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About this Book

This edited volume entitled "Bayesian Estimation and Reliability Estimation of Generalized Probability Distributions" is being published for the benefit of researchers and academicians. It contains ten different chapters covering a wide range of topics both in applied mathematics and statistics. The proofs of various theorems and examples have been given with minute details.

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Preface

Statistics is concerned with making inferences about the way the world is based upon things we observe happening. Statistical distributions are commonly applied to describe real world phenomena. Due to the usefulness of statistical distributions, this theory is widely studied, and new distributions are developed. The interest in developing more flexible statistical distributions remains strong in statistical profession. This edited book entitled "Bayesian Estimation and Reliability Estimation of Generalized Probability Distributions" is being published for the benefit of researchers and academicians. It contains ten different chapters covering wide range of topics both in Bayesian statistics and Probability distributions. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. During the preparation of the manuscript of this book, the editor has incorporated the fruitful academic suggestions provided by Dr. Peer Bilal Ahmad, Dr. Sheikh Parvaiz Ahmad, Dr. J. A. Reshi, Dr. Tanveer Ahmad Tarray, Dr. Kowsar Fatima, Dr. Ahmadur Rahman, Dr. Showkat Ahmad Lone, Mudasir Sofi, Uzma Jan, Aaliya Syed, and Dr. Humaira Sultan.

It is expected to have a good popularity due to its usefulness among its readers and users. Finally, I extend my thanks and appreciation to the authors for their continuous support in finalization of the book.

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(Editor)

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Chapter 1:

Bayesian Analysis of Zero-Inflated Generalized Power Series Distributions Under Different Loss Functions

Peer Bilal Ahmad

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Additional information is available at the end of the chapter

Introduction

As we know some of the family members of generalized power series distributions (GPSD) like binomial, negative binomial, Poisson and logarithmic series distributions are widely used for modelling count data. The properties of modality and divisibility of these distributions are known in the literature. Misra et.al (2003), Alamatsaz and Abbasi (2008), Aghababaei Jazi and Alamatsaz (2010), Abbasi et.al (2010) and Aghababaei Jazi et.al (2010) studied the stochastic ordering comparison between these distributions and their mixtures.

For modelling count data like accumulated claims in insurance and correlated count data which exhibit over-dispersion has resulted in introduction of zero-inflated and non-zero inflated parameter counterparts of the GPS distributions. Neyman (1939) and Feller (1943) studied that in some discrete data, the observed frequency for $X = 0$ is much higher than the expected frequency predicted by the assumed model. To be more specific, let us suppose that there are two machines. One of which is perfect and does not produce any defective item. The other machine produces defective items according to a Poisson distribution. We record the joint output of the two machines without knowing whether a specific item is produced by one or the other. In this case, the zero count seems to be inflated. Pandey (1964-65) studied a situation dealing with the number of flowers of plants of Primula veris. He has found that most of the plants were with eight flowers and inflated Poisson distribution (inflated at the point 8 not zero) proved to be the best model for fitting of such a data set. A similar data set on premature ventricular contractions where the distribution turns out to be inflated binomial has been analyzed by Farewell and Sprott (1988). Yip (1988) while dealing with the number of insects per leaf came to the conclusion that inflated Poisson distribution is the best fitted model for such a data set.

Martine, et al. (2005) and Kuhnert, et al. (2005) discussed the applications of zero-inflated modeling in ecology. Kolev, et al. (2000) studied the application of inflated-parameter family of generalized power series distributions in analysis of overdispersed insurance data. Patil and Shirke (2007) and Patil and Shirke (2011a, b) also studied different aspects of the zero-inflated power series distributions. From the literature it appears that majority of the study is restricted to properties and applications of inflated generalized power series distributions and relatively less work has been done on the estimation part particularly the Bayesian estimation of inflated generalized power series distributions. We also refer the readers to Winkelmann (2000), Hassan and Ahmad (2006), and Aghababaei Jazi and Alamatsaz (2011).

In this note, we studied the Bayesian analysis of zero-inflated power series distributions under different loss function i.e. squared error loss function and weighted squared error loss function. The results obtained for the zero-inflated power series distribution are then applied to its particular cases like zero-inflated Poisson distribution and zero-inflated negative binomial distribution.

Rodrigues (2003) studied zero-inflated Poisson distribution from the Bayesian perspective using data augmentation algorithm. Gosh, et al. (2006) introduced a flexible class of zero-inflated models which includes zero-inflated Poisson (ZIP) model, as special case and developed a Bayesian estimation method as an alternative to traditionally used maximum likelihood-based methods to analyze such data. As disused above, our aim is to give Bayes estimators of functions of parameters under different loss functions of zero-inflated generalized power series distribution (ZIGPSD) represented by the following probability mass function

$$P[X = x] = \begin{cases} \alpha + (1-\alpha)\frac{a(0)}{f(\theta)}, & x = 0 \\ (1-\alpha)\frac{a(x)\theta^x}{f(\theta)}, & x = 1,2,3,\ldots \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$ is the probability of inflation, $f(\theta) = \sum_x a(x)\theta^x$ is a function of parameter θ and is positive, finite and differentiable and coefficients $a(x)$ are non-negative and free of θ. It is clear that for $\alpha = 0$, the model (1.1) reduces to simple generalized power series distribution introduced by Patil (1961).

The whole article is divided in to different sections. Section 2 deals with the Bayes estimators of functions of parameters of zero-inflated generalized power series distribution (ZIGPSD) under squared error loss function and weighted square error loss function. Using different prior distributions and the results of zero-inflated GPSD, the Bayes estimators of functions of parameters of zero-inflated Poisson and zero-inflated negative binomial distributions are obtained in Sections 3 and 4 respectively. Finally, in Section 5, a numerical example is provided to illustrate the results and a goodness of fit test is done using the Bayes estimators.

Bayesian Estimation of Zero-Inflated GPSD

Let $X_1, X_2, \cdots, X_N$ be a random sample of size N drawn from the zero-inflated GPSD (1.1), then the likelihood function of $X_1, X_2, \cdots, X_N$ is given by

$$L(\theta, \alpha/\underline{x}) = \sum_{j=0}^{N_0} \binom{N_0}{j} \alpha^j (1-\alpha)^{N-j} (a(0))^{N_0-j} \prod_{i=1}^{N-N_0} a(x_i) \theta^t [f(\theta)]^{j-N} \tag{1.2}$$

where $\underline{x} = (x_1, x_2, \cdots, x_N)$, $t = \sum_{i=1}^{N-N_0} x_i$ and N_i is the number of observations in the i'th class such that.$\sum_{i\geq 1} N_i = N$.

For the Bayesian set up, we assumed that, priori, θ and α are independent, since in the zero-inflated distribution, an arbitrary probability is assigned to the zero class. As the parameter α represents the proportion of 'excess zeros', we may take $\text{Beta}\,(u, v)$ prior as a conjugate prior for α ,with prior density function

$$g(\alpha) = \frac{\alpha^{u-1}(1-\alpha)^{v-1}}{B(u,v)}, 0 < \alpha < 1, u, v > 0 \tag{1.3}$$

where, $B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$.

The prior distribution for θ is taken to be conjugate or non-conjugate prior distribution denoted by $h(\theta)$.

The Joint posterior probability density function (p.d.f) of θ and α corresponding to the prior $h(\theta)$ and $g(\alpha)$ respectively is given by

$$\Pi(\theta, \alpha/\underline{x}) = \frac{\sum_{j=0}^{N_0} \binom{N_0}{j} \alpha^{j+u-1} (1-\alpha)^{N-j+v-1} (a(0))^{N_0-j} \theta^t [f(\theta)]^{j-N} h(\theta)}{\sum_{j=0}^{N_0} \binom{N_0}{j} B(j+u, N-j+v)(a(0))^{N_0-j} \int_\Theta \theta^t [f(\theta)]^{j-N} h(\theta) d\theta} \tag{1.4}$$

The marginal posterior probability density functions of θ and α are respectively given by

$$\Pi(\theta/\underline{x}) = \frac{\sum_{j=0}^{N_0} \binom{N_0}{j} \alpha^{j+u-1} (1-\alpha)^{N-j+v-1} (a(0))^{N_0-j} \theta^t [f(\theta)]^{j-N} h(\theta)}{\sum_{j=0}^{N_0} \binom{N_0}{j} B(j+u, N-j+v)(a(0))^{N_0-j} \int_\Theta \theta^t [f(\theta)]^{j-N} h(\theta) d\theta} \tag{1.5}$$

$$\Pi(\alpha/\underline{x}) = \frac{\sum_{j=0}^{N_0} \binom{N_0}{j} \alpha^{j+u-1} (1-\alpha)^{N-j+v-1} (a(0))^{N_0-j} \int_\Theta \theta^t [f(\theta)]^{j-N} h(\theta) d\theta}{\sum_{j=0}^{N_0} \binom{N_0}{j} B(j+u, N-j+v)(a(0))^{N_0-j} \int_\Theta \theta^t [f(\theta)]^{j-N} h(\theta) d\theta} \tag{1.6}$$

The Bayes estimates $\hat{\eta}(\theta)$ of $\eta(\theta)$ and $\hat{\gamma}(\alpha)$ of $\gamma(\alpha)$ under the squared error loss function (SELF), where $\eta(\theta)$ and $\gamma(\alpha)$ are respectively the functions of θ and α are given by

$$\hat{\eta}_B = \frac{\sum_{j=0}^{N_0} \binom{N_0}{j} B(j+u, N-j+v)(a(0))^{N_0-j} \int_\Theta \eta(\theta) \theta^t [f(\theta)]^{j-N} h(\theta) d\theta}{\sum_{j=0}^{N_0} \binom{N_0}{j} B(j+u, N-j+v)(a(0))^{N_0-j} \int_\Theta \theta^t [f(\theta)]^{j-N} h(\theta) d\theta} \tag{1.7}$$

$$\hat{\gamma}_B = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}(a(0))^{N_0-j}\int_0^1\int_\Theta \gamma(\alpha)\alpha^{j+u-1}(1-\alpha)^{N-j+v-1}\theta^t[f(\theta)]^{j-N}h(\theta)d\theta d\alpha}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)(a(0))^{N_0-j}\int_\Theta \theta^t[f(\theta)]^{j-N}h(\theta)d\theta} \tag{1.8}$$

Similarly, under the weighted squared error loss function (WSELF) given by $L(\eta(\theta),d) = w(\theta)(\eta(\theta)-d)^2$ and $L(\gamma(\theta),d) = z(\alpha)(\gamma(\alpha)-d)^2$, where $w(\theta)$ is a function of θ, and $z(\alpha)$ is a function of α , d is a decision, the Bayes estimate $\hat{\eta}_w$ of $\eta(\theta)$ and $\hat{\gamma}_w$ of $\gamma(\alpha)$ are given by

$$\hat{\eta}_w = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)(a(0))^{N_0-j}\int_\Theta w(\theta)\eta(\theta)\theta^t[f(\theta)]^{j-N}h(\theta)d\theta}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)(a(0))^{N_0-j}\int_\Theta w(\theta)\theta^t[f(\theta)]^{j-N}h(\theta)d\theta} \tag{1.9}$$

$$\hat{\gamma}_w = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}(a(0))^{N_0-j}\int_0^1\int_\Theta z(\alpha)\gamma(\alpha)\alpha^{j+u-1}(1-\alpha)^{N-j+v-1}\theta^t[f(\theta)]^{j-N}h(\theta)d\theta d\alpha}{\sum_{j=0}^{N_0}\binom{N_0}{j}(a(0))^{N_0-j}\int_0^1\int_\Theta z(\alpha)\alpha^{j+u-1}(1-\alpha)^{N-j+v-1}\theta^t[f(\theta)]^{j-N}h(\theta)d\theta d\alpha} \tag{1.10}$$

Two different forms of $w(\theta)$and $z(\alpha)$ as weights has been considered and are given below:

