leafs of this symplectic form.

3.1. Equations of motion

Let $\mathcal{R}$ be a (infinite-dimensional) phase space endowed with the non-degenerate symplectic structure. For simplicity, we take it in the canonical form

$$\omega_0 = (\delta \mathbf{v}, \delta \mathbf{u}),$$

$$\mathbf{v} = (v_1, \dots, v_i \dots), \quad \mathbf{u} = (u_1, \dots, u_i \dots).$$

Consider the space of "times" $\mathcal{N} = \{\mathbf{t} = (t_1, \dots, t_a, \dots)\}$ and corresponding dual to times Hamiltonians $(H^1, \dots, H^a, \dots)$ on $\mathcal{R}$ depending on times as well. We consider the extended phase space which is the bundle $\mathcal{P}$ over $\mathcal{N}$ with fibers $\mathcal{R}$. Introduce a symplectic form on $\mathcal{P}$

$$\omega = \omega_0 - \sum_a \delta H^a \delta t_a = (\delta \mathbf{v}, \delta \mathbf{u}) - \sum_a \delta H^a \delta t_a. \tag{3.1}$$

This form is closed but degenerated on $\mathcal{P}$. The vector fields

$$\mathcal{V}^a = \sum_i (A_i^a \partial_{v_i} + B_i^a \partial_{u_i}) + \partial_{t_a}.$$

lie in the kernel of ω iff

$$A_i^a + \frac{\partial H^a}{\partial u_i} = 0, \quad -B_i^a + \frac{\partial H^a}{\partial v_i} = 0,$$

$$-A_i^a \frac{\partial H^b}{\partial v_i} - B_i^a \frac{\partial H^b}{\partial u_i} - \frac{\partial H^a}{\partial t_b} + \frac{\partial H^b}{\partial t_a} = 0. \tag{3.2}$$

Then the vector field $\mathcal{V}^a \in \mathrm{Ker}\ \omega$ takes the form

$$\mathcal{V}^a = -\frac{\partial H^a}{\partial u_i} \partial_{v_i} + \frac{\partial H^a}{\partial v_i} \partial_{u_i} + \partial_{t_a}.$$

It can be checked immediately that they commute. For any functions $f(V, U, t)$ on $\mathcal{P}$ its evolution is defined as

$$\frac{df(\mathbf{v}, \mathbf{u}, \mathbf{t})}{dt_a} = \mathcal{V}^a f(\mathbf{v}, \mathbf{u}, \mathbf{t}), \quad \mathcal{V}^a \in \mathrm{Ker}\ \omega. \tag{3.3}$$

It follows from (3.2) that the Hamiltonians subject to the classical zero curvature conditions (*the generalized Whithem hierarchy*)

$$\frac{dH^a}{dt_b} - \frac{dH^b}{dt_a} + \{H^b, H^a\}_{\omega_0} = 0, \ a, b = 1 \dots \tag{3.4}$$

Evidently, for the time independent Hamiltonians (3.3),(3.4) give the standard approach. Define the action S of the system as $\delta^{-1}\omega = \delta S$

$$\delta S = (\mathbf{v}, \delta \mathbf{u}) - \sum_a H_a \delta t_a \tag{3.5}$$

and the *τ-function*

$$\delta \log \tau = \delta S. \tag{3.6}$$

The equations of motion can be written down in the Hamilton-Jacobi form

$$\frac{\partial S}{\partial t_a} = -H_a(\frac{\delta S}{\delta \mathbf{u}}, \mathbf{u}, \mathbf{t}).$$

3.2. Moment map

The symplectic reduction can be applied to the nonautonomous system with symmetries in the standard way. We shortly repeat this approach. Let $\mathcal{G}$ be a symmetry group of the system. It means that for any $\epsilon \in$Lie($\mathcal{G}$) there exists the vector field acting on $\mathcal{P}$ such that the Lie derivative $\mathcal{L}_\epsilon$ annihilates ω (3.1)

$$\mathcal{L}_\epsilon \omega = (\delta j_\epsilon + j_\epsilon \delta)\omega = 0.$$

The vector field is called a *hamiltonian vector field* with respect to ω. Since ω is closed we can write locally $j_\epsilon \omega = \delta F_\epsilon$. If $j_\epsilon \omega$ is exact then j_ϵ is the *strictly hamiltonian vector field.* The function $F_\epsilon = F_\epsilon(\mathbf{v}, \mathbf{u}, \mathbf{t})$ is a linear function on Lie($\mathcal{G}$)

$$F_\epsilon =< \epsilon, \mathcal{J}(\mathbf{v}, \mathbf{u}, \mathbf{t}) >$$

and thereby defines the *the moment map*

$$\mathcal{J} : \mathcal{P} \to Lie^*(\mathcal{G}).$$

Assume that we put the moment constraint

$$\mathcal{J}(\mathbf{v}, \mathbf{u}, \mathbf{t}) = 0. \tag{3.7}$$

The quotient

$$\mathcal{P}^{\text{red}} = \mathcal{P}//\mathcal{G} := \mathcal{J}^{-1}(0)/\mathcal{G}$$

is a symplectic space with the symplectic form ω^{red} which is the reduction of the original form ω (3.1). It is defined by the two step procedure.
i) Fixing the gauge. In other words, there should be defined a surface in the phase space $\mathcal{P}$, which is transversal to the orbits of the gauge group $\mathcal{G}$.
ii) Solving the moment constraint equations (3.7) for the dynamical variables, restricted to the gauge fixing surface.

The main difference with the autonomous hamiltonian systems is that we demand the gauge invariance for the whole degenerate symplectic form (3.1) only and don't consider the form ω_0 and the Hamiltonians separately.

4 Symplectic reduction and factorization

4.1. Space of "times" $\mathcal{N}'$.

Let $\Sigma_{g,n}$ be a Riemann curve of genus g with n marked points . Let us fix the complex structure on $\Sigma_{g,n}$ by defining local coordinates $(z, \bar{z})$ in open maps covering $\Sigma_{g,n}$. Assume that the marked points $(x_1, \ldots, x_n)$ are in the generic position, i.e. there exists a set of their vicinities $(\mathcal{U}_1, \ldots, \mathcal{U}_n)$, such that $\mathcal{U}_a \cap \mathcal{U}_b = \emptyset$ for $a \neq b$.

The deformations of the basic complex structure are determined by the of Beltrami differentials μ, which are smooth $(-1,1)$ differentials on $\Sigma_{g,n}$. We identify this set with the space of times $\mathcal{N}'$. The Beltrami differentials can be defined in the following way. Consider a chiral smooth transformation of $\Sigma_{g,n}$, which in some local map can be represented as

$$w = z - \epsilon(z,\bar{z}), \quad \bar{w} = \bar{z} \tag{4.1}$$

Up to the conformal factor $1 - \partial\epsilon(z,\bar{z})$ the corresponding one-form dw is equal

$$dw = dz - \mu d\bar{z}, \quad \mu = \frac{\bar{\partial}\epsilon(z,\bar{z})}{1 - \partial\epsilon(z,\bar{z})}. \tag{4.2}$$

The Beltrami differential defines the new holomorphic structure - the deformed antiholomorphic operator annihilating dw, while the antiholomorphic structure is kept unchanged

$$\partial_{\bar{w}} = \bar{\partial} + \mu\partial, \quad \partial_w = \partial.$$

In addition, assume that μ vanishes in the marked points

$$\mu(z,\bar{z})|_{x_a} = 0. \tag{4.3}$$

In our construction we consider small deformations of the basic complex structure $(z,\bar{z})$. It allows to replace (4.2) by

$$\mu = \bar{\partial}\epsilon(z,\bar{z}). \tag{4.4}$$

Nevertheless, in some cases we will use the exact representation (4.2) as well.

4.2 Fibers $\mathcal{R}'$.

Let $\mathcal{E}$ be a principle stable G bundle over a Riemann curve $\Sigma_{g,n}$. Assume that G is a complex simple Lie group . The phase space $\mathcal{R}'$ is constructed by the following data:
i)the affine space $\{\mathcal{A}\}$ of Lie(G)-valued connection on $\mathcal{E}$.
It has the following component description:
a) C^∞ connection $\{\bar{A}\}$, corresponding to the $d\bar{w} = d\bar{z}$ component of $\mathcal{A}$;
b)The dual to the previous space the space $\{A\}$ of dw components of connection $\mathcal{A}$. A can have simple poles in the marked points. Moreover, assume that $A\mu$ is a well defined function ;
ii)cotangent bundles $T^*G_a = \{(p_a, g_a),\ p_a \in \text{Lie}^*(G_a),\ g_a \in G_a\},\ (a = 1,\ldots,n)$ in the points $(x_1,\ldots,x_n)$.
There is the canonical symplectic form on $\mathcal{R}'$

$$\omega_0 = \int_\Sigma < \delta A, \delta\bar{A} > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a >, \tag{4.5}$$

where $<\ ,\ >$ denotes the Killing form on Lie(G).

According with Sect.2 we can consider the elements g_a as trivializations of fibers in the marked points and p_a as the residues of holomorphic connections in the same points

$$g_a \in Isom(\mathcal{V}_a, V),\ (V \sim \text{Lie}(G)), \quad p_a = res|_{x_a} A.$$

Thus, the symplectic form (4.5) is the generalization of (2.1) on the singular curves.

4.3. Extended phase space $\mathcal{P}'$.

According to the general prescription the bundle $\mathcal{P}'$ over $\mathcal{N}'$ with $\mathcal{R}'$ as the fibers plays role of the extended phase space. Consider the degenerate form on $\mathcal{P}'$

$$\omega = \omega_0 - \frac{1}{\kappa}\int_\Sigma < \delta A, A > \delta\mu. \tag{4.6}$$

Thus, we deal with the infinite set of Hamiltonians $< A, A > (z, \bar{z})$, parametrized by points of $\Sigma_{g,n}$ and corresponding set of times $\mu(z, \bar{z})$. We apply the formalism presented in the previous section to these systems.

4.4. Equations of motion

They take the form (see (3.3),(4.6))

$$\frac{\delta A}{\delta\mu}(z,\bar{z}) = 0, \quad \kappa\frac{\delta\bar{A}}{\delta\mu}(z,\bar{z}) = A(z,\bar{z}),$$

$$\frac{\delta p_b}{\delta\mu} = 0, \quad \frac{\delta g_b}{\delta\mu} = 0, \tag{4.7}$$

We can introduce the modified connection

$$\bar{A}' = \bar{A} - \frac{1}{\kappa}\mu A. \tag{4.8}$$

In its terms (4.6) take the canonical form:

$$\omega = \int_\Sigma < \delta A, \delta\bar{A}' > + \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a > . \tag{4.9}$$

and the equations of motion (4.7) become trivial

$$\frac{\delta A}{\delta\mu}(z,\bar{z}) = 0, \quad \frac{\delta\bar{A}'}{\delta\mu}(z,\bar{z}) = 0. \tag{4.10}$$

In other words, as we discussed in Sect.2, if we forget about the complex structures of curves the bundle become trivial $\mathcal{P}' \sim \mathcal{R}' \times \mathcal{N}'$.

