Identification of a linear time-varying system using Haar wavelet

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Abstract: In this paper, Haar wavelet based identification of a continuous-time linear time-varying (LTV) system is proposed. For that purpose, input and output data are analyzed to derive an algebraic equation, leading to estimation of Haar wavelet coefficients for the impulse response. Finally, it is demonstrated that an LTV system can be effectively identified by solving the algebraic equation and by synthesizing the time-varying impulse response from the estimated wavelet coefficients.

Key-Words: Linear time-varying system, impulse response, Haar wavelet

1 Introduction
System identification has received much attention in control engineering and signal processing fields. In particular, wavelet-based approaches to identification of linear time-varying (LTV) systems have been addressed in a continuous-time domain [1]: For example, Daubechies wavelet was applied as an orthogonal basis. However, since no analytic expression exists for Daubechies wavelet, high computational burden is required for the system identification.

On the other hand, system identification of a continuous-time LTV state-space model by Haar wavelet was also presented, requiring less computational burden than that by Daubechies wavelet [2]. Moreover, some properties of Haar wavelets were established [3] and utilized for state analysis and parameter estimation of bilinear systems [4].

In this paper, a new approach to estimating the impulse response of a continuous-time LTV system is proposed by employing some useful properties of Haar wavelet, which can be achieved by solving an algebraic equation to estimate Haar wavelet coefficients for the impulse response, requiring relatively low computational burden.

2 Basic properties of Haar wavelet
Orthogonal basis functions including Haar wavelet has been utilized for the system identification [2]. In particular, the amplitude of the Haar wavelet is \( \pm 2^j \) \((j = 0,1,2,\cdots)\) in some finite intervals and zeros elsewhere (i.e., see (1)-(2)), reducing greatly the calculation process when the Haar wavelet is utilized [4]. If the scaling function and the prototype Haar wavelet are denoted by \( h_0(t) \) and \( h_1(t) \), respectively, all other wavelet bases (i.e., \( h_i(t) \)) can be generated by dilations and translations of \( h_1(t) \), and each base is normalized with unit energy [5]:

\[
\begin{align*}
h_0(t) &= 1, \quad 0 \leq t < 1, \\
h_1(t) &= \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t < 1 \end{cases} \quad (1) \\
h_n(t) &= 2^{-j/2} h_1(2^j t - k), \quad n = 2^j + k \geq 2, \quad j \geq 1, \quad 0 \leq k < 2^j \quad (2)
\end{align*}
\]

Let’s define \( h_m(t) \) as a group of the Haar wavelets:

\[
h_m(t) = [h_0(t), h_1(t), \cdots, h_m(t)]^T, \quad m = 2^i, \quad i \geq 0 \quad (3)
\]

Also, a digital representation of \( h_m(t) \) is defined by

\[
H_m = \left[ h_m\left(\frac{1}{2m}\right),\ h_m\left(\frac{3}{2m}\right),\ \cdots\ h_m\left(\frac{2m-1}{2m}\right) \right] \in R^{m} \quad (4)
\]

When \( m \) Haar bases are taken, the largest sampling time without aliasing is \( 1/m \) [2]. In general, a signal usually has some finite support, and thus, without loss of generality, the signal duration can be normalized as the time interval \( t \in [0,1) \) as in [2]. Accordingly, any square-integrable function \( y(t) \) in the interval \( 0 \leq t < 1 \) can be expressed by using the orthogonal bases \( \{h_0(t), h_1(t),\ldots,h_n(t),\ n = 1,2,\ldots,\infty\} \) [2]: i.e.,

\[
y(t) = \sum_{i=0}^{\infty} c_i h_i(t), \quad c_i = \int y(t) h_i(t) dt \quad (5)
\]

In practice, the approximation of \( y(t) \) using only \( m \) Haar wavelets is as follows:

\[
y(t) \approx \sum_{i=0}^{m} c_i h_i(t) = c_{(m)} h_m(t), \quad m = 2^i, \quad i \geq 0 \quad (6)
\]

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Another property is related with multiplication of two Haar wavelets. It can be also described by Haar bases \[4\]. That is, let $$\tau$$ be fixed at an arbitrary impulse response of an LTV system. (8) are utilized in this paper for identification of the system is unknown. Let the time $$t$$ be calculated from the following integration of $$P$$.