(i) Let $w(\theta) = \theta^{-2}, z(\alpha) = \alpha^{-2}$, The Bayes estimate $\hat{\eta}_M$ of $\eta(\theta)$and $\hat{\gamma}_M$ of $\gamma(\alpha)$ known as the minimum expected loss (MEL) estimate are given by

$$\hat{\eta}_M = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)(a(0))^{N_0-j}\int_\Theta \eta(\theta)\theta^{t-2}[f(\theta)]^{j-N}h(\theta)d\theta}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)(a(0))^{N_0-j}\int_\Theta \theta^{t-2}[f(\theta)]^{j-N}h(\theta)d\theta} \tag{1.11}$$

$$\hat{\gamma}_M = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}(a(0))^{N_0-j}\int_0^1\int_\Theta \gamma(\alpha)\alpha^{(j+u-2)-1}(1-\alpha)^{N-j+v-1}\theta^t[f(\theta)]^{j-N}h(\theta)d\theta d\alpha}{\sum_{j=0}^{N_0}\binom{N_0}{j}(a(0))^{N_0-j}B(j+u-2,N-j+v)\int_\Theta \theta^t[f(\theta)]^{j-N}h(\theta)d\theta} \tag{1.12}$$

(ii) Let $w(\theta) = \theta^{-2}e^{-\delta\theta}; \delta > 0$ and $z(\alpha) = \alpha^{-2}e^{-\lambda\alpha}; \lambda > 0$.The Bayes estimate $\hat{\eta}_E$ of $\eta(\theta)$and $\hat{\gamma}_E$ of $\gamma(\alpha)$ known as the exponentially weighted minimum expected loss (EWMEL) estimate are given by

$$\hat{\eta}_E = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)(a(0))^{N_0-j}\int_\Theta \eta(\theta)\theta^{t-2}e^{-\delta\theta}[f(\theta)]^{j-N}h(\theta)d\theta}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)(a(0))^{N_0-j}\int_\Theta \theta^{t-2}e^{-\delta\theta}[f(\theta)]^{j-N}h(\theta)d\theta} \tag{1.13}$$

$$\hat{\gamma}_E = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}(a(0))^{N_0-j}\int_0^1\int_\Theta \gamma(\alpha)e^{-\lambda\alpha}\alpha^{(j+u-2)-1}(1-\alpha)^{N-j+v-1}\theta^t[f(\theta)]^{j-N}h(\theta)d\theta d\alpha}{\sum_{j=0}^{N_0}\binom{N_0}{j}(a(0))^{N_0-j}B(j+u-2,N-j+v)M(j+u-2,N+u+v-2,-\lambda)\int_\Theta \theta^t[f(\theta)]^{j-N}h(\theta)d\theta} \tag{1.14}$$

where $M(a,b;z)$ is the confluent hypergeometric function and has a series representation given by

$$M(a,b;z) = \sum_{j=0}^{\infty}\frac{(a)_j z^j}{(b)_j j!} \text{ where } (a)_0 = 1 \tag{1.15}$$

and $(a)_j = a(a+1)(a+2)\dots\dots\dots(a+j-1)$ (1.16)

Now, we apply the above results to zero-inflated Poisson and zero-inflated negative binomial distributions which are the special cases of the p.m.f. (1.1) and obtain the corresponding Bayes estimators of parameters in each case.

Bayesian Estimation of Zero-Inflated Poisson Distribution

A discrete random variable X is said to follow zero-inflated Poisson distribution (NZIPD) if its probability mass function is given by

$$P[X=x]=\begin{cases}\alpha+(1-\alpha)\frac{e^{-\theta}}{x!}, & x=0\\ (1-\alpha)\frac{e^{-\theta}\theta^{x}}{x!}, & x=1,2,3,\ldots\end{cases} \tag{1.17}$$

where $\theta>0, o<\alpha<1$.

If $\alpha=0$, the model (1.17) reduces to classical Poisson distribution.

It is a special case of (1.1) with

$$f(\theta)=e^{\theta},\ a(x)=\frac{1}{x!}$$

In this case, the likelihood function $L(\theta,\alpha/\underline{x})$ is of the form

$$L(\theta,\alpha/\underline{x})=\sum_{j=0}^{N_0}\binom{N_0}{j}\alpha^{j}\,(1-\alpha)^{N-j}\theta^{t}e^{-\theta(N-j)} \tag{1.18}$$

With gamma prior for θ given by

$$h(\theta)=\frac{a^{b}}{\Gamma b}e^{-a\theta}\theta^{b-1},\theta,a,b>0 \tag{1.19}$$

and beta prior for α given by (1.3), the joint Posterior probability density function of θ and α is given by

$$\Pi(\theta,\alpha/\underline{x})=\frac{\sum_{j=0}^{N_0}\binom{N_0}{j}\alpha^{(j+u)-1}\,(1-\alpha)^{(N-j+v)-1}\theta^{(t+b)-1}e^{-\theta(N-j+a)}}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{\Gamma(t+b)}{(N-j+a)^{t+b}}} \tag{1.20}$$

The marginal posterior distribution of θ and α are respectively given by

$$\Pi(\theta/\underline{x})=\frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\theta^{(t+b)-1}e^{-\theta(N-j+a)}}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{\Gamma(t+b)}{(N-j+a)^{t+b}}} \tag{1.21}$$

$$\Pi(\alpha/\underline{x}) = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}\frac{1}{(N-j+a)^{t+b}}\alpha^{(j+u)-1}(1-\alpha)^{(N-j+v)-1}}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{1}{(N-j+a)^{t+b}}} \tag{1.22}$$

Under SELF, the Bayes estimate $\hat{\theta}_B^r$ of θ^r and $\hat{\alpha}_B^r$ of α^r are given by

$$\hat{\theta}_B^r = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{\Gamma(t+b+r)}{(N-j+a)^{t+b+r}}}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{\Gamma(t+b)}{(N-j+a)^{t+b}}} \tag{1.23}$$

$$\hat{\alpha}_B^r = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u+r,N-j+v)\frac{1}{(N-j+a)^{t+b}}}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{1}{(N-j+a)^{t+b}}} \tag{1.24}$$

Similarly, under WSELF, when $w(\theta) = \theta^{-2}, z(\alpha) = \alpha^{-2}$, the minimum expected loss (MEL) estimate of $\eta(\theta) = \theta^r$ and $\gamma(\alpha) = \alpha^r$ are obtained as

$$\hat{\theta}_M^r = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{\Gamma(t+b-2+r)}{(N-j+a)^{t+b-2+r}}}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{\Gamma(t+b-2)}{(N-j+a)^{t+b-2}}} \tag{1.25}$$

$$\hat{\alpha}_M^r = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u+r-2,N-j+v)\frac{1}{(N-j+a)^{t+b}}}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u-2,N-j+v)\frac{1}{(N-j+a)^{t+b}}} \tag{1.26}$$

Finally, under the weighted squared error loss function, when $w(\theta) = \theta^{-2}e^{-\delta\theta}; \delta > 0$ and $z(\alpha) = \alpha^{-2}e^{-\lambda\alpha}, \lambda > 0$, the EWMEL estimate $\eta(\theta)$ and $\gamma(\alpha)$ are given by

$$\hat{\theta}_E^r = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{\Gamma(t+b-2+r)}{(N-j+a+\delta)^{t+b-2+r}}}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\frac{\Gamma(t+b-2)}{(N-j+a+\delta)^{t+b-2}}} \tag{1.27}$$

$$\hat{\alpha}_E^r = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u+r-2,N-j+v)M(J+u+r-2,N+u+v+r-2;\lambda)\frac{1}{(N-j+a)^{t+b}}}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u-2,N-j+v)M(J+u-2,N+u+v-2;\lambda)\frac{1}{(N-j+a)^{t+b}}} \tag{1.28}$$

Bayesian Estimation of Zero Inflated Negative Binomial Distribution

A discrete random variable X is said to have zero-inflated negative binomial distribution (ZINBD) if its probability mass function is given by

$$P[X = x] = \begin{cases} \alpha + (1-\alpha)(1-\theta)^m, & x = 0 \\ (1-\alpha)\binom{m+x-1}{x}\theta^x(1-\theta)^m, & x = 1, 2, 3, \ldots\ldots. \end{cases} \tag{1.29}$$

where $0 < \theta < 1, 0 < \alpha \leq 1$

It is a special case of (1.1) with $f(\theta) = (1-\theta)^{-m}$ and $a(x) = \binom{m+x-1}{x}$.

If $\alpha = 0$, the model (1.29) reduces to binomial distribution.

In this case the likelihood function $L(\theta, \alpha/\underline{x})$ is given by

$$L(\theta,\alpha/\underline{x}) \propto \sum_{j=0}^{N_0}\binom{N_0}{j}\alpha^{j}\,(1-\alpha)^{N-j}\theta^{t}(1-\theta)^{mN-mj} \tag{1.30}$$

Since $0 < \theta < 1$, we have taken two different prior distributions for θ given below

$$h_1(\theta) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)}, 0 < \theta < 1, a, b > 0 \tag{1.31}$$

where $B(a,b) = \frac{\Gamma a \Gamma b}{\Gamma(a+b)}$ and

$$h_2(\theta) = \frac{e^{-c\theta}\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)M(a,a+b;-c)}, 0 < \theta < 1, a, b > 0, \tag{1.32}$$

where $M(a,b;z)$ is the confluent hypergeometric function and has a series representation given by (1.15) and (1.16)

The joint posterior p.d.f of θ and α corresponding to the prior $h_1(\theta)$ and $g(\alpha)$ is given by

$$\Pi_1(\theta,\alpha/\underline{x}) = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}\alpha^{(j+u)-1}\,(1-\alpha)^{(N-j+v)-1}\theta^{(t+a)-1}(1-\theta)^{mN-mj+b-1}}{\sum_{j=0}^{N_0}\binom{N_0}{j}\,B(j+u,N-j+v)\,B(t+a\,,mN-mj+b)}$$

(1.33)

The marginal posterior distribution of θ and α are respectively given by

$$\Pi_1(\theta/\underline{x}) = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,\theta^{(t+a)-1}(1-\theta)^{mN-mj+b-1}}{\sum_{j=0}^{N_0}\binom{N_0}{j}\,B(j+u,N-j+v)\,B(t+a\,,mN-mj+b)} \tag{1.34}$$

$$\Pi_1(\alpha/\underline{x}) = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}\,B(t+a\,,mN-mj+b)\alpha^{(j+u)-1}\,(1-\alpha)^{(N-j+v)-1}}{\sum_{j=0}^{N_0}\binom{N_0}{j}\,B(j+u,N-j+v)\,B(t+a\,,mN-mj+b)} \tag{1.35}$$

Similarly, the joint posterior p.d.f of θ and α corresponding to the prior $h_2(\theta)$ and $g(\alpha)$ is given by

$$\Pi_2(\theta,\alpha/\underline{x}) = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}\alpha^{(j+u)-1}\,(1-\alpha)^{(N-j+v)-1}\theta^{(t+a)-1}(1-\theta)^{mN-mj+b-1}e^{-c\theta}}{\sum_{j=0}^{N_0}\binom{N_0}{j}\,B(j+u,N-j+v)\,B(t+a\,,mN-mj+b)M(t+a,a+b+mN-mj,-c)} \tag{1.36}$$

The marginal posterior distributions of θ and α are respectively given by

$$\Pi_2(\theta/\underline{x}) = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,\theta^{(t+a)-1}(1-\theta)^{mN-mj+b-1}e^{-c\theta}}{\sum_{j=0}^{N_0}\binom{N_0}{j}\,B(j+u,N-j+v)\,B(j+a\,,mN-mj+b)M(t+a,a+b+mN-mj,-c)} \tag{1.37}$$

$$\Pi_2(\alpha/\underline{x}) = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}\,B(t+a\,,mN-mj+b)M(t+a,a+b+mN-mj,-c)\alpha^{(j+u)-1}\,(1-\alpha)^{(N-j+v)-1}}{\sum_{j=0}^{N_0}\binom{N_0}{j}\,B(j+u,N-j+v)\,B(t+a\,,mN-mj+b)M(t+a,a+b+mN-mj,-c)} \tag{1.38}$$