4.5. Symmetries

The form ω (4.6) (or (4.9)) is invariant with respect to the action of the group $\mathcal{G}_0$ of diffeomorphisms of $\Sigma_{g,n}$, which are trivial in vicinities $\mathcal{U}_a$ of marked points:

$$\mathcal{G}_0 = \{z \to N(z,\bar{z}), \bar{z} \to \bar{N}(z,\bar{z}),),\ N(z,\bar{z}) = z + o(|z - x_a|),\ z \in \mathcal{U}_a\}. \tag{4.11}$$

In particular, its action on the Beltrami differentials takes the form of the Möbius transform

$$\mu \to \frac{\frac{\partial z}{\partial \bar{N}} + \mu\frac{\partial \bar{z}}{\partial \bar{N}}}{\frac{\partial z}{\partial N} + \mu\frac{\partial \bar{z}}{\partial N}} \tag{4.12}$$

Another infinite gauge symmetry of the form (4.6) (or (4.9)) is the group

$$\mathcal{G}_1 = \{f(z,\bar{z}) \in C^\infty(\Sigma_g, G)\},$$

that acts on the dynamical fields as

$$A + \kappa\partial \to f(A+\kappa\partial)f^{-1}, \quad \bar{A} + \bar{\partial} + \mu\partial \to f(\bar{A} + \bar{\partial} + \mu\partial)f^{-1},$$
$$(\bar{A}' + \bar{\partial} \to f(\bar{A}' + \bar{\partial})f^{-1}), \tag{4.13}$$
$$p_a \to f_a p_a f_a^{-1}, \quad g_a \to g_a f_a^{-1}, \quad (f_a = \lim_{z \to x_a} f(z,\bar{z})), \quad \mu \to \mu.$$

In other words, the gauge action of $\mathcal{G}_1$ does not touch the base $\mathcal{N}'$ and transforms only the fibers $\mathcal{R}'$. The whole gauge group is the semidirect product

$$\mathcal{G}_1 \oslash \mathcal{G}_0. \tag{4.14}$$

There is an additional finite-dimensional symmetry group $\mathcal{G}_2$ which commutes with (4.14). It acts only on the singular curves in the fibers at the marked points. It is remnant of $\mathcal{G}_1$ on the desingular curves (see Sect.2).

$$\mathcal{G}_2 = \otimes_{a=1}^n G_a,$$
$$p_a \to p_a, \quad g_a \to h_a g_a, \quad h_a \in G_a, \ (a = 1, \ldots, n). \tag{4.15}$$

This action commutes with (4.14).

4.6. Symplectic reduction with respect to $\mathcal{G}_1$

The infinitesimal action of $\mathcal{G}_1$ (4.13) generates the vector field $\epsilon_1 \in \mathcal{A}^{(0)}(\Sigma_{g,n}, \mathrm{Lie}(G))$

$$j_{\epsilon_1} A = -\kappa\partial\epsilon_1 + [\epsilon_1, A],$$
$$j_{\epsilon_1} \bar{A}' = -\bar{\partial}\epsilon_1 + [\epsilon_1, \bar{A}'],$$
$$j_{\epsilon_1} p_a = [\epsilon_1(z_a, \bar{z}_a), p_a],$$
$$j_{\epsilon_1} g_a = -g_a \epsilon_1(z, \bar{z}_a),$$
$$j_{\epsilon_1} \mu = 0.$$

Let

$$F_{A,\bar{A}'} = \bar{\partial}A - \kappa\partial\bar{A}' + [\bar{A}, A] = (\bar{\partial} + \partial\mu)A - \kappa\partial\bar{A} + [\bar{A}.A].$$

Note that the action of the operator $(\bar{\partial} + \partial\mu)$ on the connection A is correctly defined.

The corresponding moment map

$$\mathcal{J}_1 : \mathcal{P} \to \mathrm{Lie}^*(\mathcal{G}_1).$$

takes the form

$$\mathcal{J}_1 = -F_{A,\bar{A}'}(z,\bar{z}) + 2\pi i \sum_{a=1}^n \delta^2(x_a^0) p_a.$$

Let $\mathcal{J}_1 = 0$. In other words, the moment constraints equation takes the form

$$F_{A,\bar{A}'}(z,\bar{z}) = 2\pi i \sum_{a=1}^{n} \delta^2(x_a^0) p_a. \tag{4.16}$$

It means that we deal with the flat connection everywhere on $\Sigma_{g,n}$ except the marked points. The holonomies of $(A, \bar{A})$ around the marked points are conjugated to $\exp 2\pi i p_a$.

Let $(L, \bar{L})$ be the gauge transformed connections

$$\bar{A} = f\bar{L}f^{-1} + f(\bar{\partial} + \mu\partial)f^{-1}, \tag{4.17}$$

$$A = fLf^{-1} + \kappa f \partial f^{-1}, \tag{4.18}$$

Then (4.16) takes the form

$$(\bar{\partial} + \partial\mu)L - \kappa\partial\bar{L} + [\bar{L}, L] = 2\pi i \sum_{a=1}^{n} \delta^2(x_a^0) p_a. \tag{4.19}$$

The gauge fixing allows to choose $\bar{A}$ in a such way that $\partial\bar{L} = 0$. In fact, the anti-holomorphity of $f^{-1}(\bar{\partial} + \mu\partial)f + f^{-1}\bar{A}f$ amounts to the classical equations of motion for the Wess-Zumino-Witten functional $S_{WZW}(f, \bar{A})$ in the external field $\bar{A}$, which does has extremal points. Then instead of (4.19) we have

$$(\bar{\partial} + \partial\mu)L + [\bar{L}, L] = \sum_{a=1}^{n} \delta^2(x_a) p_a. \tag{4.20}$$

We can rewrite it as

$$\partial_{\bar{w}} L + [\bar{L}, L] = 2\pi i \sum_{a=1}^{n} \delta^2(x_a) p_a. \tag{4.21}$$

Anyway, by choosing $\bar{L}$ we fix somehow the gauge in generic case. The last form of the moment constraint (4.21) coincides with that for the Hitchin systems [7], which allows to apply the known solutions.

4.7. Symplectic reduction with respect to $\mathcal{G}_2$

The gauge transforms $h_a \in G_a$ in the points x_a acts on T^*G_a. In the case of punctures it allows to fix p_a on some coadjoint orbit $p_a = g_a p_a^{(0)} g_a^{-1}$ and obtain the symplectic quotient $\mathcal{O}_a = T^*G_a // G_a$. In fact, the moment corresponding to this action is

$$\mu_a = g_a p_a g_a^{-1} \in \mathrm{Lie}^*(G).$$

Let $\mu_a = J_a$ be some fixed point in $\mathrm{Lie}^*(G)$. Then the gauge fixing allows to choose g_a up to the stabilizator of J_a and

$$p_a = g_a^{-1} J_a g_a \in \mathcal{O}_a.$$

Thus, in (4.19) or (4.20) p_a are elements of $\mathcal{O}_a$. The symplectic form on $\mathcal{O}_a$ keeps the same form as on T^*G_a.

Consider the double points case (the nodal singularities). Let x_1 and x_2 be preimages of the nodal point y under the normalization map. Then the symplectic form on the normalization of the singular curve

$$\omega = \delta < p_1, g_1^{-1}\delta g_1 > + \delta < p_2, g_2^{-1}\delta g_2 >$$

generates the moment

$$\mu = g_1 p_1 g_1^{-1} + g_2 p_2 g_2^{-1}.$$

Put $\mu = 0$. Then

$$p_1 = -\tilde{g}_2 p_2 \tilde{g}_2^{-1}, \quad \tilde{g}_2 = g_1^{-1} g_2.$$

The pair $(p_2, \tilde{g}_2)$ is an arbitrary element of T^*G. Therefore, in this case $T^*G_1 \oplus T^*G_2//G = T^*G$.

In what follows we will concentrate on the case of curves with punctures (without double points). Let $\mathcal{I}_{g,n}$ be the equivalence classes of the connections $(A, \bar{A})$ with respect to the gauge action (4.17),(4.18) - the moduli space of stable flat G bundles over $\Sigma_{g,n}$. It is a smooth finite dimensional space. If we fix the conjugacy classes of holonomies $(L, \bar{L})$ around marked points $\mathcal{I}_{g,n}$ becomes a symplectic manifold. It is extended here by the coadjoint orbits $\mathcal{O}_a$ in the marked points $x_a, (a = 1, \dots, n)$ in the consistent way (see (4.19)). Fixing the gauge we come to the symplectic quotient

$$\mathcal{R} \subset \mathcal{I}_{g,n} \times \prod_{a=1}^{n} \mathcal{O}_a,$$

$$\mathcal{R} = \mathcal{R}'//(\mathcal{G}_1 \oplus \mathcal{G}_2) = \mathcal{J}_1^{-1}(0)/(\mathcal{G}_1 \oplus \mathcal{G}_2).$$

It has dimension

$$\dim(\mathcal{R}) = \begin{cases} \dim(\sum_{a=1}^{n} \mathcal{O}_a//G), & g = 0, \\ 2\mathrm{rank}G + \dim(\sum_{a=1}^{n} \mathcal{O}_a//H), & g = 1 \\ (2g-2)\dim G + \dim(\sum_{a=1}^{n} \mathcal{O}_a) & g \geq 2, \end{cases}$$

where H is the Cartan subgroup $H \subset G$ and $\mathcal{O}_a//G$ and $\mathcal{O}_a//H$ are the symplectic quotients of the symplectic spaces $\mathcal{O}_a$ under the actions of the automorphisms of the bundles in the rational and the elliptic cases correspondingly. The connections $(L, \bar{L})$ in addition to $\mathbf{p} = (p_1, \dots, p_n)$ depend on a finite even number of free parameters $2r$

$$(\mathbf{v}, \mathbf{u}), \; \mathbf{v} = (v_1, \dots, v_r), \; \mathbf{u} = (u_1, \dots, u_r).$$

$$r = \begin{cases} 0 & g = 0, \\ \mathrm{rank}G, & g = 1, \\ (g-1)\dim G, & g \geq 2. \end{cases}$$

$\mathcal{R}$ is a symplectic manifold with the non degenerate symplectic form which is the reduction of (4.5)

$$\omega_0 = \int_{\Sigma} < \delta L, \delta \bar{L} > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a \delta g_a^{-1} >= \tag{4.22}$$

On this stage we come to the bundle $\mathcal{P}''$ with the finite-dimensional fibers $\mathcal{R}$ over the infinite-dimensional base $\mathcal{N}'$ with the symplectic form

$$\omega = \int_{\Sigma} < \delta L, \delta \bar{L} > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1} \delta g_a > -\kappa \int_{\Sigma} < L, \delta L > \delta \mu. \tag{4.23}$$

4.8. Factorization with respect to the diffeomorphisms $\mathcal{G}_0$

We can utilize invariance of ω with respect to $\mathcal{G}_0$ and reduce $\mathcal{N}'$ to the finite-dimensional space $\mathcal{N}$, which is isomorphic to the moduli space $\mathcal{M}_{g,n}$. Let ϵ_0 be a vector field generated by the diffeomorphisms (4.11). Consider the action of the Lie derivative $\mathcal{L}_{\epsilon_0}$ on $\mathcal{A} = (A, \bar{A})$

$$\mathcal{L}_{\epsilon_0}\mathcal{A} = dj_{\epsilon_0}\mathcal{A} + j_{\epsilon_0}d\mathcal{A} = d_{\mathcal{A}}(j_{\epsilon_0}\mathcal{A}) + j_{\epsilon_0}F_{\mathcal{A}}, \quad (F_{\mathcal{A}} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]).$$

Thus for the flat connections $F_{\mathcal{A}} = 0$ the action of diffeomorphisms $\mathcal{G}_0$ on the connection fields is generated by the gauge transforms $j_{\epsilon_0}\mathcal{A} \in \mathcal{G}_1$. But we already have performed the symplectic reduction with respect to $\mathcal{G}_1$. Therefore, j_{ϵ_0} belongs to the kernel of ω (4.23) and we can push it down on the factor space $\mathcal{P}''/\mathcal{G}_0$. Since $\mathcal{G}_0$ acts only on $\mathcal{N}'$, it can be done by fixing the dependence of μ on the coordinates in the Teichmüller space $\mathcal{T}_{g,n}$. According to (4.4) represent μ as

$$\mu = \sum_{s=1}^{l} \mu_s, \quad \mu_s = t_s\mu_s^0, \; l = \dim \mathcal{T}_{g,n}, \; \mu_s^0 = \bar{\partial} n_s. \tag{4.24}$$

The Beltrami differential (4.24) defines the tangent vector

$$\mathbf{t} = (t_1, \ldots, t_l),$$

to the Teichmüller space $\mathcal{T}_{g,n}$ at the fixed point of $\mathcal{T}_{g,n}$.