In (7), $$\Theta = \Theta_{(m)}$$ (or see (11)) should be estimated first. Also, the output $$y(t)$$ can be expressed from (10)-(12) as

$$y(t) = \int_{\tau}^{0} a_{m(t)}^T h_{(m)}(\tau) b_{(m)}^T h_{(m)}(\tau) \, d\tau$$

Since $$b_{(m)}^T h_{(m)}(\tau)$$ is a scalar, (13) can be written by

$$y(t) = \int_{\tau}^{0} a_{m(t)}^T h_{(m)}(\tau) h_{(m)}^T h_{(m)}(\tau) b_{(m)} \, d\tau$$

(14)

Note that $$h_{(m)}(\tau) h_{(m)}^T (\tau)$$ in (14) is a matrix function of $$\tau$$ and can still be described by Haar bases \[4\]. That is, from (7) and (8), there exists $$\Theta_{(m)} = \Theta_{(m)} \in R^{m \times 1}$$ satisfying $$h_{(m)}(\tau) h_{(m)}^T (\tau) b_{(m)} = \Theta_{(m)} h_{(m)}(\tau)$$. Then, (14) becomes

$$y(t) = a_{m(t)}^T \Theta_{(m)} \int_{\tau}^{0} h_{(m)}(\tau) \, d\tau$$

(15)

Since the Haar wavelet possesses finite points of discontinuity on the bounded time domain, the Haar wavelet can be integrable over the interval $$(0, t_f)$$ \[8\]. Let $$P_{(m)}$$ be calculated from the following integration of the Haar wavelet.

$$P_{(m)} = \int_{0}^{t_f} h_{(m)}(\tau) \, d\tau, \quad P_{(m)} \in R^{m \times 1}$$

(16)

By substituting (16) into (15), we have

$$y(t) = a_{m(t)}^T \Theta_{(m)} P_{(m)}$$

(17)
For a simple notation, let’s denote \( \Theta_{\tau_1}(\omega) \) by \( w(m) \in \mathbb{R}^{m \times 1} \). From (11), we can see that \( m \) unknown coefficients in \( a_{m,\tau} \) should be estimated to identify \( h(\tau, \tau) \), but we have only one equation (17). To solve such problem, \( m \) different inputs are applied to the LTV system, producing \( m \) outputs observed at \( t_k \) and leading to the following \( m \) equations to solve \( m \) unknown coefficients in \( a_{m,\tau} \).

Furthermore, (18) can be described in the following matrix form:

\[
Y(t_k) = W_{(m)}^T, \quad a_{m,\tau}
\]

(19)

\[
Y(t_k) = \begin{bmatrix} y_1(t_k), & y_2(t_k), & \cdots & y_m(t_k) \end{bmatrix}^T \in \mathbb{R}^{m \times 1}
\]

\[
W_{(m)} = \begin{bmatrix} w_{(m),1}, & w_{(m),2}, & \cdots & w_{(m),m} \end{bmatrix} \in \mathbb{R}^{m \times m}
\]

Accordingly, \( h(t_k, \tau) \) can be achieved from (19) and (11) if \( W_{(m)} \) is of full rank \( m \). When \( W_{(m)} \) is not of full rank, we need to set up input data until \( m \) column vectors of \( W_{(m)} \) are linearly independent.

4 Simulation results

To demonstrate the performance of the proposed approach, three LTV systems with different (or not necessarily separable with respect to its arguments) impulse responses are considered.

**Example 1**: Consider an LTV system whose impulse response is given by

\[
h(t, \tau) = \cos(5\pi(\tau^2 + t\tau))
\]

(20)

For this simulation, a piecewise-constant function was applied as the input to the LTV system (20) and the output was obtained from (9). Fig. 1 illustrates the true impulse responses (at \( t = 0.7 \) and \( t = 0.3 \)) and their approximations estimated by the proposed Haar wavelet-based approach (here, \( m = 32 \)).

**Example 2**: Consider another LTV system whose impulse response is given by

\[
h(t, \tau) = \sin(10\pi(\tau + t))e^{-10(\tau + t)}
\]

(21)

As in Example 1 and Example 2, the same input data are also utilized. In Fig. 3, the true impulse response at \( t = 0.7 \) and its approximation (obtained by the proposed approach) are shown, from which it can be seen that multiresolution analysis by Haar wavelet with larger number of bases (e.g., \( m = 32 \)) yields a better approximation to the true impulse response than ones by Haar wavelet with smaller number of bases (e.g., \( m = 8, 16 \)).

Fig. 1. The time-varying function and its approximation: (a) \( \tau = 0.7 \) and (b) \( \tau = 0.3 \)
Fig. 2. A time-varying function and its approximation: (a) $t=0.9$ and (b) $t=0.4$

Fig. 3. Approximations and multiresolution analysis by Haar wavelet with (a) $m=8$, (b) $m=16$, and (c) $m=32$.

5 Conclusions
In this paper, the problem of identifying a LTV system from input and output data is considered, whereby Haar wavelet is employed to form an algebraic equation for the system identification, from which Haar wavelet coefficients for the impulse response are estimated. Also, since the Haar wavelet possesses a finite value in a bounded time-domain and with unit energy, the proposed approach yields better computational efficiency than that by other wavelet or by square functions such as Walsh’s. Future research includes further extension of the proposed approach to the identification of nonlinear LTV systems.

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