Under SELF, the Bayes estimate of θ^r and α^r corresponding to the posterior density (1.34) and (1.35) respectively, are given by

$$\hat{\theta}_{1B}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,B(t+a+r\,,mN-mj+b)}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,B(t+a\,,mN-mj+b)} \tag{1.39}$$

$$\hat{\alpha}_{1B}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u+r,N-j+v)\,B(t+a\,,mN-mj+b)}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,B(t+a\,,mN-mj+b)} \tag{1.40}$$

Under WSELF, when $w(\theta) = \theta^{-2}, z(\alpha) = \alpha^{-2}$, the minimum expected loss (MEL) estimate of θ^r and α^r corresponding to the posterior density (1.34) and (1.35) respectively, are given by

$$\hat{\theta}_{1M}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,B(t+a+r-2\,,mN-mj+b)}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,B(t+a-2\,,mN-mj+b)} \tag{1.41}$$

$$\hat{\alpha}_{1M}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u-2+r,N-j+v)\,B(t+a\,,mN-mj+b)}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u-2,N-j+v)\,B(t+a\,,mN-mj+b)} \tag{1.42}$$

Finally under WSELF, when $w(\theta) = \theta^{-2}e^{-\delta\theta}; \delta > 0$, and $z(\alpha) = \alpha^{-2}e^{-\lambda\alpha}, \lambda > 0$, the EWMEL estimate of θ^r and α^r corresponding to the posterior density (1.34) and (1.35) respectively, are given by

$$\hat{\theta}_{1E}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,B(t+a+r-2\,,mN-mj+b)M_1}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,B(t+a-2\,,mN-mj+b)M_2} \tag{1.43}$$

where $M_1 = M(a+t-2+r, a+b+t+mN-mj-2+r, -\delta)$

$M_2 = M(a+t-2, a+b+t+mN-mj-2, -\delta)$

$$\hat{\alpha}_{1E}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u-2+r,N-j+v)\,B(t+a\,,mN-mj+b)M_3}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u-2,N-j+v)\,B(t+a\,,mN-mj+b)M_4} \tag{1.44}$$

where $M_3 = M(j+u-2+r, N+u+v-2+r, -\lambda)$

$M_4 = M(j+u-2, N+u+v-2, -\lambda)$

Also, SELF, the Bayes estimate of θ^r and of α^r corresponding to the posterior density (1.37) and (1.38) respectively, are given by

$$\hat{\theta}_{2B}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,B(t+a+r\,,mN-mj+b)M_5}{\sum_{j=0}^{N_0}\binom{N_0}{j}B(j+u,N-j+v)\,B(t+a\,,mN-mj+b)M(t+a,a+b+mN-mj,-c)} \tag{1.45}$$

where, $M_5 = M(a+t+r, a+b+t+mN-mj+r, -c)$

$$\hat{\alpha}_{2B}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u+r,N-j+v)\, B(t+a\,,mN-mj+b) M(t+a,a+b+mN-mj,-c)}{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u,N-j+v)\, B(t+a\,,mN-mj+b) M(t+a,a+b+mN-mj,-c)} \tag{1.46}$$

Under WSELF, when $w(\theta) = \theta^{-2}, z(\alpha) = \alpha^{-2}$, the MEL estimate of θ^r and α^r corresponding to the posterior density (1.37) and (1.38) respectively, are given by

$$\hat{\theta}_{2M}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u,N-j+v)\, B(t+a+r-2\,,mN-mj+b) M_6}{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u,N-j+v)\, B(t+a-2\,,mN-mj+b) M_7} \tag{1.47}$$

where, $M_6 = M(a+t+r-2, a+b+t+mN-mj+r-2, -c)$,

$$M_7 = M(a+t-2, a+b+t+mN-mj-2, -c)$$

$$\hat{\alpha}_{2M}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u+r-2,N-j+v)\, B(t+a\,,mN-mj+b) M(t+a,a+b+mN-mj,-c)}{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u-2,N-j+v)\, B(t+a\,,mN-mj+b) M(t+a,a+b+mN-mj,-c)} \tag{1.48}$$

Finally under WSELF, when $w(\theta) = \theta^{-2}e^{-\delta\theta}; \delta > 0$, and $z(\alpha) = \alpha^{-2}e^{-\lambda\alpha}, \lambda > 0$, the EWMEL estimate θ^r and α^r corresponding to the posterior density (1.37) and (1.38) respectively, are given by

$$\hat{\theta}_{2E}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u,N-j+v)\, B(t+a+r-2\,,mN-mj+b) M_8}{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u,N-j+v)\, B(t+a-2\,,mN-mj+b) M_9} \tag{1.49}$$

where $M_8 = M(a+t-2+r, a+b+t+mN-mj-2+r, -(c+\delta))$

$$M_9 = M(a+t-2, a+b+t+mN-mj-2, -(c+\delta))$$

$$\hat{\alpha}_{2M}^{r} = \frac{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u+r-2,N-j+v)\, B(t+a\,,mN-mj+b) M(t+a,a+b+mN-mj,-c) M_{10}}{\sum_{j=0}^{N_0}\binom{N_0}{j} B(j+u-2,N-j+v)\, B(t+a\,,mN-mj+b) M(t+a,a+b+mN-mj,-c) M_{11}}$$

(1.50)

$$M_{10} = M(j+u+r-2, N+u+v+r-2, -\lambda)$$

$$M_{11} = M(j+u-2, N+u+v-2, -\lambda)$$

An Illustrative Example

In order to demonstrate the practical applications of the above-mentioned results, we fitted the classical Poisson distribution and zero-inflated Poisson distribution to the data pertaining to the number of strikes in 4-weaks in Vehicle Manufacturing Industry in the United Kingdom during 1948-1958 (Kendall (1961)). The expected frequencies of classical Poisson distribution are obtained by maximum likelihood estimator, while the expected frequencies of zero-inflated

Poisson distribution are obtained by using Bayes estimators, obtained under square error loss function (SELF) and two different weighted square error loss functions. The prior values used for the beta distribution (2.2) will be $u = 3, v = 1$, while those used for the gamma distribution (3.3) will be $a = 0.25$, $b = 1$ and for exponentially weighted minimum expected loss (EWMEL) estimates (3.11) and (3.12) will be $\delta = \lambda = 0.25$. The values for the prior parameters $a, b, u, v, \delta, \lambda$ were chosen so that the posterior distribution would reflect the data as much, and the prior information as little, as possible. The observed frequencies, expected frequencies, the value of Pearson's chi-square statistics is given in table-I.

Table 1: *Number of outbreaks of Strike in Vehicle manufacturing Industry in the U.K. during 1948-1958*

No. of Outbreaks	Observed Frequency	Expected Frequency (Poisson Distribution)	Expected Frequency (Zero-inflated Poisson Distribution)		
			SELF	WSELF	
				MEL	EWMEL
0	110	103.5	110.3	107.9	108.0
1	33	42.5	30.4	33.1	33.1
2	9	8.7	11.7	11.7	11.7
3	3	1.2	3.0	2.8	2.7
4	1	0.1	0.6	0.5	0.5
Total	156	156	156	156	156
χ^2		3.4317	0.5690	0.3079	0.2796
Estimated value					
θ		0.4103	0.7673	0.7089	0.7038
α			0.4532	0.3927	0.3910

Conclusion and Comments

The values of the expected frequencies and the corresponding χ^2 value clearly shows that the zero-inflated Poisson distribution provided a closer fit than that provided by the classical Poisson distribution. It is also clear from the table that the Bayes estimators obtained under

weighted squared error loss functions (WSELF) gives closer fits than the Bayes estimator obtained under squared error loss function (SELF). Also, the exponentially minimum expected loss (EWMEL) estimates gives better fits than the minimum expected loss (MEL) estimates. Keeping in view the importance of count data modeling it is recommended that whenever the experimental number of zeros are more than that given by the model, the model should be adjusted accordingly to account for the extra zeros.

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Chapter 2:

Parameter Estimation of Weighted New Weibull Pareto Distribution

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Additional information is available at the end of the chapter

Introduction

Weighted distributions occur commonly in studies related to reliability, survival analysis, biomedicine, ecology, analysis of family data, and several other areas. There are number of authors worked on weighted distributions among them are Monsef and Ghoneim (2015) proposed weighted Kumaraswamy distribution for modeling some biological data, Sofi Mudasir and Ahmad (2015) study the length biased Nakagami distribution, Jan et al. (2017) studied the weighted Ailamujia distribution and find its applications to real data sets, Sofi Mudasir and Ahmad (2017) estimate the scale parameter of weighted Erlang distribution through classical and Bayesian methods of estimation, Dar et al. studied the characterization and estimation of Weighted Maxwell distribution (2018).

If $V \geq 0$ is a random variable with density function $f(v)$ and $w(v,\theta) \geq 0$ is a weight function, then the weighted random variable V_W has the probability density function given by

$$f_w(v) = Zw(v,\theta) f(v) \tag{2.1}$$

Where Z is the normalizing constant.

When $w(v,\theta) = v^{\theta}, \theta > 0,$ then the distribution is called the weighted distribution of order θ. The probability density function of WNWP distribution is obtained by using (2.1) and is given by

$$f_w(v) = \frac{\beta \eta^{\frac{\theta}{\beta}+1}}{\alpha^{\beta+\theta}\Gamma\left(\frac{\theta}{\beta}+1\right)} v^{\beta+\theta-1} \exp\left(-\eta\left(\frac{v}{\alpha}\right)^{\beta}\right), v \geq 0; \alpha, \beta, \eta, \theta > 0. \tag{2.2}$$

The corresponding cumulative distribution function is

$$F(v) = \frac{1}{\Gamma\left(\frac{\theta}{\beta}+1\right)} \gamma\left(\frac{\theta}{\beta}+1, \eta\left(\frac{v}{\alpha}\right)^{\beta}\right) \tag{2.3}$$

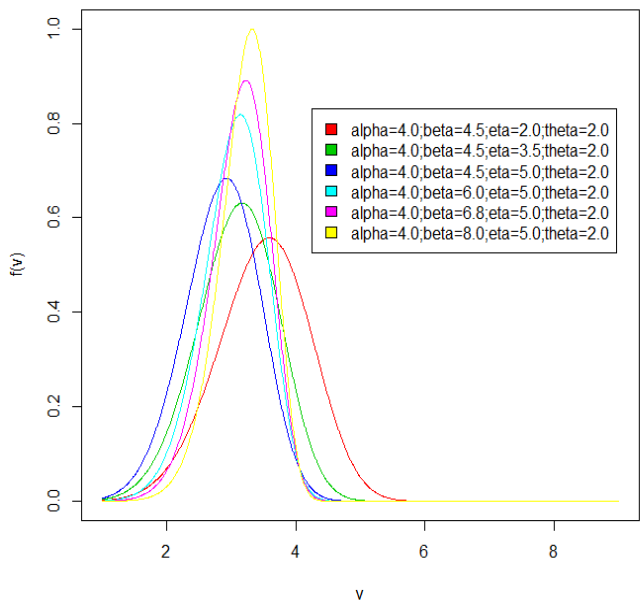

Figure 2.1: *Graph of Probability density function of Weighted Weibull Pareto distribution with different values of parameters*

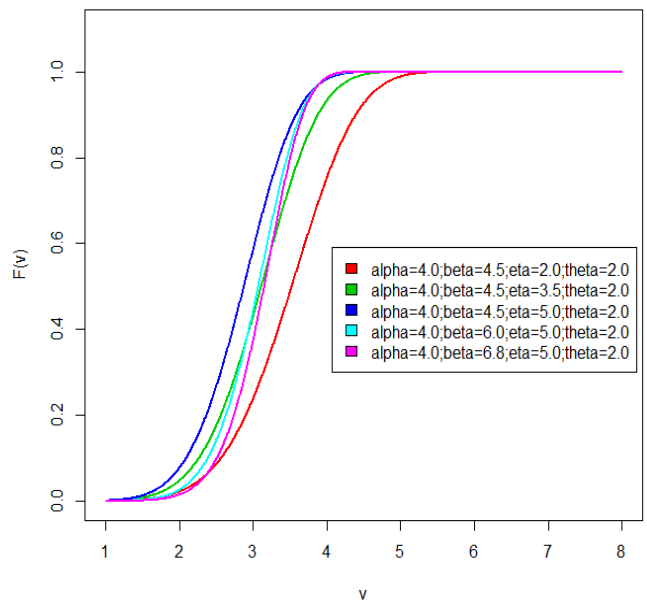

Figure 2.2: *Graph of Cumulative distribution function of Weibull Pareto distribution with different values of parameters*

Estimation Procedures

This section is devoted to three parameter estimation procedures: method of moments (MOM), maximum likelihood method of estimation (MLE), and Bayesian method of estimation.