We specify the dependence of μ on the positions of the marked points in the following way. Let $\mathcal{U}'_a \supset \mathcal{U}_a$ be two vicinities of the marked point x_a such that $\mathcal{U}'_a \cap \mathcal{U}'_b = \emptyset$ for $a \neq b$. Let $\chi_a(z, \bar{z})$ be a smooth function

$$\chi_a(z, \bar{z}) = \begin{cases} 1, & z \in \mathcal{U}_a \\ 0, & z \in \Sigma_g \setminus \mathcal{U}'_a. \end{cases} \tag{4.25}$$

Introduce times related to the positions of the marked points $t_a = x_a - x_a^0$. Then

$$\mu_a^0 = \bar{\partial} n_a(z, \bar{z}), \quad n_a(z, \bar{z}) = (1 + c_a(z - x_a))\chi_a(z, \bar{z}) \tag{4.26}$$

In other words $n_a(z, \bar{z})$ defines a local vector field deforming the complex coordinates only in $\mathcal{U}'_a$

$$w = z - t_a n_a(z, \bar{z}).$$

The action of $\mathcal{G}_0$ on the phase space $\mathcal{P}''$ reduces the infinite-dimensional component $\mathcal{N}'$ to $\mathcal{T}_{g,n}$. After the reduction we come to the bundle with base $\mathcal{T}_{g,n}$. Substituting

$$\delta\mu = \sum_{s=1}^{l} \mu_s^0 \delta t_s, \quad (\partial_s = \partial_{t_s}). \tag{4.27}$$

in(4.23) we obtain

$$\omega = \omega_0(\mathbf{v}, \mathbf{u}, \mathbf{p}, \mathbf{t}) - \frac{1}{\kappa}\sum_{s=1}^{l} \delta H_s(\mathbf{v}, \mathbf{u}, \mathbf{p}, \mathbf{t})\delta t_s, \tag{4.28}$$

where ω_0 is defined by (4.22), and H_s are the Hamiltonians

$$H_s = \int_\Sigma < L, L > \partial_s \mu \tag{4.29}$$

In fact, we still have a remnant discrete symmetry, since ω is invariant under the mapping class group $\pi_0(\mathcal{G}_0)$. Eventually, we come to the moduli space $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\pi_0(\mathcal{G}_0)$.

Summarizing, we have defined the extended phase space (the bundle $\mathcal{P}$) as the result of the symplectic reduction with respect to the $\mathcal{G}_1 \oplus \mathcal{G}_2$ action and subsequent factorization under the $\mathcal{G}_0$ action. We can write symbolically

$$\mathcal{P} = (\mathcal{P}''//\mathcal{G}_1 \oplus \mathcal{G}_2)/\mathcal{G}_0.$$

It is endowed with the symplectic form (4.28).

4.9. The hierarchies of the isomonodromic deformations (HID)

The equations of motion can be extracted from the symplectic form (4.28), as it was described in Section 3 (see (3.3)). They will be referred as *the hierarchies of the isomonodromic deformations* (HID). This notion will be justify later. In terms of the local coordinates (3.3) takes the form

$$\kappa \partial_s \mathbf{v} = \{H_s, \mathbf{v}\}_{\omega_0}, \quad \kappa \partial_s \mathbf{u} = \{H_s, \mathbf{u}\}_{\omega_0}, \quad \kappa \partial_s \mathbf{p} = \{H_s, \mathbf{p}\}_{\omega_0} \tag{4.30}$$

The Poisson bracket $\{\cdot, \cdot\}_{\omega_0}$ is the inverse tensor to ω_0. We also has the *Whitham hierarchy* (3.4) accompanying (4.30). It follows from (4.22) that the Hamiltonians H_s (4.29) commute.

$$\{H_r, H_s\}_{\omega_0} = 0.$$

Therefore,

$$\partial_s H_r - \partial_r H_s = 0, \tag{4.31}$$

and there exists the one form on $\mathcal{M}_{g,n}$ defining *the tau function* of the hierarchy of isomonodromic deformations

$$\delta \log \tau = -\frac{1}{\kappa} \sum H_s dt_s. \tag{4.32}$$

The following three statements are valid for the hierarchy of isomonodromic deformations (4.30):

Proposition 4.1 *The flatness condition (4.19) and HID (4.30) are equivalent to the consistent system of linear equations*

$$(\kappa \partial + L)\Psi = 0, \tag{4.33}$$

$$(\kappa \partial_s + M_s)\Psi = 0, \quad (s = 1, \ldots, l = \dim \mathcal{M}_{g,n}) \tag{4.34}$$

$$(\bar{\partial} + \mu \partial + \bar{L})\Psi = 0, \tag{4.35}$$

where M_s is a solution to the linear equation

$$\partial_{\bar{w}} M_s - [M_s, \bar{L}] = \kappa \partial_s \bar{L} - L\mu^0. \tag{4.36}$$

Proposition 4.2 *The linear conditions (4.34) provide the isomonodromic deformations of the linear system (4.33), (4.35) with respect to change the "times" on* $\mathcal{M}_{g,n}$.

Therefore, HID (4.30) are the monodromy preserving conditions for the linear system (4.33),(4.35).

The presence of derivative with respect to the spectral parameter $w \in \Sigma_{g,n}$ in the linear equation (4.33) is a distinguish feature of the isomonodromy preserving equations. It plagues the application of the inverse scattering method to this type of systems. The later means the solving the Riemann-Hilbert problem which amounts the reconstructing the pair $(L, \bar{L})$ from the monodromy data. In [35] this technique was applied to calculate asymptotics of solutions for bundles over rational curves. Nevertheless, in general case we have in some sense the explicit form of solutions:

Proposition 4.3 (The projection method.) *The solution of the Cauchy problem of (4.30) for the initial data* $\mathbf{v}^0, \mathbf{u}^0, \mathbf{p}^0$ *at the time* $\mathbf{t} = \mathbf{t}^0$ *is defined in terms of the elements* $L^0, \bar{L}^0$ *as the gauge transform*

$$\bar{L}(\mathbf{t}) = f^{-1}(L^0(\mu(\mathbf{t}) - \mu(\mathbf{t}^0)) + \bar{L}^0)f + f^{-1}(\bar{\partial} + \mu(\mathbf{t})\partial)f, \tag{4.37}$$

$$L(\mathbf{t}) = f^{-1}(\partial + L^0)f, \quad \mathbf{p}(\mathbf{t}) = f^{-1}(\mathbf{p}^0)f, \tag{4.38}$$

where $f = f(z, \bar{z})$ *is a smooth* G*-valued functions on* $\Sigma_{g,n}$ *fixing the gauge.*

It means that solutions of HID are gauge transformations of free motion in the upstairs system. Equations (4.37), (4.38) look like the dressing transform of the free motion. To find solutions one should know the gauge transform f from the upstairs system to a fixed gauge. For example, in case of genus zero we consider the holomorphic solutions Ψ. Therefore, f should kill the $\bar{L}$ operator in (4.37) (see below).

Proofs.
To prove first statement represent A as (4.18). The first equation in (4.7) $\partial_s A = 0$ means that

$$\kappa\partial_s L - \kappa\partial M_s + [M_s, L] = 0. \tag{4.39}$$

Then (4.39) is the consistency condition for the linear system (4.33), (4.34). On this stage M_s is defined as $M_s = \kappa f^{-1}\partial_s f$. To find the linear equation (4.36), defining M_s, it is necessary to substitute the gauge transformed form of $\bar{A}$ (4.17) together with (4.18) in the equation of motion in the form $\kappa\partial_s\bar{A} = A\partial_s\mu$. This equality is the same as

$$\kappa(\partial_s\mu\partial + \partial_s\bar{L}) - \partial_{\bar{w}} M_s + [M_s, \bar{L}] = \partial_s\mu(\kappa\partial + L).$$

The later equation coincides with (4.36). Eventually,

$$\begin{array}{ccc} \text{the compatibility } (4.33), (4.34) & \longleftrightarrow & \text{the Lax equations } (4.39), \\ \text{the compatibility } (4.33), (4.35) & \longleftrightarrow & \text{the flatness} (4.19), \\ \text{the compatibility } (4.34), (4.35) & \longleftrightarrow & (4.36). \end{array}$$

This concludes the first statement.

To prove the second statement note that (4.33),(4.35) are equivalent to

$$(\kappa\partial + A)\Psi^f = 0,$$

$$(\bar{\partial} + \mu\partial + \bar{A})\Psi^f = 0, \quad (\Psi^f = f^{-1}\Psi).$$

Due to the equations of motion ($\partial_s A = 0,\ \partial_s \bar{A} = \frac{1}{\kappa} A\partial_s\mu$) the monodromies of this system are independent on moduli, i.e. $\partial_s \Psi^f = 0$. The monodromies of the reduced system (4.33),(4.35) are conjugate to the monodromies of the later one. Thus, we come to the second statement.

To derive the expressions for dynamical variables in the projection method we lift the initial data $L^0, \bar{L}^0, p_1^0, \ldots, p_n^0$ from the reduced phase space $\mathcal{R}$ in the point t^0 to $\mathcal{R}'$ by the trivial gauge transform. Due to the equations of motion (4.7), the evolution in $\mathcal{R}$ is trivial

$$A(t) = L^0, \quad \bar{A} = \kappa L^0(\mu(t) - \mu(t^0)) + (\bar{L}^0)$$

and can be push back on the reduced phase space $\mathcal{R}$ by the gauge transform to the fixed gauge. This procedure is reflected in projection method formulae (4.37),(4.38).

5 Remarks about the Hitchin systems and the KZB equations

5.1. Scaling limit.

Consider our system in the limit $\kappa \to 0$. We will prove that in this limit we come to the Hitchin systems, which are living on the cotangent bundles to the moduli space of holomorphic G-bundles over $\Sigma_{g,n}$. The value $\kappa = 0$ is called critical and looks singular (see (4.6), (4.28)). To get around we rescale the times

$$\mathbf{t} = \mathbf{T} + \kappa\mathbf{t}^{\mathbf{H}}, \tag{5.1}$$

where $\mathbf{t}^H$ are the fast (Hitchin) times and $\mathbf{T}$ are the slow times. Therefore,

$$\delta\mu(\mathbf{t}) = \kappa\sum_s \mu_s^0 \delta t_s^H, \quad (\mu_s^0 = \bar{\partial} n_s).$$

After this rescaling the forms (4.6),(4.28) become regular. The rescaling procedure means that we blow up a vicinity of the fixed point (5.1) in $\mathcal{M}_{g,n}$ and the whole dynamic of the Hitchin systems is developed in this vicinity [2]. For example, we have instead (4.5) and (4.6)

$$\omega = \int_\Sigma < \delta A, \delta\bar{A} > + 2\pi i \sum_{a=1}^n \delta < p_a, g_a^{-1}\delta g_a > - \sum_s \int_\Sigma < \delta A, A > \partial_s\mu^0 \delta t^H. \tag{5.2}$$

If $\kappa = 0$ the connection A behaves as the one-form $A \in \Omega^{(1,0)}(\Sigma_{g,n}, \mathrm{Lie}(G))$ (see (4.13)). It is the so called the Higges field in terms of [3]. An important point is that the Hamiltonians now become the times independent. The form (5.2) is the starting point in the derivation

[2] We are grateful to A.Losev for elucidating this point.

of the Hitchin systems via the symplectic reduction [3, 7]. Essentially, it is the same procedure as described above. Namely, we obtain the same moment constraint (4.20) and the same gauge fixing (4.17). But now we are sitting in a fixed point $\mu(\mathbf{t}^0)$ of the moduli space $\mathcal{M}_{g,n}$ and don't need the factorization under the action of the diffeomorphisms and, thereby, do not worry about the modular properties of solutions $(L, \bar{L})$ of the moment constraint (4.20). This only difference between the solutions $(L, \bar{L})$ and the quadratic Hamiltonians H_s in the Hitchin systems and in HID. The symplectic reduction allows to identify the phase space $\mathcal{R}$ with the cotangent bundle to the moduli of holomorphic stable bundles.

Propositions 4.1 and 4.3 are valid for the Hitchin systems in a slightly modified form.

Proposition 5.1 *There exists the consistent system of linear equations*

$$(\lambda + L)\Psi = 0, \quad \lambda \in \mathbf{C} \tag{5.3}$$

$$(\partial_s + M_s)\Psi = 0, \quad \partial_s = \frac{\partial}{\partial t_s^H}, \ (s = 1, \ldots, l = \dim \mathcal{M}_{g,n}) \tag{5.4}$$

$$(\bar{\partial} + \bar{L})\Psi = 0, \ \bar{\partial} = \partial_{\bar{z}}, \tag{5.5}$$

where M_s is a solution to the linear equation

$$\bar{\partial} M_s - [M_s, \bar{L}] = \partial_s \bar{L} - L\mu^0. \tag{5.6}$$

Here we have

The compatibility (5.3), (5.4)	$\longleftrightarrow$	the Lax equation $\partial_{T_s} L = [L, M_s]$,
the compatibility (5.3), (5.5)	$\longleftrightarrow$	the moment equation (4.20),
the compatibility (5.4), (5.5)	$\longleftrightarrow$	(5.6).

To derive these equations from the general case (4.33), (4.34), and (4.35) we use the WKB approximation

$$\Psi = \Phi \exp \frac{\mathcal{S}}{\kappa},$$

where Φ is a group valued function and

$$\mathcal{S} = \mathcal{S}(\mathbf{t}, w) = \mathcal{S}^0(\mathbf{T}, w^0) + \kappa \mathcal{S}^1, \quad w^0 = z - \sum_s T_s n_s(z, \bar{z}),$$

$$\mathcal{S}^1 = \sum_s t_s^H \left(\frac{\partial}{\partial_{T_s}} - n_s(z, \bar{z})\partial\right)\mathcal{S}^0. \tag{5.7}$$

In the first order in κ the equations (4.33),(4.34),(4.35) just gives (5.3),(5.4),(5.5) if

$$\partial \mathcal{S}^0 = \lambda, \tag{5.8}$$

$$\frac{\partial}{\partial \bar{w}_0}\mathcal{S}^0 = 0, \quad \frac{\partial}{\partial t_s^H}\mathcal{S}^0 = 0.$$

Therefore, as soon as $\mathcal{S}^0$ satisfies these equation, Ψ provides solutions to the linearization of Hitchin systems. The detaile analyses of the perturbation in the rational case was undertaken in [36].

Equation (5.3) allows to introduce the fixed spectral curve

$$\mathcal{C}: \ \det(\lambda \cdot Id + L) = 0, \ \ \mathcal{C} \in \mathbf{P}(T^*\Sigma \oplus 1)$$

where λ is a coordinate in the cotangent space. The Hitchin phase space $\mathcal{R}$ has the "spectral" description. It is the bundle

$$\pi : \mathcal{R} \to \mathcal{M}_C$$

over the moduli $\mathcal{M}_C$ of spectral curves with abelian varieties as generic fibers. The map π acts from the pair $(A, \bar{A}) \sim (L, \bar{L})$, ($L$ now is the Higgs field $L \in \Omega^{(1,0)}(\Sigma_{g,n}, \mathrm{Lie}^*(G))$) to the set of coefficients of the characteristic polynomial $\det(\lambda \cdot Id + L)$. The one-form $\theta = \lambda dw^0$ being integrated over corresponding cycles in $\mathcal{C}$ gives rise to the action variables. The angle variables can be also extracted from the spectral curve. All together defines the symplectic structure on the Hitchin phase space in the spectral picture. This original Hitchin construction is working in the singular case as well [7]. The symplectic structure of this type connected with hyperelliptic curves was introduced originally in the soliton theory by Novikov and Veselov [37]. In terms of 4d gauge theories θ is the Seiberg-Witten differential. It follows from (5.8) that $\theta = d\mathcal{S}^0$ and along with (5.7) it defines the first order approximation to the solutions of the linear form (4.33),(4.34),(4.35) of HID.

It is possible to define the dynamic of the spectral curve beyond the crtical level as it was done for the Painlevé equations in [38]. It follows from the Lax representation (4.39) that

$$\partial_s \det(\lambda \cdot Id + L) = \mathrm{tr}\partial_w M_s(\lambda \cdot Id + L)^{-1}.$$

Note, that it defines the motion of $\mathcal{C}$ only within the subset $\mathcal{M}_{g,n} \subset \mathcal{M}_C$.

When L and thereby M can be find explicitly the simplified form of (4.20) allows to apply the inverse scattering method to find solutions of the Hitchin hierarchy as it was done for $\mathrm{SL}(N, \mathbf{C})$ holomorphic bundles over $\Sigma_{1,1}$ [39], corresponding to the elliptic Calogero system. We present the alternative way to describe the solutions:

Proposition 5.2 (The projection method.)

$$\bar{L}(t_s) = f^{-1}(L^0(t_s - t_s^0)\partial_s\mu^0 + \bar{L}^0)f + f^{-1}\bar{\partial}f,$$

$$L(t) = f^{-1}L^0 f, \ \ p_a(t) = f^{-1}(p_a^0)f$$

The degenerate version of these expressions was known for a long time [40].

5.2. About KZB

The Hitchin systems are the classical limit of the KZB equations on the critical level [7, 10]. The later has the form of the Schrödinger equations, which is the result of geometric quantization of the moduli of flat G bundles [16, 15]. The conformal blocks of the WZW theory on $\Sigma_{g,n}$ with vertex operators in marked points are the ground state wave functions

$$\hat{H}_s F = 0, \quad (s = 1, \ldots, l).$$

The classical limit means that one replaces operators on their symbols and generators of finite-dimensional representations in the vertex operators by the corresponding elements of coadjoint orbits.

Generically, for the quantum level $\kappa^{quant} \neq 0$ the KZB equations can be written in the form of the nonstationar Schrödinger equations [15, 10]

$$(\kappa^{quant} \partial_s + \hat{H}_s) F = 0.$$

To pass to the classical limit in this equation we replace the conformal block by its quasiclassical expression

$$F = \exp \frac{iS}{\hbar},$$

where S is the classical action ($S = \log \tau$ (4.32)) and renormalize

$$\kappa = \frac{\kappa^{quant}}{\hbar}.$$

The classical limit $\hbar \to 0$, $\kappa^{quant} \to 0$ leads the Hamilton-Jacobi equations for S, which are equivalent to the HID (4.30).

Summarizing, we arrange these quantum and classical systems in the commutative diagram. The vertical arrows denote to the classical limit and mean the simultaneous rescaling of the quantum level, while the limit $\kappa^{quant} \to 0$ ($\kappa \to 0$) on the horizontal arrows includes also the rescaling the moduli of complex structures. The examples in the bottom of the diagram will be considered in next sections.

$$\left\{\begin{array}{c}\text{KZB eqs.}, \ (\kappa, \mathcal{M}_{g,n}, G) \\ (\kappa^{quant} \partial_{t_a} + \hat{H}_a) F = 0, \\ (a = 1, \ldots, \dim \mathcal{M}_{g,n})\end{array}\right\} \xrightarrow{\kappa^{quant} \to 0} \left\{\begin{array}{c}\text{KZB eqs. on the critical level}, \\ (\mathcal{M}_{g,n}, G), \ (\hat{H}_a) F = 0, \\ (a = 1, \ldots, \dim \mathcal{M}_{g,n})\end{array}\right\}$$

$$\downarrow \hbar \to 0 \qquad\qquad\qquad\qquad \downarrow \hbar \to 0$$

$$\left\{\begin{array}{c}\text{Hierarchies of Isomonodromic} \\ \text{deformations on } \mathcal{M}_{g,n}\end{array}\right\} \xrightarrow{\kappa \to 0} \left\{\begin{array}{c}\text{Hitchin systems} \\ \end{array}\right\}$$

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$$\left\{\begin{array}{c}\text{Schlesinger eqs.} \\ \text{Painlevé type eqs.} \\ \text{Elliptic Schlesinger eqs.}\end{array}\right\} \xrightarrow{\kappa \to 0} \left\{\begin{array}{c}\text{Classical Gaudin eqs.} \\ \text{Calogero eqs.} \\ \text{Elliptic Gaudin eqs.}\end{array}\right\}$$

6 Genus zero - Schlesinger equations

6.1. Derivation of equations

Consider $\mathbf{C}P^1$ with n punctures $(x_1, \ldots, x_n | x_a \neq x_b)$. The Beltrami differential μ is related only to the positions of marked points. Then from (4.26)

$$\delta\mu_a = \bar{\partial}(1 + c_a \bar{\partial}(z - x_a)\chi_a(z, \bar{z}))\delta t_a, \quad (\delta t_a = \delta x_a). \tag{6.1}$$

On $\mathbf{C}P^1$ the gauge transform (4.17) allows to choose $\bar{A}$ to be identically zero. After the gauge fixing

$$\bar{A} = f\partial_{\bar{w}} f^{-1}, \tag{6.2}$$
$$A = fLf^{-1} + \kappa f \partial_w f^{-1},$$

the moment equation takes the form

$$\partial_{\bar{w}} L = 2\pi i \sum_{a=1}^{n} \delta^2(x_a) p_a.$$

It allows to find L

$$L = \sum_{a=1}^{n} \frac{p_a}{w - x_a}. \tag{6.3}$$

Then we have from (6.1)

$$\frac{1}{2}\delta \int_{\mathbf{C}P^1} < L, L > \delta\mu =$$
$$= \frac{1}{2}\sum_{b,a} \delta \int_{\mathbf{C}P^1} \frac{< p_a, p_b >}{(w - x_b)(w - x_a)} \sum_c \bar{\partial}(1 + c_c(w - x_c))\chi_c \delta x_c = \sum_a (\delta H_{a,1} + \delta H_{a,0})\delta x_a,$$

where

$$H_{a,1} = \sum_{b \neq a} \frac{< p_a, p_b >}{x_a - x_b}, \tag{6.4}$$

and

$$H_{2,a} = c_a < p_a, p_a > .$$

$H_{1,a}$ are precisely the Schlesinger's Hamiltonians. On the symplectic quotient ω (4.23) takes the form

$$\omega = \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a > -\frac{1}{\kappa}\sum_{b=1}^{n}(\delta H_{b,1} + \delta H_{b,0})\delta x_b.$$

Note, that we still have a gauge freedom with respect to the coordinate independent G action, since this action does not change our gauge fixing (6.2). The corresponding moment constraint means that the sum of residues of L vanishes:

$$\sum_{a=1}^{n} p_a = 0. \tag{6.5}$$

While $H_{2,a}$ are the Casimirs and lead to the trivial equations, the equation of motion for $H_{1,a}$ are the Schlesinger equations

$$\kappa \partial_b p_a = \frac{[p_a, p_b]}{x_a - x_b}, \quad (a \neq b),$$

$$\kappa\partial_a p_a = -\sum_{b\neq a}\frac{[p_a,p_b]}{x_a-x_b}.$$

As by product we obtain by this procedure the corresponding linear problem (4.33),(4.34) and (4.35) with $L=(6.3)$, $\bar{L}=0$ and

$$M_{a,1} = -\frac{p_a}{w-x_a}$$

as a solution to (4.36). The tau-function (4.32) for the Schlesinger equations has the form [2]

$$\delta\log\tau = -\sum_{c\neq b} < p_b, p_c > \delta\log(x_c-x_b).$$

6.2. Solutions via the projection method

We will find the dressing transform defining the evolution on the coadjoint orbits

$$p_a(\mathbf{t}) = f^{-1}(z,\bar{z},\mathbf{t})p_a^0 f(z,\bar{z},\mathbf{t}). \tag{6.6}$$

Recall that the times in the Schlesinger equations are $t_a = x_a - x_a^0$ and assume that $t_a^0 = 0$. We have from (6.3)

$$L(\mathbf{t}_0 = 0) = L^0 = \sum_{a=1}^{n}\frac{p_a^0}{z-x_a^0}.$$

It follows from (6.2) that $\bar{L}=0$. Then the projection method (4.37) gives in this case

$$f^{-1}(z,\bar{z},\mathbf{t})L^0\mu(\mathbf{t})f(z,\bar{z},\mathbf{t}) + f^{-1}(z,\bar{z},\mathbf{t})(\bar{\partial}+\mu(\mathbf{t})\partial)f(z,\bar{z},\mathbf{t}) = 0.$$

In other words, the gauge transform $f(z,\bar{z},\mathbf{t})$ defining the evolution of solutions (the dressing transform) can be found from the equation

$$[\bar{\partial} + \sum_k t_k\bar{\partial}\chi_k(z,\bar{z})(\partial + L^0)]f(z,\bar{z},\mathbf{t}) = 0. \tag{6.7}$$

We seek for smooth solutions to this equation assuming that the times $t_s = x_s - x_s^0$ are small. To this end consider the perturbative series

$$f(z,\bar{z},\mathbf{t}) = id + \sum_k t_k a_k + \sum_{j\leq k} t_j t_k a_{jk} + \sum_{i\leq j\leq k} t_i t_j t_k a_{ijk} + \ldots, \tag{6.8}$$

where $a_k, a_{jk}, a_{ijk}, \ldots$ are smooth maps $\mathbf{C}P^1 \to$ Universal enveloping algebra(G). Then (6.7) leads to the system

$$\begin{aligned}
&1)\bar{\partial}a_k = -L^0\bar{\partial}\chi_k\\
&2)\bar{\partial}a_{jk} = -(\partial+L^0)(\bar{\partial}\chi_j a_k + \bar{\partial}\chi_k a_j)\\
&3)\bar{\partial}a_{ijk} = -(\partial+L^0)\bar{\partial}\chi_{[j}a_{jk]}\\
&4)\ldots,
\end{aligned}$$

where $_{[\ldots]}$ means the symmetrization. All equations have the same structure - their solutions depends only on the previous step. Since $(0,1)$-forms on $\mathbf{C}P^1$ are exact the equations can be integrated and solutions are found step by step.