Method of Moments (MOM)

Method of moments is a popular technique for parameter estimation. The moment estimator for the scale parameter α can be obtained by equating the first sample moment to the corresponding population moment and is given by

$$\hat{\alpha} = \frac{\eta^{\frac{1}{\beta}} \rho_{\theta}}{\rho_{\theta+1}} \bar{v} .$$

where $\rho_{\theta+s} = \Gamma\left(\frac{\theta+s}{\beta}+1\right)$.

Method of Maximum Likelihood Estimation (MLE)

Let $v_1, v_2, ..., v_n$ be a random sample from the WNWP distribution with parameter vector $\Theta = (\alpha, \beta, \eta, \theta)$. By considering (1), the likelihood function is given by

$$L(\Theta) = \left(\frac{\beta\eta^{\frac{\theta}{\beta}+1}}{\alpha^{\beta+\theta}\Gamma\left(\frac{\theta}{\beta}+1\right)}\right)^n \left(\prod_{i=1}^{n} v_i^{\beta+\theta-1}\right)\exp\left(-\frac{\eta}{\alpha^{\beta}}t\right).$$

The log-likelihood function can be expressed as

$$l(\Theta) = n\log\left(\frac{\beta\eta^{\frac{\theta}{\beta}+1}}{\alpha^{\beta+\theta}\Gamma\left(\frac{\theta}{\beta}+1\right)}\right) + (\beta+\theta-1)\sum_{i=1}^{n}\log(v_i) - \frac{\eta}{\alpha^{\beta}}t. \tag{2.4}$$

In order to estimate α, differentiate eq.(4) w.r.t. α and equate to zero, we get

$$\hat{\alpha} = \left(\frac{\beta\eta t}{n(\beta+\theta)}\right)^{\frac{1}{\beta}}.$$

Where $t = \sum_{i=1}^{n} v_i^{\beta}$.

Bayesian Method of Estimation

Here we try to find Bayes estimator for the scale parameter α for the pdf defined in (2.2). We use different priors and different loss functions.

Posterior Distribution Under the Assumption of Extension of Jeffrey's Prior

The extension of Jeffrey's prior relating scale parameter α is given as

$$\pi_1(\alpha) \propto \frac{1}{\alpha^{2c_1}}, \alpha > 0, c_1 \in R^+$$

Remark 1:

If $c_1 = 0,$ we get uniform prior, i.e.,

$\pi_{11}(\alpha) = q$, where q is constant of proportionality.

Remark 2:

If $c_1 = \frac{1}{2},$ we have $\pi_{12}(\alpha) \propto \frac{1}{\alpha}$ which is Jeffrey's prior.

Remark 3:

If $c_1 = \frac{3}{2},$ we get Hartigan's prior, i.e.,

$$\pi_{13}(\alpha) \propto \frac{1}{\alpha^3}.$$

The posterior distribution of scale parameter α under extension of Jeffrey's prior is given as

$$P_1(\alpha \mid \underline{v}) = \frac{\beta(\eta t)^{\frac{n\theta+2c_1-1}{\beta}+n} \exp\left(-\frac{\eta}{\alpha^\beta} t\right)}{\Gamma\left(\frac{n\theta+2c_1-1}{\beta}+n\right)\alpha^{n\theta+n\beta+2c_1}}. \tag{2.5}$$

Posterior Distribution Under the Assumption of Quasi Prior

The quasi prior relating to the scale parameter α is given as

$$\pi_2(\alpha) \propto \frac{1}{\alpha^{d_1}}, \alpha > 0, d_1 > 0.$$

The posterior distribution under quasi prior is given as

$$P_2(\alpha \mid \underline{v}) = \frac{\beta(\eta t)^{\frac{n\theta+d_1-1}{\beta}+n} \exp\left(-\frac{\eta}{\alpha^\beta} t\right)}{\Gamma\left(\frac{n\theta+d_1-1}{\beta}+n\right)\alpha^{n\theta+n\beta+d_1}}. \tag{2.6}$$

Bayes Estimator Under Squared Error Loss Function (SELF) Using Extension of Jeffrey's Prior

The SELF relating to the parameter α is defined as

$$L(\hat{\alpha} - \alpha) = b(\hat{\alpha} - \alpha)^2$$

Where b is a constant and $\hat{\alpha}$ is the estimator of α.

Risk function under SELF using extension of Jeffrey's prior is given by

$$R(\hat{\alpha}) = \int_0^\infty b(\hat{\alpha} - \alpha)^2 P_1(\alpha \mid \underline{v}) d\alpha .$$

$$= b\hat{\alpha}^2 + b(\eta t)^{\frac{2}{\beta}} \frac{\Gamma\left(\frac{n\theta + 2c_1 - 3}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - 1}{\beta} + n\right)} - 2b(\eta t)^{\frac{1}{\beta}} \frac{\Gamma\left(\frac{n\theta + 2c_1 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - 1}{\beta} + n\right)} \hat{\alpha} .$$

Minimization of risk function w.r.t. $\hat{\alpha}$ gives us the Bayes estimator as

$$\hat{\alpha} = \frac{\Gamma\left(\frac{n\theta + 2c_1 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - 1}{\beta} + n\right)} (\eta t)^{\frac{1}{\beta}} .$$

Bayes Estimator Under the Combination of Quadratic Loss Function (QLF) And Extension of Jeffrey's Prior

Risk function under QLF using extension of Jeffrey's prior is given by

$$R(\hat{\alpha}) = \int_0^\infty \left(\frac{\hat{\alpha} - \alpha}{\alpha}\right)^2 P_1(\alpha \mid \underline{v}) d\alpha$$

$$= \frac{\Gamma\left(\frac{n\theta + 2c_1 + 1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - 1}{\beta} + n\right)(\eta t)^{\frac{2}{\beta}}} \hat{\alpha}^2 + 1 - 2 \frac{\Gamma\left(\frac{n\theta + 2c_1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - 1}{\beta} + n\right)(\eta t)^{\frac{1}{\beta}}} \hat{\alpha} .$$

Now the solution of $\frac{\partial(R(\hat{\alpha}))}{\partial\hat{\alpha}} = 0$ is the required Bayes estimator and is given by

$$\hat{\alpha} = \frac{\Gamma\left(\frac{n\theta + 2c_1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 + 1}{\beta} + n\right)} (\eta t)^{\frac{1}{\beta}}.$$

Bayes Estimator Under the Combination of Al-Bayyati's Loss Function (ALF) and Extension of Jeffrey's Prior

The risk function under the combination of ALF and extension of Jeffrey's prior is

$$R(\hat{\alpha}) = \int_0^\infty \alpha^{c_2} (\hat{\alpha} - \alpha)^2 P_1(\alpha \mid \underline{v}) d\alpha$$

$$= \frac{\Gamma\left(\frac{n\theta + 2c_1 - c_2 - 1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - 1}{\beta} + n\right)} (\eta t)^{\frac{c_2}{\beta}} \hat{\alpha}^2 + \frac{\Gamma\left(\frac{n\theta + 2c_1 - c_2 - 3}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - 1}{\beta} + n\right)} (\eta t)^{\frac{c_2+2}{\beta}} -$$

$$2\frac{\Gamma\left(\frac{n\theta + 2c_1 - c_2 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - 1}{\beta} + n\right)} (\eta t)^{\frac{c_2+1}{\beta}} \hat{\alpha}.$$

On solving $\frac{\partial(R(\hat{\alpha}))}{\partial\hat{\alpha}} = 0$ for $\hat{\alpha}$, we get the Bayes estimator given as

$$\hat{\alpha} = \frac{\Gamma\left(\frac{n\theta + 2c_1 - c_2 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - c_2 - 1}{\beta} + n\right)} (\eta t)^{\frac{1}{\beta}}$$

Bayes Estimator Under Squared Error Loss Function (SELF) Using Quasi Prior

Under the combination of SELF and quasi prior, the risk function is given by

$$R(\hat{\alpha}) = \int_0^\infty b(\hat{\alpha} - \alpha)^2 P_2(\alpha \mid \underline{v}) d\alpha. \tag{2.7}$$

After substituting the value of eq. (2.6) in eq. (2.7) and simplification, we get

$$R(\hat{\alpha}) = b\hat{\alpha}^2 + b(\eta t)^{\frac{2}{\beta}} \frac{\Gamma\left(\frac{n\theta + d_1 - 3}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 - 1}{\beta} + n\right)} - 2b(\eta t)^{\frac{1}{\beta}} \frac{\Gamma\left(\frac{n\theta + d_1 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 - 1}{\beta} + n\right)} \hat{\alpha}.$$

The solution of $\frac{\partial(R(\hat{\alpha}))}{\partial\hat{\alpha}} = 0$ is the required Bayes estimator and is given by

$$\hat{\alpha} = \frac{\Gamma\left(\frac{n\theta + d_1 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 - 1}{\beta} + n\right)} (\eta t)^{\frac{1}{\beta}}$$

Bayes Estimator Under Quadratic Loss Function (QLF) Using Quasi Prior

The risk function under the combination of QLF and quasi prior is given by

$$R(\hat{\alpha}) = \frac{\Gamma\left(\frac{n\theta + d_1 + 1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 - 1}{\beta} + n\right)(\eta t)^{\frac{2}{\beta}}} \hat{\alpha}^2 + 1 - 2\frac{\Gamma\left(\frac{n\theta + d_1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 - 1}{\beta} + n\right)(\eta t)^{\frac{1}{\beta}}} \hat{\alpha}$$

.

Minimization of risk function w.r.t. $\hat{\alpha}$ gives us the Bayes estimator as

$$\hat{\alpha} = \frac{\Gamma\left(\frac{n\theta + d_1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 + 1}{\beta} + n\right)} (\eta t)^{\frac{1}{\beta}}$$

Bayes Estimator Under the Combination of Al-Bayyati's Loss Function (ALF) and Quasi Prior

The risk function under the combination of ALF and quasi prior is given by

$$R(\hat{\alpha}) = \frac{\Gamma\left(\frac{n\theta + d_1 - c_2 - 1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 - 1}{\beta} + n\right)} (\eta t)^{\frac{c_2}{\beta}} \hat{\alpha}^2 + \frac{\Gamma\left(\frac{n\theta + d_1 - c_2 - 1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 - 1}{\beta} + n\right)} (\eta t)^{\frac{c_2}{\beta}} -$$

$$2\frac{\Gamma\left(\frac{n\theta + d_1 - c_2 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 - 1}{\beta} + n\right)} (\eta t)^{\frac{c_2+1}{\beta}} \hat{\alpha}.$$

On solving $\frac{\partial(R(\hat{\alpha}))}{\partial\hat{\alpha}} = 0$ for $\hat{\alpha}$, we get the Bayes estimator given as

$$\hat{\alpha} = \frac{\Gamma\left(\frac{n\theta + d_1 - c_2 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + d_1 - c_2 - 1}{\beta} + n\right)} (\eta t)^{\frac{1}{\beta}}$$

The Bayes estimates using different priors under different loss functions are given below in table 2.1.