In the first order one has

$$a_k(z,\bar z) = -\sum_{a=1}^{n}\frac{p_a^0}{z-x_a^0}\chi_k^a(z,\bar z), \quad (\chi_k^a(z,\bar z)=\chi_k(z,\bar z)-\chi_k(x_a,\bar x_a)). \tag{6.9}$$

Note that $a_k(z,\bar z)$ is a nonsingular function on $\mathbf{C}P^1$ due to the definition of $\chi_k^a(z,\bar z)$. Consider now the second order approximation.

$$\bar\partial a_{jk}(z,\bar z) = \sum_{a=1}^{n}\partial(\frac{p_a^0}{z-x_a^0})\bar\partial\chi_{[j}(z,\bar z)\chi_{k]}^a(z,\bar z) + \sum_{a=1}^{n}\frac{p_a^0}{z-x_a^0}\bar\partial\chi_{[j}(z,\bar z)\partial\chi_{k]}^a(z,\bar z)+$$
$$+\sum_{a=1}^{n}\frac{p_a^0}{z-x_a^0}\sum_{b=1}^{n}\frac{p_b^0}{z-x_b^0}\bar\partial\chi_{[j}(z,\bar z)\chi_{k]}^b(z,\bar z).$$

The result of integration is

$$a_{jk}(z,\bar z) = \sum_{a=1}^{n}p_a^0(\frac{\psi_j^a(z,\bar z)\delta_{jk}}{z-x_a^0} - \frac{\chi_j^a(z,\bar z)\chi_k^a(z,\bar z)}{(z-x_a^0)^2})+ \tag{6.10}$$
$$\sum_{a=1}^{n}\frac{(p_a^0)^2}{(z-x_a^0)^2}\chi_j(z,\bar z)\chi_k^a(z,\bar z) - \sum_{a\neq b}^{n}\frac{p_a^0p_b^0}{z-x_b^0}(\chi_j^a(z,\bar z)\chi_k^a(z,\bar z)-\chi_j^a(x_b^0)\chi_k^a(x_b^0)).$$

Here $\psi_j^a(z,\bar z)$ is defined as the result of integration

$$\bar\partial\psi_j^a(z,\bar z) = 2\chi_j(z,\bar z)\chi_j^a(z,\bar z), \quad (\psi_j^a(x_a)=0). \tag{6.11}$$

It provides the absence of poles in $z=x_a$ in the first term in the right hand side of (6.10). Due to the subtraction of $\chi_j^a(x_b^0)\chi_k^a(x_b^0)$ in the last term we kill the pole of $a_{jk}(z,\bar z)$ in $z=x_b$. Thus, in second order approximation we obtain the regular solution as well. It is defined almost explicitly up to the integration (6.11).

Therefore, we have defined the dressing transformation (6.8),(6.9),(6.10) up to the third order of the initial data (6.6). The calculations of the higher order corrections reproduce the same procedure as on the second order and we can repeat them step by step.

7 Genus one- elliptic Schlesinger, Painlevé VI...

The genus one case is still feasible to write down the explicit formulae for the Hamiltonians and the equations of motion.

7.1. Deformations of elliptic curves

In addition to the moduli coming from the positions of the marked points there is the elliptic module τ, $Im\tau>0$ of curves $\Sigma_{1,n}$. As in (4.24),(4.26) we take the Beltrami differential in the form

$$\mu=\sum_{a=1}^{n}\mu_a+\mu_\tau, \quad (\mu_a=t_a\bar\partial n_a)$$

where $n_a(z,\bar z)$ is the same as in (4.26) and

$$n_\tau=(\bar z-z)(1-\sum_{a=1}^{n}\chi_a(z,\bar z)). \tag{7.1}$$

We replace

$$t_\tau \to \frac{t_\tau}{\rho}, \quad t_\tau = \tau - \tau_0, \quad \rho = \tau_0 - \bar{\tau}_0.$$

Here τ_0 defines the reference complex structure on the curve

$$T_0^2 = \{0 < x \le 1,\ 0 < y \le 1,\ z = x + \tau_0 y,\ \bar{z} = x + \bar{\tau}_0 y\}.$$

For small t_τ from (7.1)

$$\mu_\tau = \tilde{\mu}_\tau \bar{\partial}(\bar{z} - z)(1 - \sum_{a=1}^{n} \chi'_a(z, \bar{z})), \quad (\tilde{\mu}_\tau = \frac{t_\tau}{\tau - \tau_0}), \tag{7.2}$$

or

$$\mu_\tau = \frac{t_\tau}{\rho} \bar{\partial}(\bar{z} - z)(1 - \sum_{a=1}^{n} \chi_a(z, \bar{z})). \tag{7.3}$$

As we assumed from the very beginning, μ_τ vanishes in the marked points. It describes not only a small vicinity of T_0^2 in $\mathcal{M}_{1,1}$, but also the whole Teichmüller space as well. In terms of μ_τ the Teichmüller space is the unity disk $|\mu_\tau| < 1$ (see (4.12) for the transformation law). In terms of τ it is the upper half plane $\mathrm{Im}\tau > 0$. On the other hand, the times $\mathbf{t} = (t_\tau, t_1 = x_1 - x_1^0, \dots, t_n = x_n - x_n^0)$ define deformations

$$T_0^2(x_1^0, \dots, x_n^0) \xrightarrow{\mathbf{t}} T_\tau^2(x_1, \dots, x_n)$$

in a vicinity of T_0^2 in $\mathcal{M}_{1,1}$. Eventually, for small t_τ and t_a we have from (7.3)

$$\delta\mu = \delta\tilde{\mu}_\tau \bar{\partial}(\bar{z} - z)(1 - \sum_{a=1}^{n} \chi_a(z, \bar{z})) + \sum_{a=1}^{n} \delta t_a \bar{\partial}(1 + c_a(z - x_a))\chi_a(z, \bar{z})) \tag{7.4}$$

7.2. Flat bundles on a family of elliptic curves

Note first, that $\bar{A}$ as a $\bar{\partial}$-connection determines a holomorphic G bundle $\mathcal{E}$ over T_τ^2. For stable bundles $\bar{A}$ can be gauge transformed by (4.17) to the Cartan $(z, \bar{z})$-independent form $\bar{L}$

$$\bar{A} = f\bar{L}f^{-1} + f(\bar{\partial} + \mu\partial)f^{-1},$$

$$\bar{L} \in \mathcal{H} - \text{Cartan subalgebra of Lie}(G).$$

Therefore, a stable bundle $\mathcal{E}$ is decomposed into the direct sum of line bundles

$$\mathcal{E} = \oplus_{k=1}^{r} \mathcal{L}_k, \quad r = \mathrm{rank}(G).$$

The set of gauge equivalent connections represented by $\{\bar{L}\}$ can be identified with the r power of the Jacobian of T_τ^2, factorized by the action of the Weyl group W of G. Put

$$\bar{L} = 2\pi i \frac{1 - \tilde{\mu}_\tau}{\rho} \mathbf{u}, \quad \mathbf{u} \in \mathcal{H}, \quad (\frac{1 - \tilde{\mu}_\tau}{\rho} = \frac{1}{\tau - \bar{\tau}_0}). \tag{7.5}$$

It means that

$$\int_{T_\tau^2} \bar{L} dw = \mathbf{u}. \tag{7.6}$$

Let $\bar{L} = \bar{\partial} \log \phi$. Then the integral

$$\int_{T^2_\tau} \bar{L} dw = \int_{P_0}^{P} \log \phi dw.$$

defines the Abel map $P \in T^2_\tau \to \mathbf{u}$. We will come again to this point later.

The flatness condition (the moment constraints (4.21)) for the gauge transformed connections $(L, \bar{L})$ takes the form

$$\partial_{\bar{w}} L + [\bar{L}, L] = 2\pi i \sum_{a=1}^{n} \delta^2(x_a^0,) p_a. \tag{7.7}$$

Let $R = \{\alpha\}$ be the root system of of Lie$(G) = \mathcal{G}$ and

$$\mathcal{G} = \mathcal{H} \oplus_{\alpha \in R} \mathcal{G}_\alpha$$

be the root decomposition. Impose the vanishing of the residues in (7.7)

$$\sum_{a=1}^{n} p_a|_{\mathcal{H}} = 0, \tag{7.8}$$

where $p_a|_{\mathcal{H}}$ is the Cartan component of p_a and we have identified $\mathcal{G}$ with its dual space $\mathcal{G}^*$. This condition is similar to (6.5) and has the sense of the moment constraints for the remnant gauge action (see below **7.3**).

We will parametrized the set of solutions of (7.7) by two elements $\mathbf{v}, \mathbf{u} \in \mathcal{H}$. Let $E_1(w)$ be the Eisenstein function of module τ (A.2) and $(p_a)_{\mathcal{H}}$ and $(p_a)_\alpha$ are the Cartan and the root component of $p_a \in \mathcal{O}_a$,

Lemma 7.1 *Solutions of the moment constraint equation (7.7) have the form*

$$L = P + X, \quad P \in \mathcal{H}, \quad X = \sum_{\alpha \in R} X_\alpha. \tag{7.9}$$

$$P = 2\pi i (\frac{\mathbf{v}}{1 - \tilde{\mu}_\tau} - \kappa \frac{\mathbf{u}}{\rho} + \sum_{a=1}^{n} (p_a)_{\mathcal{H}} E_1(w - x_a)), \tag{7.10}$$

$$X_\alpha = \sum_{a=1}^{n} X_\alpha^a, \tag{7.11}$$

$$X_\alpha^a = \frac{(p_a)_\alpha}{1 - \tilde{\mu}_\tau} \exp 2\pi i \{ \frac{(w - x_a) - (\bar{w} - \bar{x}_a)}{\tau - \bar{\tau}_0} \alpha(\mathbf{u}) \} \phi(\alpha(\mathbf{u}), w - x_a),$$

where $\phi(u, w)$ is defined in (A.4).

Proof. Consider first the Cartan component. Since $\bar{L} \in \mathcal{H}$

$$\partial_{\bar{w}} P = 2\pi i \sum_{a=1}^{n} \delta^2(x_a) p_a.$$

From (A.28),(A.30) we obtain (7.10). The special choice of the constant part of P will be explained later. Note, that $\mathbf{v} \in \mathcal{H}$ is a new parameter.

For the root components (7.7) takes the form

$$(\partial_{\bar{w}} + 2\pi i \frac{1-\tilde{\mu}_\tau}{\rho}\alpha(\mathbf{u}))X_\alpha = 2\pi i \sum_{a=1}^{n} \delta^2(x_a,)(p_a)_\alpha .$$

Comparing it with (A.31) and its solution (A.32) we come to (7.11).□

Therefore we have found the flat connections $L = L(\mathbf{v}, \mathbf{u}), \bar{L} = \bar{L}(\mathbf{u})$.