Table 2.1: *Bayes estimators under different combinations of loss functions and prior distributions*

Prior	Loss function	Estimator
Extension of Jeffrey's prior	Squared error	$\hat{\alpha} = \frac{\Gamma\left(\frac{n\theta + 2c_1 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - 1}{\beta} + n\right)} (\eta t)^{\frac{1}{\beta}}$
	Quadratic	$\hat{\alpha} = \frac{\Gamma\left(\frac{n\theta + 2c_1}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 + 1}{\beta} + n\right)} (\eta t)^{\frac{1}{\beta}}$
	Al-Bayyati's	$\hat{\alpha} = \frac{\Gamma\left(\frac{n\theta + 2c_1 - c_2 - 2}{\beta} + n\right)}{\Gamma\left(\frac{n\theta + 2c_1 - c_2 - 1}{\beta} + n\right)} (\eta t)^{\frac{1}{\beta}}$

Quasi prior	Squared error	$\hat{\alpha}=\dfrac{\Gamma\left(\dfrac{n\theta+d_1-2}{\beta}+n\right)}{\Gamma\left(\dfrac{n\theta+d_1-1}{\beta}+n\right)}(\eta t)^{\frac{1}{\beta}}$
	Quadratic	$\hat{\alpha}=\dfrac{\Gamma\left(\dfrac{n\theta+d_1}{\beta}+n\right)}{\Gamma\left(\dfrac{n\theta+d_1+1}{\beta}+n\right)}(\eta t)^{\frac{1}{\beta}}$
	Al-Bayyati's	$\hat{\alpha}=\dfrac{\Gamma\left(\dfrac{n\theta+d_1-c_2-2}{\beta}+n\right)}{\Gamma\left(\dfrac{n\theta+d_1-c_2-1}{\beta}+n\right)}(\eta t)^{\frac{1}{\beta}}$
Hartigan's prior	Squared error	$\hat{\alpha}=\dfrac{\Gamma\left(\dfrac{n\theta+1}{\beta}+n\right)}{\Gamma\left(\dfrac{n\theta+2}{\beta}+n\right)}(\eta t)^{\frac{1}{\beta}}$
	Quadratic	$\hat{\alpha}=\dfrac{\Gamma\left(\dfrac{n\theta+3}{\beta}+n\right)}{\Gamma\left(\dfrac{n\theta+4}{\beta}+n\right)}(\eta t)^{\frac{1}{\beta}}$
	Al-Bayyati's	$\hat{\alpha}=\dfrac{\Gamma\left(\dfrac{n\theta-c_2+1}{\beta}+n\right)}{\Gamma\left(\dfrac{n\theta-c_2+2}{\beta}+n\right)}(\eta t)^{\frac{1}{\beta}}$

Data Analysis

In this subdivision we analyze two real life data sets for illustration of the proposed procedure. The first data set represents the exceedances of flood peaks (m^3/s) of the Wheaton river near car cross in Yukon territory, Canada. The data set consists of 72 exceedances for the year 1958-1984, rounded to one decimal place. The second data set represents the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli.

Data set 2.1. Exceedances of flood peaks (m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consists of 72 exceedances for the year 1958-1984, rounded to one decimal place as shown below.

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7
1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1
0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9
0.9	1.7	7.0	20.1	0.4	2.8	14.1	9.9
5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3
1.9	10.4	13.0	10.7	12.0	30.0	9.3	3.6
2.5	27.6	14.4	36.4	1.7	2.7	37.6	64.0
1.7	9.7	0.1	27.5	1.1	2.5	0.6	27.0

Data set 2.2. The data set is from Kundu & Howlader (2010), the data set represents the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli. The regimen number is the common logarithm of the number of bacillary units per 0.5 ml. (log (4.0) 6.6). Corresponding to regimen 6.6, there were 72 observations listed below:

12	15	22	24	24	32	32	33	34	38	38	43	44	48
52	53	54	54	55	56	57	58	58	59	60	60	60	60
61	62	63	65	65	67	68	70	70	72	73	75	76	76
81	83	84	85	87	91	95	96	98	99	109	110	121	127
129	131	143	146	146	175	175	211	233	258	258	263	297	341
341	376												

Table 2.2: *Estimates and (posterior risk) under extension of Jeffrey's prior using different loss functions using data set 1.*

θ	β	η	B	c_1	c_2	MOM	MLE	SELF	QLF	ALF
1.0	4.0	1.0	1.5	1.0	1.0	12.36694	169.11393	169.29035 (30.030345)	169.05506 (0.0006941971)	169.40861 (3403.479)
	4.5	1.5	2.0	1.5	1.5	13.63276	105.17297	105.21717 (12.467706)	105.09922 (0.0005599442)	105.30611 (6766.528)
	5.5	2.5	2.5	2.5	2.0	14.81888	49.80533	49.79078 (2.407005)	49.75225 (0.0003865627)	49.82950 (2401.806)
	6.0	3.0	3.0	3.0	2.5	15.08626	37.15070	37.13227 (1.365364)	37.10784 (0.0003286734)	37.16296 (3851.661)
2.0	4.0	1.0	1.5	1.0	1.0	11.65966	161.57869	161.71912 (22.814399)	161.53183 (0.0005785327)	161.81316 (2468.287)
	4.5	1.5	2.0	1.5	1.5	12.98237	101.34020	101.37625 (9.788270)	101.28007 (0.0004739584)	101.44871 (5019.687)
	5.5	2.5	2.5	2.5	2.0	14.30474	48.52619	48.51391 (1.980545)	48.48136 (0.0003352443)	48.54660 (1874.647)
	6.0	3.0	3.0	3.0	2.5	14.63197	36.33304	36.31726 (1.143087)	36.29634 (0.0002878088)	36.34353 (3047.827)
3.0	4.0	1.0	1.5	1.0	1.0	11.11300	155.47028	155.58608 (18.087358)	155.43165 (0.0004959065)	155.66358 (1881.713)
	4.5	1.5	2.0	1.5	1.5	12.46849	98.16827	98.19853 (7.956599)	98.11778 (0.0004108653)	98.25934 (3887.504)
	5.5	2.5	2.5	2.5	2.0	13.88447	47.43435	47.42376 (1.669945)	47.39567 (0.0002959546)	47.45196 (1509.457)
	6.0	3.0	3.0	3.0	2.5	14.25562	35.62675	35.61301 (0.977227)	35.59476 (0.0002559819)	35.63591 (2479.360)

Table 2.3: *Estimates and (posterior risk) under Quasi prior using different loss functions using data set 1.*

θ	β	η	**b**	c_2	d_1	**MOM**	**MLE**	**SELF**	**QLF**	**ALF**
1.0	4.0	1.0	1.0	1.0	1.0	12.36694	169.11393	169.40861 (30.156580)	169.17250 (0.0006961301)	169.52728 (262.2463)
	4.5	1.5	1.5	1.5	1.5	13.63276	105.17297	105.30611 (12.536442)	105.18761 (0.0005620686)	105.39546 (662.6073)
	5.5	2.5	2.5	2.0	2.5	14.81888	49.80533	49.83921 (2.424638)	49.80044 (0.0003886283)	49.87818 (342.8307)
	6.0	3.0	3.0	2.5	3.0	15.08626	37.15070	37.16912 (1.376253)	37.14453 (0.0003306295)	37.20003 (637.4309)
2.0	4.0	1.0	1.0	1.0	1.0	11.65966	161.57869	161.8132 (22.894200)	161.62534 (0.0005798746)	161.90749 (194.5057)
	4.5	1.5	1.5	1.5	1.5	12.98237	101.34020	101.4487 (9.833878)	101.35215 (0.0004754796)	101.52146 (500.4294)
	5.5	2.5	2.5	2.0	2.5	14.30474	48.52619	48.5548 (1.993109)	48.52207 (0.0003367968)	48.58767 (270.8228)
	6.0	3.0	3.0	2.5	3.0	14.63197	36.33304	36.3488 (1.151059)	36.32775 (0.0002893075)	36.37523 (509.4890)
3.0	4.0	1.0	1.0	1.0	1.0	11.11300	155.47028	155.66358 (18.1415295)	155.50877 (0.0004968921)	155.74127 (151.1310)
	4.5	1.5	1.5	1.5	1.5	12.46849	98.16827	98.25934 (7.9887023)	98.17831 (0.0004120079)	98.32035 (393.5805)
	5.5	2.5	2.5	2.0	2.5	13.88447	47.43435	47.45903 (1.6792879)	47.43080 (0.0002971639)	47.48736 (220.3956)
	6.0	3.0	3.0	2.5	3.0	14.25562	35.62675	35.64050 (0.9832819)	35.62215 (0.0002571668)	35.66352 (418.1959)

Table 2.4: *Estimates and (posterior risk) under Hartigan's prior using different loss functions using data set 1.*

θ	β	η	b	c_2	d_1	MOM	MLE	SELF	QLF	ALF
1.0	4.0	1.0	1.0	1.0	1.0	12.36694	169.11393	169.17250 (29.904988)	168.93803 (0.0006922748)	169.29035 (259.8740)
	4.5	1.5	1.5	1.5	1.5	13.63276	105.17297	105.21717 (12.467706)	105.09922 (0.0005599442)	105.30611 (658.4081)
	5.5	2.5	2.5	2.0	2.5	14.81888	49.80533	49.82950 (2.421093)	49.79078 (0.0003882134)	49.86842 (342.2278)
	6.0	3.0	3.0	2.5	3.0	15.08626	37.15070	37.16912 (1.376253)	37.14453 (0.0003306295)	37.20003 (637.4309)
2.0	4.0	1.0	1.0	1.0	1.0	11.65966	161.57869	161.6253 (22.735061)	161.43860 (0.0005771970)	161.71912 (193.0399)
	4.5	1.5	1.5	1.5	1.5	12.98237	101.34020	101.3762 (9.788270)	101.28007 (0.0004739584)	101.44871 (497.7475)
	5.5	2.5	2.5	2.0	2.5	14.30474	48.52619	48.5466 (1.990585)	48.51391 (0.0003364852)	48.57944 (270.4103)
	6.0	3.0	3.0	2.5	3.0	14.63197	36.33304	36.3488 (1.151059)	36.32775 (0.0002893075)	36.37523 (509.4894)
3.0	4.0	1.0	1.0	1.0	1.0	11.11300	155.47028	155.50877 (18.0334558)	155.35472 (0.0004949247)	155.58608 (150.1550)
	4.5	1.5	1.5	1.5	1.5	12.46849	98.16827	98.19853 (28.1517503)	98.11778 (0.0004108653)	98.25934 (391.7533)
	5.5	2.5	2.5	2.0	2.5	13.88447	47.43435	47.45196 (1.6774122)	47.42376 (0.0002969213)	47.48027 (220.0996)
	6.0	3.0	3.0	2.5	3.0	14.25562	35.62675	35.64050 (0.9832819)	35.62215 (0.0002571668)	35.66352 (418.1959)

Table 2.5: *Estimates and (posterior risk) under extension of Jeffrey's prior using different loss functions using data set 2.*