7.3. Symmetries.

The remnant gauge transforms do not change the gauge fixing and thereby preserve the chosen Cartan subalgebra $\mathcal{H} \subset G$. These transformations are generated by the Weyl subgroup W of G and elements $f(w, \bar{w}) \in \mathrm{Map}(T^2_\tau, \mathrm{Cartan}(G))$. Let Π be the system of simple roots, $R^\vee = \{\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}\}$, is the dual root system, and $\mathbf{m} = \sum_{\alpha \in \Pi} m_\alpha \alpha^\vee$ be the element from the dual root lattice $\mathbf{Z}R^\vee$. Then the Cartan valued harmonics

$$f_{\mathbf{m},\mathbf{n}} = \exp 2\pi i (\mathbf{m}\frac{w-\bar{w}}{\tau-\bar{\tau}_0} + \mathbf{n}\frac{\tau\bar{w}-\bar{\tau}_0 w}{\tau-\bar{\tau}_0}), \quad (\mathbf{m},\mathbf{n} \in R^\vee) \tag{7.12}$$

generate the basis in the space of gauge transforms. They act as

$$\bar{L} \to \bar{L} + 2\pi i \frac{\mathbf{m}-\mathbf{n}\tau}{\tau-\bar{\tau}_0}.$$

$$P \to P + 2\pi i \kappa \frac{-\mathbf{m}+\mathbf{n}\bar{\tau}_0}{\tau-\bar{\tau}_0},$$

$$X^a_\alpha \to X^a_\alpha \varphi(m_\alpha, n_\alpha),$$

$$\varphi(m_\alpha, n_\alpha) = \exp \frac{4\pi i}{\tau-\bar{\tau}_0}[(m_\alpha + n_\alpha\bar{\tau}_0)(w-x_a) - (m_\alpha + n_\alpha\tau)(\bar{w}-\bar{x}_a)]. \tag{7.13}$$

In terms of the new variables $\mathbf{v}$ and $\mathbf{u}$ they take especial simple form

$$\mathbf{u} \to \mathbf{u} + \mathbf{m} - \mathbf{n}\tau, \quad \mathbf{v} \to \mathbf{v} - \kappa\mathbf{n}. \tag{7.14}$$

The whole discrete gauge symmetry is the semidirect product $\hat{W}$ of the Weyl group W and the lattice $\mathbf{Z}R^\vee \oplus \tau\mathbf{Z}R^\vee$. It is the Bernstein-Schvartsman complex crystallographic group [42]. The factor space $\mathcal{H}/\hat{W}$ is the genuine space for the coordinates $\mathbf{u}$, that we discussed above (see (7.5) and (7.6)).

The transformations (7.12) according with (4.13) act also on $p_a \in \mathcal{O}_a$. This action leads to the symplectic quotient $\mathcal{O}_a//H$ and generates the moment equation (7.8).

The modular group $\mathrm{PSL}_2(\mathbf{Z})$ s a subgroup of mapping class group for the Teichmüller space $\mathcal{T}_{1,n}$. We don't consider here the action of the permutation of the marked points on dynamical variables $(\mathbf{v}, \mathbf{u}, \mathbf{p}, \tau, x_a)$. Due to (4.12) and (7.2) its action on τ takes the standard form

$$\tau \to \gamma\tau = \frac{a\tau+b}{c\tau+d}, \quad \gamma \in \mathrm{PSL}_2(\mathbf{Z}).$$

We summarize the action of the Bernstein-Schvartsmann group and the modular group on the dynamical variables:

	W={s}	$\mathbf{Z}R^\vee \oplus \tau\mathbf{Z}R^\vee$	$\mathrm{PSL}_2(\mathbf{Z})$
$\mathbf{v}$	$s\mathbf{v}$	$\mathbf{v}+\kappa\mathbf{n}$	$\mathbf{v}(c\tau+d)-\kappa c\mathbf{u}$
$\mathbf{u}$	$s\mathbf{u}$	$\mathbf{u}-\mathbf{m}+\mathbf{n}\tau$	$\mathbf{u}(c\tau+d)^{-1}$
$(p_a)_{\mathcal{H}}$	$s(p_a)_{\mathcal{H}}$	p_a	p_a
$(p_a)_\alpha$	$(p_a)_{s\alpha}$	$\varphi(m_\alpha,n_\alpha)(p_a)_\alpha$	$(p_a)_\alpha$
τ	τ	τ	$\frac{a\tau+b}{c\tau+d}$
x_a	x_a	x_a	$\frac{x_a}{c\tau+d}$

Here $\varphi(m_\alpha,n_\alpha)$ is defined by (7.13).

7.4 Symplectic form.

The set $(\mathbf{v},\mathbf{u})\in\mathcal{H},\mathbf{p}=(p_1,\dots,p_n)\in\oplus_{a=1}^n\mathcal{O}_a$ of dynamical variables along with the times $\mathbf{t}=(t_\tau,t_1,\dots,t_n)$ describe local coordinates in the total space of the bundle $\mathcal{P}$. According with the general prescription, we can define the hamiltonian system on this set. The main statement, formulated in terms of the theta-functions and the Eisenstein functions (see Appendix), takes the form

Proposition 7.1 *The symplectic form ω (4.28) on $\mathcal{P}$ is*

$$\frac{1}{4\pi^2}\omega=(\delta\mathbf{v},\delta\mathbf{u})+\sum_{a=1}^n\delta<p_a,g_a^{-1}\delta g_a>-\frac{1}{\kappa}(\sum_{a=1}^n\delta H_{2,a}+\delta H_{1,a})\delta t_a-\frac{1}{\kappa}\delta H_\tau\delta\tau, \tag{7.15}$$

with the Hamiltonians

$$H_{2,a}=c_a<p_a,p_a>; \tag{7.16}$$

$$H_{1,a}==2(\frac{\mathbf{v}}{1-\tilde{\mu}_\tau}-\kappa\frac{\mathbf{u}}{\rho},p_a|_{\mathcal{H}})+\sum_{b\neq a}(p_a|_{\mathcal{H}},p_b|_{\mathcal{H}})E_1(x_a-x_b)+ \tag{7.17}$$

$$\sum_{b\neq a}\sum_\alpha(p_a|_\alpha,p_b|_{-\alpha})\phi(\alpha(\mathbf{u}),x_a-x_b);$$

$$H_\tau=\frac{(\mathbf{v},\mathbf{v})}{2}- \tag{7.18}$$

$$\frac{1}{4\pi^2}\sum_{a=1}^n\sum_\alpha(p_a|_\alpha,p_a|_{-\alpha})E_2(\alpha(\mathbf{u}))+\sum_{a\neq b}^n(p_a|_{\mathcal{H}},p_b|_{\mathcal{H}})(E_2(x_a-x_b)-E_1^2(x_a-x_b))-$$

$$\frac{1}{4\pi^2}\sum_{a\neq b}\sum_\alpha(p_a|_\alpha,p_b|_{-\alpha})\phi(-\alpha(\mathbf{u}),x_a-x_b)(E_1(\alpha(\mathbf{u}))-E_1(x_b-x_a+\alpha(\mathbf{u})));$$

$$\phi(\alpha(\mathbf{u}),x_a-x_b)=\frac{\theta(\alpha(\mathbf{u})+x_a-x_b)\theta'(0)}{\theta(\alpha(\mathbf{u}))\theta(x_a-x_b)}.$$

Proof. The form we have to calculate is (4.23)

$$\omega=\int_\Sigma<\delta L,\delta\bar{L}>+2\pi i\sum_{a=1}^n\delta<p_a,g_a^{-1}\delta g_a>-\frac{1}{\kappa}\int_\Sigma<L,\delta L>\delta\mu$$

with $\delta\mu$ (7.4). First, we gauge transform $(L,\bar{L})$ by

$$f(w,\bar{w})=\prod_{a=1}^n\exp(2\pi i\frac{w-\bar{w}}{\tau-\tau_0}\chi'_a(w,\bar{w})\mathbf{u}),$$

where we choose $\chi'_a(w,\bar{w})$ in a such way that

$$\mathrm{supp}\chi'_a(w,\bar{w})\subset \mathrm{supp}\chi_a(w,\bar{w}), \tag{7.19}$$

$$\mathrm{supp}\chi'_a(w,\bar{w})\cap \mathrm{supp}\bar{\partial}\chi_a(w,\bar{w})=\emptyset,$$

and $\chi_a(w,\bar{w})$ related to the moduli curves (7.1). In fact, the first condition follows from the second.

As we know the gauge transformations do not change ω, but do change relations between its summands. Instead of (7.5).(7.10) and (7.11) we obtain

$$\bar{L}=2\pi i\frac{1-\tilde{\mu}_\tau}{\rho}\mathbf{u}\partial_{\bar{w}}(\bar{w}-w)(1-\sum_{a=1}^{n}\chi'_a(w,\bar{w})), \tag{7.20}$$

$$P=2\pi i(\frac{\mathbf{v}}{1-\tilde{\mu}_\tau}-\kappa\frac{\mathbf{u}}{\rho}+\sum_{a=1}^{n}(p_a)_{\mathcal{H}}E_1(w-x_a))-$$

$$-\frac{1-\tilde{\mu}_\tau}{\rho}\mathbf{u}\partial_w(\bar{w}-w)\sum_{a=1}^{n}\chi'_a(w,\bar{w}), \tag{7.21}$$

$$X_\alpha=\frac{1}{1-\tilde{\mu}_\tau}\exp\{\frac{w-\bar{w}}{\tau-\bar{\tau}_0}\alpha(\mathbf{u})\}\sum_{a=1}^{n}(p_a)_\alpha\phi(\alpha(\mathbf{u}),w-x_a), \tag{7.22}$$

Taking into account the explicit form of $\bar{L}$ (7.5) we obtain

$$<\delta L,\delta\bar{L}>=\frac{(\delta\mathbf{v},\delta\mathbf{u})}{\rho}+S(\delta\tau,\delta\mathbf{t}), \tag{7.23}$$

where $S(\delta\tau,\delta\mathbf{t})$ is a sum of terms with a linear dependence on the "time" differentials. It is compensated by terms coming from

$$-\frac{1}{4\pi^2\kappa}\int_{T^2_\tau}<L,\delta L>\delta\mu.$$

Let us calculate the Hamiltonians

$$-\frac{1}{4\pi^2}<L,L>=(\frac{\mathbf{v}}{1-\tilde{\mu}_\tau}-\kappa\frac{\mathbf{u}}{\rho}+(\sum_{a=1}^{n}p_a|_{\mathcal{H}}E_1(w-x_a))^2-$$

$$-\frac{4\pi^2}{1-\tilde{\mu}_\tau}\sum_{a,b}\sum_{\alpha\in R}(p_a|_\alpha,p_b|_{-\alpha})\phi(-\alpha(\mathbf{u}),w-x_a)\phi(\alpha(\mathbf{u}),w-x_b)- \tag{7.24}$$

$$\frac{1}{2\pi i}(\frac{\mathbf{v}}{1-\tilde{\mu}_\tau}-\kappa\frac{\mathbf{u}}{\rho}+\sum_{a=1}^{n}(p_a)_{\mathcal{H}}E_1(w-x_a)),\frac{1-\tilde{\mu}_\tau}{\rho}\mathbf{u})\partial_w(\bar{w}-w)\sum_{a=1}^{n}\chi'_a(w,\bar{w}).$$

To take the integral over T^2_0 we have to couple this two form with $\delta\mu$ (7.4)

$$-\frac{1}{4\pi^2}\int_{T^2_\tau}<L,L>\delta\mu=$$

$$-\frac{1}{4\pi^2}\int_{T^2_\tau} < L, L > \delta\tilde{\mu}_\tau\bar{\partial}(\bar{w}-w)(1-\sum_{a=1}^n \chi_a(w,\bar{w})) + \sum_{a=1}^n \delta t_a \partial_{\bar{w}}(1+c_a(w-x_a))\chi_a(w,\bar{w})).$$

Due to our choice of $\chi'_a(w,\bar{w})$ (7.19), the last line of (7.24) does not contribute in the integral. Therefore, we leave with the holomorphic double periodic part of two-form $< L, L >$. Using (A.10) we rewrite (7.24) as

$$-\frac{1}{4\pi^2} < L, L >= (\frac{\mathbf{v}}{1-\tilde{\mu}_\tau} - \kappa\frac{\mathbf{u}}{\rho}, \frac{\mathbf{v}}{1-\tilde{\mu}_\tau} - \kappa\frac{\mathbf{u}}{\rho}) +$$

$$2\sum_{a=1}^n (\frac{\mathbf{v}}{1-\tilde{\mu}_\tau} - \kappa\frac{\mathbf{u}}{\rho}, p_a|_{\mathcal{H}}) E_1(w-x_a) +$$

$$\sum_{a\neq b}^n (p_a|_{\mathcal{H}}, p_b|_{\mathcal{H}})(E_1(w-x_a) - E_1(w-x_b)) -$$

$$-\frac{4\pi^2}{1-\tilde{\mu}_\tau}\sum_{a\neq b}\sum_{\alpha\in R}(p_a|_\alpha, p_b|_{-\alpha})\phi(-\alpha(\mathbf{u}), w-x_a)\phi(\alpha(\mathbf{u}), w-x_b) -$$

$$-\frac{4\pi^2}{1-\tilde{\mu}_\tau}\sum_a\sum_{\alpha\in R}(E_2(w-x_a) - E_2(\alpha(\mathbf{u})).$$

Since L has only first order poles (7.21),(7.22), we expand it on the deformed torus according with(A.34)

$$-\frac{1}{4\pi^2} < L, L >= (\sum_{a=1}^n H_{2,a}E_2(w-x_a) + H_{1,a}E_1(w-x_a)) + h_0. \qquad (7.25)$$

Due to (A.34),(A.39) and (A.40)

$$-\frac{1}{4\pi^2}\int_{T^2_\tau} < L, L > \delta\mu =$$

$$\sum_{a=1}^n (\int_{T^2_\tau}(H_{2,a}E_2(w-x_a)\partial_{\bar{w}}(w-x_a)\chi_a(w,\bar{w}) + H_{1,a}E_1(w-x_a))\partial_{\bar{w}}\chi_a(w,\bar{w}))\delta t_a$$

$$+h_0\delta\tilde{\mu}_\tau\bar{\partial}(\bar{w}-w)(1-\sum_{a=1}^n \chi_a(w,\bar{w})).$$

Taking into account and (A.35),(A.36) we find

$$H_{2,a} = res|_{x_a} < L, L > (w-x_a) = c_a < p_a, p_a > +const.,$$

$$H_{1,a} = res|_{x_a} < L, L >= (7.17).$$

The constant term in (7.25) h_0 is defined by (A.37). To find it we use (A.10) and (A.38). Then $H_\tau = h_0\partial_\tau\tilde{\mu}_\tau$. After some algebra we obtain (7.18). □

7.5 Example 1. $PVI_{\frac{\nu^2}{4},-\frac{\nu^2}{4},\frac{\nu^2}{4},\frac{1}{2}-\frac{\nu^2}{4}}$.

Consider SL$(2,\mathbf{C})$ bundles over the family of $\Sigma_{1,1}$. Then (7.5) takes the form

$$\bar{L} = 2\pi i\frac{1-\tilde{\mu}_\tau}{\rho}\mathrm{diag}(u,-u). \tag{7.26}$$

In this case the position of the marked point is no long the module and we put $x_1 = 0$. We have from (7.2)

$$w = z - \frac{\tau-\tau_0}{\rho}(\bar{z}-z),\ \ \bar{w} = \bar{z},$$

$$\partial_{\bar{w}} = \bar{\partial} + \frac{\tau-\tau_0}{\tau-\bar{\tau}_0}\partial.$$

Since $\dim\mathcal{O} = 2$ the orbit degrees of freedom can be gauged away by the hamiltonian action of the diagonal group. We assume that

$$p = \nu[(1,1)^T \otimes (1,1) - Id].$$

Then we have from(7.9),(7.10),(7.11)

$$L = \begin{pmatrix} 2\pi i(\frac{v}{1-\tilde{\mu}_\tau} - \kappa\frac{u}{\rho}) & x(2u,w,\bar{w}) \\ x(-2u,w,\bar{w}) & 2\pi i(-\frac{v}{1-\tilde{\mu}_\tau} + \kappa\frac{u}{\rho}) \end{pmatrix}. \tag{7.27}$$

$$x(u,w,\bar{w}) = \frac{\nu}{2\pi i(1-\tilde{\mu})_\tau}\exp 2\pi i\{(w-\bar{w})u\frac{1-\tilde{\mu}_\tau}{\rho}\}\phi(u,w).$$

The symplectic form (7.15)

$$-\frac{1}{8\pi^2}\omega = (\delta v,\delta u) - \frac{1}{\kappa}\delta H_\tau\delta\tau,$$

and

$$H_\tau = \frac{v^2}{2} + U(u|\tau),\ \ U(u|\tau) = -(\frac{\nu}{2\pi i})^2 E_2(2u|\tau) - \frac{\nu^2}{2\pi i}.$$

Then the equations of motion are

$$\kappa\frac{\partial u}{\partial\tau} = v, \tag{7.28}$$

$$\kappa\frac{\partial v}{\partial\tau} = \frac{\nu^2}{(2\pi i)^2}\frac{\partial}{\partial u}E_2(2u|\tau). \tag{7.29}$$

Due to (A.10) we obtain

$$\kappa^2\frac{\partial^2 u}{\partial\tau^2} = \frac{\nu^2}{(2\pi i)^2}\frac{\partial}{\partial u}\wp(2u|\tau).$$

which coincides with (1.12). This equation provides the isomonodromic deformation for the linear system (4.33),(4.35) with L (7.27) and $\bar{L}$ (7.26) with respect to change the module τ. The Lax pair is given by L (7.27) and M_τ

$$M_\tau = \begin{pmatrix} 0 & y(2u,w,\bar{w}) \\ y(-2u,w,\bar{w}) & 0 \end{pmatrix},$$

where $y(u,w,\bar{w})$ is defined by the equation (see (4.36)

$$(\partial_{\bar{w}}+\frac{2\pi i}{\tau-\bar{\tau}_0}u)y(u,w,\bar{w})=-\frac{\rho}{\kappa(\tau-\bar{\tau}_0)^2}x(u,w,\bar{w}). \tag{7.30}$$

Using the representation (A.33) for $x(u,w,\bar{w})=g_2(u,w,\bar{w})$ we find

$$y(u,w,\bar{w})=\frac{\rho}{2\pi i\kappa(\tau-\bar{\tau}_0)}\partial_u x(u,w,\bar{w}). \tag{7.31}$$

The equivalence of the Lax equation

$$\partial_\tau L-\kappa\partial_w M+[M,L]=0$$

to the equations of motion (7.28),(7.29) can be checked by substituting L (7.27) and M with $y(u,w,\bar{w})$‘ (7.31). Details of this procedure will be replenished for the SL$(N,\mathbf{C})$ bundles in the next example.

The projection method determines solutions of (7.28),(7.29) as a result of diagonalization of L (7.27) by the gauge transform on the deformed curve T^2_τ

$$\frac{1}{\tau-\bar{\tau}_0}\mathrm{diag}(u(\tau),-u(\tau))=$$

$$f(z,\bar{z},\tau)(\frac{2\pi i(\tau-\tau_0)}{\rho}\left(\begin{array}{cc} v^0-\kappa\frac{u^0}{\rho} & x(2u^0,z,\bar{z})\\ x(-2u^0,z,\bar{z}) & -v^0+\kappa\frac{u^0}{\rho}\end{array}\right)+\frac{1}{\rho}\mathrm{diag}(u^0,-u^0))f^{-1}(z,\bar{z},\tau)+$$

$$+f^{-1}(z,\bar{z},\tau)(\bar{\partial}+\frac{\tau-\tau_0}{\tau-\bar{\tau}_0}\partial)f(z,\bar{z},\tau).$$

On the critical level ($\kappa=0$) we come to the two-body elliptic Calogero system.

7.6. Example 2.

For flat G bundles over $\Sigma_{1,1}$ we obtain PVI-type equations, related to arbitrary root systems. They are described by the system of second order differential equations for the $\mathbf{u}=(u_1,\ldots,u_r)$, ($r$ =rank$G$) variables. In addition, there are the orbit variables $\mathbf{p}\in\mathcal{O}(G)$ satisfying the Euler top equations. Consider in detail the SL$(N,\mathbf{C})$ case with the most degenerate orbits $\mathcal{O}=T^*\mathbf{C}P^{N-1}$. They have dimension $2N-2$. The orbit variables can be gauged away by the diagonal gauge transforms and we are left with the coupling constant ν. We already have the $\bar{L}$ and L matrices (see (7.5) and (7.9))

$$L=P+X,$$

$$P=2\pi i(\frac{\mathbf{v}}{1-\tilde{\mu}_\tau}-\kappa\frac{\mathbf{u}}{\rho}),$$

$$X=\{x_\alpha\}=(\tau-\bar{\tau}_0)\nu\exp 2\pi i\{\frac{w-\bar{w}}{\tau-\bar{\tau}_0}\alpha(\mathbf{u})\}\phi(\alpha(\mathbf{u}),w).$$

Here $\alpha=e_j-e_k,\ \alpha(\mathbf{u})=u_j-u_k,\ (j\neq k)$, and

$$\mathbf{u}=\mathrm{diag}(u_1,\ldots,u_N),\quad \mathbf{v}=\mathrm{diag}(v_1,\ldots,v_N).$$

The equations of motion take the form

$$\kappa \frac{du_j}{d\tau} = v_j, \tag{7.32}$$

$$\kappa \frac{dv_j}{d\tau} = -\partial_{u_j} U(\mathbf{u}|\tau), \tag{7.33}$$

$$U(\mathbf{u}|\tau) = \frac{1}{(2\pi i)^2} \sum_{j \neq k} p_{j,k} p_{k,j} E_2(u_j - u_k|\tau).$$

On the critical level this Painlevé type system degenerates into N-body elliptic Calogero system.

Let us check the consistency of the equations of motion with the Lax representation

$$\partial_\tau L - \kappa \partial_w M + [M, L] = 0. \tag{7.34}$$

Put the M operator in the form

$$M = -D + Y,$$

where

$$Y = \{y(u_j - u_k)\} = \{y_\alpha\},$$

and D is a diagonal matrix. We have already found y_α (7.31) using (4.36). But the diagonal part D is not fixed by (4.36) and should be found from the consistency of the Lax equation with the equations of motion. We take it in the same form as in the Calogero system [41]

$$D = \mathrm{diag}(d_1, \ldots, d_N), \quad d_j = \sum_{i \neq j}^{N} s(u_j - u_i).$$

First, we will prove few facts concerning the matrix elements of L and M. We will prove that they satisfy the following functional equation

$$x(u, z, \bar{z}) y(v, z, \bar{z}) - x(v, z, \bar{z}) y(u, z, \bar{z}) = (s(v) - s(u)) x(u + v, z, \bar{z}). \tag{7.35}$$

In particular, its solution $s(u, w)$ is w independent

$$s(u) = \frac{1}{\kappa} \wp(u) + const. \tag{7.36}$$

The relation (7.35) is the so called Calogero functional equation. Due to (7.31) we can put in it the derivatives of x instead of y. The exponential factor in the expression of x cancels and (7.35) take the form of the addition formula (A.26). Simultaneously, we obtain (7.36). We also need the following identity

$$(\partial_\tau - \frac{\rho}{2\pi i} \partial_z \partial_u + \frac{u}{\tau - \bar{\tau}_0} \partial_u) x(u, z, \bar{z}) = 0. \tag{7.37}$$

It can be derived from the representation (A.33) for $x(u, z, \bar{z}) = g_2(u, z, \bar{z})$.

Consider now the Lax equation (7.34). Separation the diagonal and nondiagonal terms leads to the system

$$\frac{d}{d\tau} P + \kappa \partial_w D + \sum_\alpha (y_\alpha x_{-\alpha} - y_{-\alpha} x_\alpha) = 0, \tag{7.38}$$

$$\frac{d}{d\tau}x_\alpha - \kappa\partial_w y_\alpha - \alpha(D)x_\alpha + [Y,X]_\alpha = 0. \qquad (7.39)$$

Due to (7.36) and (7.32) the first equation (7.38) can be rewritten as

$$2\pi i\frac{d}{d\tau}v_k = \frac{(1-\tilde{\mu}_\tau)^2}{2\pi i\kappa}\sum_{j\neq k}(x(u_j-u_k)x'(u_k-u_j) - x(u_k-u_j)x'(u_j-u_k)) =$$

$$\sum_{j\neq k}[x(u_j-u_k)x(u_k-u_j)]'.$$

But

$$x(u)x(-u) = \frac{\nu^2}{(1-\mu)^2}\phi(u,w)\phi(-u,w) = \frac{\nu^2}{(1-\mu)^2}(E_2(w)-E_2(u)),$$

(see (A.10)). Thereby we come to the (7.33).