θ	β	η	B	c_1	c_2	MOM	MLE	SELF	QLF	ALF
1.0	4.0	1.0	1.0	1.0	1.0	102.0919	2473.4000	2475.9802 (6423.77429)	2472.5390 (0.0006941971)	2477.7098 (10647998)
	4.5	1.5	1.5	1.5	1.5	112.5415	1141.7260	1142.2058 (1469.27156)	1140.9254 (0.0005599442)	1143.1713 (28521214)
	5.5	2.5	2.5	2.5	2.0	122.3332	350.4549	350.3525 (119.17625)	350.0814 (0.0003865627)	350.6250 (5887940)
	6.0	3.0	3.0	3.0	2.5	124.5405	222.1827	222.0725 (48.83544)	221.9264 (0.0003286734)	222.2561 (12050165)
2.0	4.0	1.0	1.0	1.0	1.0	96.25317	2363.1922	2365.2461 (4880.21539)	2362.5070 (0.0005785327)	2366.6216 (7722190)
	4.5	1.5	1.5	1.5	1.5	107.17238	1100.1187	1100.5099 (1153.51019)	1099.4659 (0.0004739584)	1101.2966 (21158203)
	5.5	2.5	2.5	2.5	2.0	118.08883	341.4543	341.3678 (98.06123)	341.1388 (0.0003352443)	341.5979 (4595628)
	6.0	3.0	3.0	3.0	2.5	120.79026	217.2926	217.1983 (40.88520)	217.0732 (0.0002878088)	217.3554 (9535319)
3.0	4.0	1.0	1.0	1.0	1.0	91.74031	2273.8528	2275.5464 (3869.05665)	2273.2878 (0.0004959065)	2276.6800 (26855836)
	4.5	1.5	1.5	1.5	1.5	102.93017	1065.6851	1066.0136 (937.65483)	1065.1370 (0.0004108653)	1066.6737 (64929980)
	5.5	2.5	2.5	2.5	2.0	114.61948	333.7715	333.6970 (82.68276)	333.4994 (0.0002959546)	333.8955 (11511443)
	6.0	3.0	3.0	3.0	2.5	117.68336	213.0687	212.9864 (34.95281)	212.8773 (0.0002559819)	213.1234 (21993921)

Table 2.6: *Estimates and (posterior risk) under Quasi prior using different loss functions using data set 2.*

θ	β	η	b	c_2	d_1	MOM	MLE	SELF	QLF	ALF
1.0	4.0	1.0	1.0	1.0	1.0	102.0919	2473.4000	2477.7098 (6450.77712)	2474.2566 (0.0006961301)	2479.4455 (214534.4)
	4.5	1.5	1.5	1.5	1.5	112.5415	1141.7260	1143.1713 (1477.37186)	1141.8850 (0.0005620686)	1144.1413 (847675.2)
	5.5	2.5	2.5	2.0	2.5	122.3332	350.4549	350.6933 (120.04928)	350.4205 (0.0003886283)	350.9675 (316830.4)
	6.0	3.0	3.0	2.5	3.0	124.5405	222.1827	222.2929 (49.22489)	222.1458 (0.0003306295)	222.4778 (815467.2)
2.0	4.0	1.0	1.0	1.0	1.0	96.25317	2363.1922	2366.6216 (4897.28552)	2363.8745 (0.0005798746)	2368.0012 (159118.2)
	4.5	1.5	1.5	1.5	1.5	107.17238	1100.1187	1101.2966 (1158.88497)	1100.2484 (0.0004754796)	1102.0863 (640200.6)
	5.5	2.5	2.5	2.0	2.5	118.08883	341.4543	341.6556 (98.68333)	341.4253 (0.0003367968)	341.8869 (250283.6)
	6.0	3.0	3.0	2.5	3.0	120.79026	217.2926	217.3869 (41.17032)	217.2611 (0.0002893075)	217.5450 (651791.2)
3.0	4.0	1.0	1.0	1.0	1.0	91.74031	2273.8528	2276.6800 (3880.64442)	2274.4157 (0.0004968921)	2277.8163 (123634.9)
	4.5	1.5	1.5	1.5	1.5	102.93017	1065.6851	1066.6737 (941.43805)	1065.7941 (0.0004120079)	1067.3360 (503508.5)
	5.5	2.5	2.5	2.0	2.5	114.61948	333.7715	333.9452 (83.14533)	333.7466 (0.0002971639)	334.1446 (203680.8)
	6.0	3.0	3.0	2.5	3.0	117.68336	213.0687	213.1509 (35.16938)	213.0411 (0.0002571668)	213.2885 (534999.0)

Table 2.7: *Estimates and (posterior risk) under Hartigan's prior using different loss functions using data set 2.*

θ	β	η	b	c_2	d_1	MOM	MLE	SELF	QLF	ALF
1.0	4.0	1.0	1.0	1.0	1.0	102.0919	2473.4000	2474.2566 (6396.95932)	2470.8273 (0.0006922748)	2475.9802 (212593.6)
	4.5	1.5	1.5	1.5	1.5	112.5415	1141.7260	1142.2058 (1469.27156)	1140.9254 (0.0005599442)	1143.1713 (842303.2)
	5.5	2.5	2.5	2.0	2.5	122.3332	350.4549	350.6250 (119.87379)	350.3525 (0.0003882134)	350.8989 (316273.3)
	6.0	3.0	3.0	2.5	3.0	124.5405	222.1827	222.2929 (49.22489)	222.1458 (0.0003306295)	222.4778 (815467.2)
2.0	4.0	1.0	1.0	1.0	1.0	96.25317	2363.1922	2363.8745 (4863.24419)	2361.1433 (0.0005771970)	2365.246 (157919.1)
	4.5	1.5	1.5	1.5	1.5	107.17238	1100.1187	1100.5099 (1153.51019)	1099.4659 (0.0004739584)	1101.297 (636769.9)
	5.5	2.5	2.5	2.0	2.5	118.08883	341.4543	341.5979 (98.55836)	341.3678 (0.0003364852)	341.829 (249902.5)
	6.0	3.0	3.0	2.5	3.0	120.79026	217.2926	217.3869 (41.17032)	217.2611 (0.0002893075)	217.5450 (651791.2)
3.0	4.0	1.0	1.0	1.0	1.0	91.74031	2273.8528	2274.4157 (3857.52644)	2272.1627 (0.0004949247)	2275.5464 (122836.5)
	4.5	1.5	1.5	1.5	1.5	102.93017	1065.6851	1066.0136 (937.65483)	1065.1370 (0.0004108653)	1066.6737 (501170.9)
	5.5	2.5	2.5	2.0	2.5	114.61948	333.7715	333.8955 (83.05246)	333.6970 (0.0002969213)	334.0946 (203407.2)
	6.0	3.0	3.0	2.5	3.0	117.68336	213.0687	213.1509 (35.16938)	213.0411 (0.0002571668)	213.2885 (534999.0)

Conclusion

In this chapter, method of moments, maximum likelihood and Bayesian methods of estimation were studied for estimating the scale parameter of the WNWP distribution. Bayes estimators are obtained using different loss functions under different types of priors. For comparison of different loss functions and different types of priors, two real life data sets are used, and the outcomes are obtained through R-software. On equating the posterior risk obtained under different loss functions, it is clear from the above tables that QLF has minimum value of posterior risk and is thus preferable as compared to other loss functions used in this paper. It is also observed that as we increase the value of weighted parameter θ, the posterior risk decreases. Also, from tables 2.2 to 2.7, it is clear that in order to estimate the said parameter combination of quadratic loss function and extension of Jeffrey's prior can be preferred.

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Chapter 3:

Mathematical Model of Accelerated Life Testing Plan Using Geometric Process

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Introduction

In ALT analysis, the life-stress relationship is generally used to estimate the parameters of failure time distribution at use condition which is just a re-parameterization of original parameters but from statistical point of view, it is easy and reasonable to deal with original parameters directly instead of developing inferences for the parameters of the life stress relationship. It can be seen that the original parameter of the distribution can be directly handled by the assumption that the lifetime of the items formd GP at increasing level of stress. The concept of geometric process in accelerated life testing was first introduced by Lam (1988) in repair replacement problem. Since then many authors have studied maintenance problem and system reliability by using GP model. Lam (2007) used geometric model to study a multistate system and inferred a policy for replacement that minimizes the average cost per unit time for long run. After that a lot of works have been done and the available literature shows that the GP model is one of the simplest among the available models for the study of data with a single or multiple trend, e.g., Lam and Zhang (1996), Lam (2005). Zhang (2008) studied repairable system with delayed repair by using the GP repair model. Huang (2011) analyzed the complete and censored data for exponential distribution applying the model of GP. Zhou et al. (2012) extended the GP model for Rayleigh distribution for the progressive type I hybrid censored data in ALT. Kamal et al. (2013) analyzed the complete samples for Pareto distribution with constant stress accelerated life testing plan by using geometric process model. Anwar et al. (2013) used the process to analyze the model of ALT for Marshal-Olkin extended exponential distribution, then extended her work for type I censored data (Anwar et al., 2014).

This chapter deals with constant stress accelerated life testing for generalized exponential distribution using geometric process with complete data. Estimates of parameters are obtained by maximum likelihood estimation technique and confidence intervals for parameters are obtained by using the asymptotic properties. Lastly, statistical properties of estimates and confidence intervals are examined through a simulation study.

The Model

The Geometric Process

A stochastic process $\{X_n, n = 1,2,...\}$ is said to be a geometric process if there is a real valued $\lambda(>0)$ in such a way that $\{\lambda^{n-1}X_n, n = 1,2,...\}$ forms a renewal process. It can be shown that if $\{X_n, n = 1,2,...\}$ is a GP and $f(x)$ is the probability density function with mean α and variance σ^2 then $\lambda^{n-1} f(\lambda^{n-1}x)$ will be the probability density function of X_n with mean $E(X_n) = \frac{\alpha}{\lambda^{n-1}}$ and $\text{var}(X_n) = \frac{\sigma^2}{\lambda^{2(n-1)}}$. Thus, the important parameters of GP are λ, α and σ^2 are to be estimated.

Generalized Exponential Life Distribution

The probability density function (pdf) of a generalized exponential distribution is given by

$$f(x,\alpha,\beta) = \frac{\alpha}{\beta} e^{-\frac{x}{\beta}} \left(1 - e^{-\frac{x}{\beta}}\right)^{\alpha-1}, \alpha, \beta, x > 0 \qquad (3.1)$$

where $\alpha,\ \beta > 0$ and are the shape and scale parameter of the life distribution respectively. The above discussed life distribution can be abbreviated as $GE(\alpha,\beta)$ with the shape α and the scale parameter β. If $\alpha = 1$, then it is written as $GE(1,\beta)$ and shows the exponential distribution with scale parameter β. The generalized exponential distribution with two parameters may be used for analyzing lifetime data, particularly, in places of Gamma and Weibull distributions with two parameters. Its shape parameter depicts the behavior of failure rate: which may be increasing or decreasing depending on the values of shape parameters. The distribution function of life distribution of the items is given as follows

$$F(x,\alpha,\beta) = \left(1 - e^{-\frac{x}{\beta}}\right)^{\alpha}, \alpha, \beta, x > 0 \qquad (3.2)$$

The survival function of the items takes the following form

$$S(x,\alpha,\beta)=1-\left(1-e^{-\frac{x}{\beta}}\right)^{\alpha} \tag{3.3}$$

The hazard rate function is given as follows

$$h(x,\alpha,\beta)=\frac{\frac{\alpha}{\beta}e^{-\frac{x}{\beta}}\left(1-e^{-\frac{x}{\beta}}\right)^{\alpha-1}}{1-\left(1-e^{-\frac{x}{\beta}}\right)^{\alpha}} \tag{3.4}$$

The shape of the failure rate function depends only on the values of parameter α. For $\alpha>1$, the GE distribution has a log concave density and for $\alpha\le 1$ it has log-convex. Therefore, when $\alpha>1$ and β is constant, then it has an increasing failure rate and when $\alpha<1$ with constant β, then it has decreasing failure rate. The failure rate function of GE distribution and Gamma distribution, with two parameters behaves in same way, while the failure rate function of Weibull distribution behaves in totally different way.

Assumptions and Method of Test

1. The failure time of items follows generalized exponential distribution $GE(\alpha,\beta)$ at all the stresses.
2. Suppose a life test with different S stress level (i.e. increasing stress level) is conducted. Under each stress level, n items from a random sample have been put on test at the same time. Let x_{ki}, $i=1,2,3...,n$ $k=1,2,...,s$ be the failure time of ith item under k^{th} stress level. We will remove the failed the items from thetest and it would run till the whole random sample exhausted (complete sample).
3. There is a linear relationship between the log of shape parameter and stress i.e., $\log(\beta_k)=p+qS_k$, where p and q are unknown, and their values depends on the nature of products and method.
4. The lifetimes of items under each stress level is denoted by the random variables $X_0,X_1,X_2,...,X_S$, where X_0 is the lifetime of the items under normal stress (or design stress) at which the items will run normally and sequence $\{X_k,k=1,2,...s\}$ forms a GP with parameter $\lambda>0$.

The assumptions discussed above are very common in ALT literature except the last one, i.e. assumption 4. It is the assumption of geometric process which is simply better among the

available usual methods without increasing the level of difficulty in calculations. By assuming the linear relationship between the log of life and stress (i.e. assumption 3), the assumption can be shown by the following theorem.

Theorem 3.1: If the level of stress in an ALT is increasing with a constant difference then under each stress level the lifetimes of items forms a GP, i.e. if $S_{k+1} - S_k$ is constant for $k = 1,2,...,s-1$, then $\{X_k, k = 1,2,...,s\}$ forms a GP.

Proof: We get the following equation from assumption (3)

$$\log\left(\frac{\beta_{k+1}}{\beta_k}\right) = q(S_{k+1} - S_k) = q\Delta S \tag{3.5}$$

From the above equation we can see that the increased stress levels form a sequence with a difference ΔS, this sequence is called arithmetic sequence that is formed with constant difference ΔS.

Here, we can write the equation (3.5) as follows

$$\frac{\beta_{k+1}}{\beta_k} = e^{b\Delta S} = \frac{1}{\lambda}(say) \tag{3.6}$$

It is obvious from equation (3.6) that

$$\beta_k = \frac{1}{\lambda}\beta_{k-1} = \frac{1}{\lambda^2}\beta_{k-2} = ... = \frac{1}{\lambda^k}\beta$$

Therefore, the probability density function of the lifetimes at the kth stress level is given as

$$f_{X_k}(x) = \frac{\alpha}{\beta_k} e^{\frac{-x}{\beta_k}} \left(1 - e^{\frac{-x}{\beta_k}}\right)^{\alpha-1}$$

$$= \frac{\alpha\lambda^k}{\beta} e^{\frac{-\lambda^k x}{\beta}} \left(1 - e^{\frac{-\lambda^k x}{\beta}}\right)^{\alpha-1} \tag{3.7}$$

and the cdf is

$$F_{x_k}(x) = \left(1 - e^{\frac{-\lambda^k x}{\beta}}\right)^{\alpha} \tag{3.8}$$

Eq. (3.7) implies that

$$f_{X_k}(x) = \lambda^k f_{X_0}(\lambda^k x) \tag{3.9}$$

Now, by the definition of geometric process and the equation (3.6) we can see that if probability density function of the lifetime of the products at usual stress level i.e. X_0 , is $f_{X_0}(x)$, then the probability density function of the lifetimes at the kth stress level i.e. X_k is given by $\lambda^k f_{X_0}(\lambda^k x),\ k = 1,2,...,s$. Hence, it is obvious that failure times of the products form a geometric process with parameter λ under a sequence of arithmetically increasing stress levels.

Expression (3.7) shows that if the lifetimes of products under a sequence of increasing stress level form a geometric process with ratio λ and if the lifetime of the items at normal stress level follows generalized exponential distribution with characteristic β , then the lifetime distribution of the test items at k^{th} stress level will also be generalized exponential with characteristic life $\frac{\beta}{\lambda^k}$.

Estimation Process

Maximum likelihood estimation (MLE) is one of the extensively used methods among all estimation methods. It can be applied to any probability distribution while other methods are somewhat restricted. The use of MLE in ALT is difficult and mathematically very complex and, even most of the times the closed form estimates of parameters do not exist. Therefore, Newton Raphson method is used to estimate the numerical values of them. The likelihood function for constant stress ALT for complete case generalized exponential failure data using GP for s stress levels is given by:

$$L = \prod_{k=1}^{s}\prod_{i=1}^{n} \frac{\alpha\,\lambda^k}{\beta} e^{-\frac{\lambda^k x_{ki}}{\beta}} \left(1 - e^{-\frac{\lambda^k x_{ki}}{\beta}}\right)^{(\alpha-1)} \tag{3.10}$$

Now take the log of the above function and rewrite as follows;

$$l = \sum\sum\left[\log\alpha + k\log\lambda - \log\beta - \frac{\lambda^k x_{ki}}{\beta} + (\alpha-1)\log(1 - e^{-\frac{\lambda^k x_{ki}}{\beta}})\right] \tag{3.11}$$

The MLEs of the parameters $\alpha,\ \beta$ and λ can be obtained by solving the following normal equations $\frac{\partial l}{\partial \alpha} = 0, \frac{\partial l}{\partial \beta} = 0$ and $\frac{\partial l}{\partial \lambda} = 0$.

$$\frac{\partial l}{\partial \alpha} = \frac{ns}{\alpha} + \sum_{k=1}^{s}\sum_{i=1}^{n} \log(1-z) = 0 \tag{3.12}$$

$$\frac{\partial l}{\partial \beta} = \frac{-ns}{\beta} + \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{\lambda^k x_{ki}}{\beta^2} - \frac{\lambda^k x_{ki}}{\beta^2} \times (\alpha-1)\frac{Z}{(1-Z)} = 0\right] = 0 \tag{3.13}$$

$$\frac{\partial l}{\partial \lambda} = \frac{kns}{\lambda} - \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{k\lambda^{(k-1)} x_{ki}}{\beta} - k\lambda^{k-1} x_{ki}\frac{(\alpha-1)}{\beta}\frac{Z}{1-Z}\right] = 0 \tag{3.14}$$

where $Z = e^{-\frac{\lambda^k x_{ki}}{\phi}}$

We use above equations namely (3.12), (3.13) and (3.14) to find the estimate of α, β and λ.

Fisher Information Matrix and Asymptotic Confidence Interval

The asymptotic Fisher Information matrix is given by:

$$\begin{bmatrix} -\frac{\partial^2 l}{\partial \alpha^2} & -\frac{\partial^2 l}{\partial \alpha \partial \beta} & -\frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 l}{\partial \beta \partial \alpha} & -\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 l}{\partial \lambda \partial \alpha} & -\frac{\partial^2 l}{\partial \beta \partial \lambda} & -\frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}$$

The elements of Fisher Information matrix can be obtained by putting a negative sign before double and partial derivatives of the parameters, which are given as follows;

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{ns}{\alpha^2}$$

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{ns}{\beta^2} - \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{2\lambda^2 x_{ki}}{\beta^3} + (\alpha-1)\lambda^2 x_{ki}\left\{\frac{Z\lambda^k x_{ki} - 2Z\beta(1-Z)}{\beta^4(1-Z)}\right\}\right]$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{kns}{\lambda^2} - \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{k(k-1)\lambda^{k-2} x_{ki}}{\beta} - \frac{kZ(\alpha-1)x_{ki}}{\beta}\left\{\frac{(k-1)\lambda^{k-2}}{(1-Z)} - \frac{k\lambda^{2(k-1)} x_{ki}}{\beta(1-Z^2)}\right\}\right]$$

$$\frac{\partial^2 l}{\partial\alpha\,\partial\beta}=\frac{\partial^2 l}{\partial\beta\,\partial\alpha}=-\sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{Z\lambda^k x_{ki}}{\beta^2(1-Z)}\right]$$

$$\frac{\partial^2 l}{\partial\alpha\,\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\,\partial\alpha}=\sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{kZ\lambda^{k-1}x_{ki}}{\beta(1-Z)}\right]$$

$$\frac{\partial^2 l}{\partial\beta\,\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\,\partial\beta}=\sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{k\lambda^{k-1}x_{ki}}{\beta^2}-\frac{(\alpha-1)Zkx_{ki}}{\beta^2}\left\{\frac{\lambda^{k-1}}{(1-Z)}-\frac{\lambda^{2k-1}x_{ki}}{\beta(1-Z)^2}\right\}\right]$$

The confidence interval for parameters α, β and λ are given as follows:

$\hat{\alpha}+Z_{1-\frac{\phi}{2}}(SE(\hat{\alpha})), \hat{\beta}+Z_{1-\frac{\phi}{2}}\left(SE\left(\hat{\beta}\right)\right)$, and $\hat{\lambda}+Z_{1-\frac{\phi}{2}}\left(SE\left(\hat{\lambda}\right)\right)$ respectively.

Simulation Study

Simulation is used for observing the statistical properties of parameters. It is an attempt to model an assumed condition to study the behaviour of function.

To perform the study, we first generate a random sample from Uniform distribution by using optim() function in R software.

Now we use inverse cdf method to transform equation (3.8) in terms of u and get the expression of x_{ki} , k=1,2,3,...,s and i=1,2,3,...,n.

$$x_{ki}=-\beta\frac{(1-u^{1/\alpha})}{\lambda^k},\quad k=1,2,\cdots,s\;\; i=1,2,\cdots,n$$

- Now take the random samples of size 20,40,60,80 and 100 from the generalized exponential distribution and replicate them 1000 times.
- The values of parameters and numbers of the stress levels are chosen to be $\alpha=1.2,\beta=2.8,\lambda=1.1,s=4or6$.
- optim() function is used to obtain the ML estimates, relative absolute bias(RAB), the mean squared error(MSE), relative error(RE) and lower and upper bound of 95% and 99% confidence intervals for different sample sizes.

The outcome obtained in the above study are summarized in Table3.1 and Table 3.2.

Table 3.1: *Simulation results of Generalised exponential using GP at* $\alpha = 1.2, \beta = 2.8,$ $\lambda = 1.1, s = 4$

Sample	Estimates	Mean	SE	√MSE	RABias	RE	Lower BOund	Upper Bound
20	α	1.0772	0.1990	0.1923	0.1022	0.1602	0.6870 0.5636	1.4674 1.5908
	β	3.0781	0.3191	0.0950	0.0993	0.0339	2.4526 2.2547	3.7036 3.9015
	λ	1.0500	0.1034	0.0999	0.0908	0.0864	0.7971 0.7329	1.2028 1.2670
40	α	1.1675	0.1200	0.1159	0.02707	0.0966	0.9322 0.8578	1.4027 1.4771
	β	3.0397	0.2565	0.0614	0.0856	0.0219	2.5369 2.3779	3.5424 3.7015
	λ	0.9952	0.1035	0.1000	0.0909	0.0558	0.7970 0.7329	1.2028 1.2670
60	α	1.2113	0.1201	0.1160	0.0094	0.0967	0.9758 0.9014	1.4467 1.5212
	β	3.0038	0.2417	0.0545	0.0728	0.0194	2.5299 2.3800	3.4777 3.6276
	λ	1.0000	0.1034	0.0999	0.0908	0.0495	0.7971 0.7330	1.2028 1.2670
80	α	1.1826	0.0805	0.0777	0.0144	0.0648	1.0248 0.9749	1.3404 1.3904
	β	3.0491	0.2645	0.0653	0.0889	0.0233	2.5306 2.3666	3.5676 3.7316
	λ	0.9924	0.1035	0.1000	0.0909	0.0593	0.7970 0.7328	1.2029 1.2671
100	α	1.1579	0.0867	0.0838	0.0350	0.0698	0.9878 0.9340	1.3280 1.3818
	β	3.0515	0.2661	0.0661	0.0898	0.0236	2.5299 2.3649	3.5731 3.7381
	λ	1.0041	0.1035	0.0999	0.0909	0.0600	0.7971 0.7329	1.2028 1.2670

Table 3.2: *Simulation results of Generalised exponential using GP at* $\alpha = 1.2, \beta = 2.8,$ $\lambda = 1.1, s = 6$.