Now check the off-diagonal part (7.39). It is convenient to go back from

$$w = (1+\frac{\tau-\tau_0}{\rho})z - \frac{\tau-\tau_0}{\rho}\bar{z}, \quad \bar{w} = \bar{z}$$

to $(z,\bar{z})$ variables. Then (7.39) takes the form

$$\frac{d}{d\tau}x(u_j-u_k) - \kappa\partial_z y(u_j-u_k) - 2\pi i(\frac{v_j-v_k}{1-\tilde{\mu}_\tau} - \kappa\frac{u_j-u_k}{\rho})y(u_j-u_k) -$$

$$-x(u_j-u_k)[\sum_{i\neq j}^N s(u_j-u_i) - \sum_{i\neq k}^N s(u_k-u_i)] +$$

$$+\sum_{i=1}^N y(u_j-u_i)x(u_i-u_k) - y(u_i-u_k)x(u_j-u_i) = 0.$$

It follows from (7.31),(7.32), and (7.37) that

$$\frac{d}{d\tau}x(u_j-u_k) - \kappa\partial_z y(u_j-u_k) - 2\pi i(\frac{v_j-v_k}{1-\tilde{\mu}_\tau} - \kappa\frac{u_j-u_k}{\rho})y(u_j-u_k) = 0.$$

On the other hand the equality

$$x(u_j-u_k)(s(u_j-u_i) - s(u_k-u_i)) + y(u_j-u_i)x(u_i-u_k) - y(u_i-u_k)x(u_j-u_i) = 0.$$

is the addition formula (7.35). It concludes the proof of the equivalence of the Lax equation and the equations of motion.

The new ingredients of this construction in compare with the original form [41] are the dependence the L-matrix on the spectral parameter[3] , the presence of derivative $\partial_z M$ in the Lax equation, and the replacement of the external time t on the modular parameter τ. Nevertheless, the form of the Lax matrices is defined as for the Calogero system by the same functional equation (7.35). It turns out that its solutions satisfy additional differential equations (7.36), (7.37), which allows to apply them in the isomonodromic

[3] The dependence on the spectral parameter first introduced in [39] for the Calogero system in a slightly different form.

situation as well. Again, the solutions of the equations of motion (7.32), (7.33) are obtained by the diagonalization

$$2\pi i\frac{1-\tilde{\mu}_\tau}{\rho}\mathrm{diag}(u_1,\ldots,u_N)=\tilde{\mu}_\tau f^{-1}(L(\mathbf{v}^0,\mathbf{u}^0,\tau^0)f+$$

$$f^{-1}(\bar{\partial}+\tilde{\mu}_\tau\partial)f.$$

Appendix A

We summarize the main formulae for elliptic functions, borrowed mainly from [43] . We assume that $q=\exp 2\pi i\tau$, and the curve T^2_τ is $\mathbf{C}/\mathbf{Z}+\tau\mathbf{Z}$ factor of $\mathbf{C}$ under the shifts generated by $(1,\tau)$.

The basic element is the theta function:

$$\theta(z|\tau)=q^{\frac{1}{8}}\sum_{n\in\mathbf{Z}}(-1)^n e^{\pi i(n(n+1)\tau+2nz)}= \tag{A.1}$$

$$q^{\frac{1}{8}}e^{-\frac{i\pi}{4}}(e^{i\pi z}-e^{-i\pi z})\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{2i\pi z})(1-q^n e^{-2i\pi z})$$

The Eisenstein functions.

$$E_1(z|\tau)=\partial_z\log\theta(z|\tau),\quad E_1(z|\tau)\sim\frac{1}{z}+\ldots, \tag{A.2}$$

$$E_2(z|\tau)=-\partial_z E_1(z|\tau)=\partial_z^2\log\theta(z|\tau),\quad E_2(z|\tau)\sim\frac{1}{z^2}+\ldots. \tag{A.3}$$

The next important function is

$$\phi(u,z)=\frac{\theta(u+z)\theta'(0)}{\theta(u)\theta(z)}. \tag{A.4}$$

It has a pole at $z=0$ and

$$res|_{z=0}\phi(u,z)=1. \tag{A.5}$$

Relations to the Weierstrass functions.

$$\zeta(z|\tau)=E_1(z|\tau)+2\eta_1(\tau)z, \tag{A.6}$$

$$\wp(z|\tau)=E_2(z|\tau)-2\eta_1(\tau), \tag{A.7}$$

where

$$\eta_1(\tau)=\zeta(\frac{1}{2})= \tag{A.8}$$

$$\frac{3}{\pi^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty\,\prime}\frac{1}{(m\tau+n)^2}=\frac{24}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)},$$

where

$$\eta(\tau)=q^{\frac{1}{24}}\prod_{n>0}(1-q^n).$$

is the Dedekind function.

$$\phi(u,z) = \exp(-2\eta_1 uz)\frac{\sigma(u+z)}{\sigma(u)\sigma(z)}. \tag{A.9}$$

$$\phi(u,z)\phi(-u,z) = \wp(z) - \wp(u) = E_2(z) - E_2(u). \tag{A.10}$$

Series representations

$$E_1(z|\tau) = -2\pi i(\frac{1}{2} + \sum_{n\neq 0}\frac{e^{2\pi iz}}{1-q^n}) = \tag{A.11}$$

$$-2\pi i(\sum_{n<0}\frac{1}{1-q^n e^{2\pi iz}} + \sum_{n\geq 0}\frac{q^n e^{2\pi iz}}{1-q^n e^{2\pi iz}} + \frac{1}{2}).$$

$$E_2(z|\tau) = -4\pi^2\sum_{n\in\mathbf{Z}}\frac{q^n e^{2\pi iz}}{(1-q^n e^{2\pi iz})^2}. \tag{A.12}$$

$$\phi(u,z) = 2\pi i\sum_{n\in\mathbf{Z}}\frac{e^{-2\pi inz}}{1-q^n e^{-2\pi iu}}. \tag{A.13}$$

Parity.

$$\theta(-z) = -\theta(z) \tag{A.14}$$

$$E_1(-z) = -E_1(z) \tag{A.15}$$

$$E_2(-z) = E_2(z) \tag{A.16}$$

$$\phi(u,z) = \phi(z,u) = -\phi(-u,-z) \tag{A.17}$$

Behaviour on the lattice

$$\theta(z+1) = -\theta(z), \quad \theta(z+\tau) = -q^{-\frac{1}{2}}e^{-2\pi iz}\theta(z), \tag{A.18}$$

$$E_1(z+1) = E_1(z), \quad E_1(z+\tau) = E_1(z) - 2\pi i, \tag{A.19}$$

$$E_2(z+1) = E_2(z), \quad E_2(z+\tau) = E_2(z), \tag{A.20}$$

$$\phi(u+1,z) = \phi(u,z), \quad \phi(u+\tau,z) = e^{-2\pi iz}\phi(u,z). \tag{A.21}$$

Modular properties

$$\theta(\frac{z}{c\tau+d}|\frac{a\tau+b}{c\tau+d}) = \epsilon e^{\frac{\pi i}{4}}(c\tau+d)^{\frac{1}{2}}\exp(\frac{i\pi cz^2}{c\tau+d})\theta(z|\tau), \quad (\epsilon^8 = 1). \tag{A.22}$$

$$E_1(\frac{z}{c\tau+d}|\frac{a\tau+b}{c\tau+d}) = (c\tau+d)E_1(z|\tau) + 2\pi iz. \tag{A.23}$$

$$E_2(\frac{z}{c\tau+d}|\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2E_2(z|\tau) + 2\pi i(c\tau+d). \tag{A.24}$$

Addition formula

$$\phi(u,z)\partial_v\phi(v,z) - \phi(v,z)\partial_u\phi(u,z) = (E_2(v) - E_2(u))\phi(u+v,z), \tag{A.25}$$

or

$$\phi(u,z)\partial_v\phi(v,z)-\phi(v,z)\partial_u\phi(u,z)=(\wp(v)-\wp(u))\phi(u+v,z). \qquad (A.26)$$

The proof of (A.25) is based on (A.5),(A.17), and (A.21).

In fact, $\phi(u,z)$ satisfies more general relation which follows from the Fay three-section formula

$$\phi(u_1,z_1)\phi(u_2,z_2)-\phi(u_1+u_2,z_1)\phi(u_2,z_2-z_1)-\phi(u_1+u_2,z_2)\phi(u_1,z_1-z_2)=0 \quad (A.27)$$

Green functions

The Green functions are $(1,0)$ forms on T^2_τ. The first $g_1(z)$ is defined by the equation

$$\bar\partial g_1(z)=2\pi i\sum_a p_a\delta^2(x_a), \quad (\sum_a p_a=0), \qquad (A.28)$$

where

$$\delta^2(x_a)=\sum_{m,n\in\mathbf{Z}} f_{m,n}(z-x_a,\bar z-\bar x_a), \qquad (A.29)$$

$$f_{m,n}(z,\bar z)=\exp\frac{2\pi i}{\rho}\{m(z-\bar z)+n(\tau\bar z-\bar\tau z)\}, \quad (\rho=\tau-\bar\tau),$$

$$g_1(z)=\sum_a p_aE_1(z-x_a)+const. \qquad (A.30)$$

For the equation

$$(\bar\partial+\frac{2\pi i}{\rho}u)g_2(u,z)=2\pi i\delta^2(0), \qquad (A.31)$$

the Green function is

$$g_2(u,z)=\frac{1}{\rho}e^{2\pi i\frac{z-\bar z}{\rho}u}\phi(u,z). \qquad (A.32)$$

$$g_2(u,z)=\rho\sum_{m,n\in\mathbf{Z}}\frac{f_{m,n}(z,\bar z)}{u-m+n\tau} \qquad (A.33)$$

Expansion of elliptic functions.

Let M_2 be the space of meromorphic elliptic function on T^2_0 with poles of order two or less. Then any $f(z)\in M_2$ can be decomposed in the sum of the Eisenstein functions

$$f(z)=\sum_a(c_{2,a}E_2(z-x_a|\tau)+c_{1,a}E_1(z-x_a|\tau))+c_0, \qquad (A.34)$$

where

$$\sum_a c_{1,a}=0, \qquad (A.35)$$

$$c_{1,a}=res|_{x_a}f(z), \quad c_{2,a}=res|_{x_a}(z-x_a)f(z), \qquad (A.36)$$

and

$$c_0=\text{const. part}[f(z)-\sum_a(c_{2,a}E_2(z-x_a|\tau)+c_{1,a}E_1(z-x_a|\tau)]. \qquad (A.37)$$

In particular, for $a\neq b$

$$\text{const. part}[\phi(u,z-x_a)\phi(-u,z-x_b)]=\phi(-u,x_a-x_b)[E_1(u)-E_1(u+x_b-x_a)]. \quad (A.38)$$

According with (A.34) denote

$$e_{2,a} = E_2(z - x_a), \quad e_{1,a} = E_1(z - x_a), \quad e_0 = 1 \tag{A.39}$$

the basis in M_2. The dual basis with respect to the integration on T_0^2 is

$$f_{2,a} = \bar{\partial}(z - x_a)\chi_a(z, \bar{z}), \quad f_{1,a} = \bar{\partial}\chi_a(z, \bar{z}), \quad f_0 = \bar{\partial}(\bar{z} - z)(1 - \sum_{a=1}^{n} \chi_a(z, \bar{z})), \tag{A.40}$$

where χ_a is the characteristic function of vicinity $\mathcal{U}_a$ of x_a (see (4.25)).

Integrals.

$$\sum_a c_{1,a} \int_{T_\tau^2} E_1(z - x_a|\tau) = \sum_a c_{1,a}(x_a - \bar{x}_a). \tag{A.41}$$

$$\int_{T_\tau^2} E_2(z - x_a|\tau) = -2\pi i \tag{A.42}$$

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