Sample	Estimates	Mean	SE	√MSE	RABias	RE	Lower BOund	Upper Bound
20	α	1.2838	0.1409	0.1361	0.0698	0.1134	1.0076 0.9202	1.5601 1.6474
	β	2.9798	0.2183	0.0444	0.0642	0.0158	2.5518 2.4165	3.4077 3.5431
	λ	0.9950	0.1035	0.1000	0.0909	0.0404	0.7970 0.7328	1.2029 1.2671
40	α	1.1598	0.1451	0.1402	0.0334	0.1168	0.8753 0.7853	1.4442 1.5342
	β	3.0598	0.2899	0.0784	0.0928	0.0280	2.4916 2.3118	3.6281 3.8079
	λ	1.0489	0.1035	0.1000	0.0909	0.0713	0.7971 0.7329	1.2028 1.2670
60	α	1.2019	0.0968	0.0935	0.0015	0.0779	1.0120 0.9519	1.3917 1.4517
	β	3.0650	0.2777	0.0719	0.0946	0.0257	2.5207 2.3485	3.6092 3.7814
	λ	0.9923	0.1035	0.1000	0.0909	0.0654	0.7970 0.7329	1.2028 1.2670
80	α	1.1868	0.0892	0.0861	0.0110	0.0718	1.0119 0.9566	1.3616 1.4169
	β	3.0600	0.2708	0.0684	0.0928	0.0244	2.5291 2.3612	3.5909 3.7588
	λ	0.9973	0.1035	0.1000	0.0909	0.0622	0.7970 0.7328	1.2029 1.2671
100	α	1.1977	0.0812	0.0785	0.0019	0.0654	1.0384 0.9880	1.3570 1.4073
	β	3.0587	0.2718	0.0689	0.0923	0.0246	2.5260 2.3574	3.5914 3.7600
	λ	1.0500	0.1035	0.1	0.0909	0.0626	0.7971 0.7329	1.2028 1.2670

Conclusion

In this study, geometric process is introduced for the study of accelerated life testing plan under constant stress when the failure time data are from a generalized exponential model. It is better choice for life testing because of its simplicity in nature. The mean, SE, MSE, RAB and RE of the parameters are obtained and based on the asymptotic normality, the 95% and 99% confidence intervals of the parameters are also obtained. The outcome in Table 3.1 and Table 3.2 show that the estimated values of α , β and λ are very close to true (or initial) values with very small SE and MSE. As sample size increases, the value of SE and MSE decreases and the confidence interval become narrower. For the Table 3.2, the maximum likelihood estimators have good statistical properties than the Table 3.1 for all sample size. The future research should extend the GP model into ALTg for different life distribution. Introducing the GP model into ALT with other test plans or censoring techniques is another object of the future research.

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Chapter 4:

Bayesian Inference of Ailamujia Distribution Using Different Loss Functions

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Additional information is available at the end of the chapter

Introduction

Ailmujia distribution is proposed by Lv et al. (2002). Pan et al. (2009) studied the interval estimation and hypothesis test of Ailamujia distribution based on small sample. Uzma et al. (2017) studied the weighted version Ailamujia distribution. The cumulative distribution function of Ailamujia distribution is given by

$$F(x;\theta,\alpha) = 1-(1+2\theta x)e^{-2\theta x} \quad , x \geq 0, \theta > 0 \tag{4.1}$$

and the probability density function (pdf) corresponding to (4.1) is

$$f(x;\theta,\alpha) = 4x\theta^2 e^{-2\theta x} \qquad , x \geq 0 \ , \theta > 0 \tag{4.2}$$

Our objective in this study is to find the Bayes estimators of the parameter of Ailamujia distribution using non-informative Jeffery's prior and informative Gamma prior under squared error loss function, Entropy loss function and LINEX loss function. Finally, an application is considered to equate the performance of these estimates under different loss functions by manipulative posteriors risk using R Software.

Material and Methods

Recently Bayesian estimation technique has established great contemplation by most researchers. Bayesian analysis is a significant approach to statistics, which properly seeks use of prior information and Bayes Theorem provides the formal basis for using this information.

In this paper we consider the Jeffrey's prior proposed by Al-Kutubi (2005) as:

$$g(\theta) \propto \sqrt{I(\theta)} \tag{4.3}$$

where $[I(\theta)] = -nE\left[\frac{\partial^2 \log f(x;\theta)}{\partial \theta^2}\right]$ is the Fisher's information matrix. For the model (4.2),

$g(\theta) = k\frac{1}{\theta}$, where k is a constant.

The second prior which we have used is gamma prior i.e

$$g(\theta) \propto \frac{\alpha^{\beta}}{\Gamma\beta} e^{-\alpha\theta} \theta^{\beta-1} \tag{4.4}$$

with the above priors, we use three different loss functions for the model (4.2), viz squared error loss function which is symmetric, and Entropy and LINEX loss function which are asymmetric loss functions.

Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from Ailamujia distribution, then the log likelihood function can be written as

$$\log L(\theta, \lambda) = n\log 4 + 2n\log\theta + \sum_{i=1}^{n} x_i - 2\theta\sum_{i=1}^{n} x_i \tag{4.5}$$

the ML estimator of θ is obtained by solving the equation

$$\frac{\partial \log L(\theta)}{\partial \theta} = 0$$

$$\Rightarrow \frac{2n}{\theta} - 2\sum_{i=1}^{n} x_i = 0 \qquad \Rightarrow \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i}$$

Bayesian estimation of Ailamujia distribution under Assumption of Jeffrey's prior

Consider n recorded values, $x = (x_1, x_2, \ldots, x_n)$ having probability density function as

$$f(x;\theta,\alpha) = 4x\theta^2 e^{-2\theta x}$$

we consider the prior distribution of θ to be Jeffrey's prior i.e $g(\theta) \propto \frac{1}{\theta}$

The posterior distribution of θ under the assumption of Jeffrey's prior is given by

$$\pi(\theta/x) \propto L(x/\theta)\, g(\theta)$$

$$\Rightarrow \pi(\theta/x) \propto (4\theta)^n \prod_{i=1}^{n} x_i\, e^{-2\theta\sum_{i=1}^{n} x_i}\, \frac{1}{\theta}$$

$$\Rightarrow \pi(\theta/x) = k\,\theta^{2n-1}\, e^{-2\theta\sum_{i=1}^{n} x_i}$$

where k is independent of θ

and $k^{-1} = \int_0^\infty \theta^{2n-1}\, e^{-2\theta\sum_{i=1}^{n} x_i}\, d\theta \qquad \Rightarrow k^{-1} = \dfrac{\Gamma 2n}{\left[2\sum_{i=1}^{n} x_i\right]^{2n}}$

Hence posterior distribution of θ is given by

$$\pi(\theta/x) = \frac{\left[2\sum_{i=1}^{n} x_i\right]^{2n}}{\Gamma 2n}\, \theta^{2n-1}\, e^{-2\theta\sum_{i=1}^{n} x_i}$$

$$\pi(\theta/x) = \frac{t^{2n}}{\Gamma 2n}\, \theta^{2n-1}\, e^{-t\theta} \tag{4.6}$$

where $t = 2\sum_{i=1}^{n} x_i$

Estimator Under Squared Error Loss Function

By using squared error loss function $l(\hat{\theta},\theta) = c_1(\hat{\theta}-\theta)^2$ for some constant c_1 the risk function is given by

$$R(\hat{\theta},\theta) = E\left[l(\hat{\theta},\theta)\right]$$

$$= \int_0^\infty c_1(\hat{\theta}-\theta)^2\, \frac{t^{2n}}{\Gamma 2n}\, \theta^{2n-1}\, e^{-t\theta}\, d\theta$$

$$=\frac{c_1 t^{2n}}{\Gamma 2n}\left[\hat{\theta}^2\frac{\Gamma 2n}{t^{2n}}+\frac{\Gamma(2n+2)}{t^{2n+2}}-2\hat{\theta}\frac{\Gamma(2n+1)}{t^{2n+1}}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we obtain the Baye's estimator as

$$\hat{\theta}_{Js}=\frac{n}{t},\quad \text{where } t=2\sum_{i=1}^{n}x_i \tag{4.7}$$

Estimator Under Entropy Loss Function

Using entropy loss function $L(\delta)=a[\delta-\log(\delta)-1]\,;a>0,\ \delta=\frac{\hat{\theta}}{\theta}$, the risk function is given by

$$R(\hat{\theta},\theta)=\int_0^\infty a[\delta-\log(\delta)-1]\ \frac{(t)^{2n}}{\Gamma(2n)}\theta^{2n-1}e^{-t\theta}\ d\theta$$

$$=\frac{a\,t^{2n}}{\Gamma(2n)}\left[\hat{\theta}\frac{\Gamma(2n-1)}{(t)^{2n-1}}-\log\hat{\theta}\frac{\Gamma(2n)}{(t)^{2n}}+\frac{\Gamma'(2n)}{t^{2n}}-\frac{\Gamma 2n}{t^{2n}}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we obtain the Baye's estimator as

$$\hat{\theta}_{JE}=\frac{2n-1}{t},\quad \text{where } t=2\sum_{i=1}^{n}x_i \tag{4.8}$$

Estimator Under LINEX Loss Function

Using LINEX loss function $l(\theta,\hat{\theta})=\exp\{b(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1$ for some constant b the risk function is given by

$$R(\hat{\theta},\theta)=\int_0^\infty\left(\exp\{b_1(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1\right)\frac{t^{2n}}{\Gamma(2n)}\theta^{2n-1}\ e^{-t\theta}\ d\theta$$

$$=\frac{t^{2n}}{\Gamma(2n)}\left[e^{b\hat{\theta}}\frac{\Gamma(2n)}{(b+t)^{2n}}-b\hat{\theta}\frac{\Gamma(2n)}{t^{2n}}+b\frac{\Gamma(2n+1)}{t^{2n+1}}-\frac{\Gamma(2n)}{t^{2n}}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 0$, we obtain the Bayes estimator as

$$\hat{\theta}_{JL} = \frac{1}{b}\log\left(\frac{b+t}{t}\right)^{2n} \tag{4.9}$$

Bayesian Estimation of Ailamujia Distribution Under Assumption of Gamma Prior

Consider n recorded values, $x = (x_1, x_2, ..., x_n)$ having probability density function as

$$f(x;\theta,\alpha) = 4x\theta^2 e^{-2\theta x}$$

we consider the prior distribution of θ to be Gamma prior i.e $g(\theta) \propto \frac{\alpha^\beta}{\Gamma\beta} e^{-\alpha\theta}\theta^{\beta-1}$

The posterior distribution of θ under the assumption of Gamma prior is given by

$$\pi(\theta/x) \propto L(x/\theta)\, g(\theta)$$

$$\Rightarrow \pi(\theta/x) \propto (4\theta)^n \prod_{i=1}^{n} x_i\, e^{-2\theta\sum_{i=1}^{n} x_i} \frac{\alpha^\beta}{\Gamma\beta} e^{-\alpha\theta}\theta^{\beta-1}$$

$$\Rightarrow \pi(\theta/x) = k\,\theta^{2n+\beta-1} e^{-\left(\alpha+2\sum_{i=1}^{n} x_i\right)\theta}$$

where k is independent of θ

and $k^{-1} = \int_0^\infty \theta^{2n+\beta-1} e^{-\left(\alpha+2\sum_{i=1}^{n} x_i\right)\theta} d\theta \quad \Rightarrow k^{-1} = \frac{\Gamma(2n+\beta)}{\left(\alpha+2\sum_{i=1}^{n} x_i\right)^{2n+\beta}}$

Henceforth posterior distribution of θ is given by

$$\pi(\theta/x) = \frac{\left(\alpha+2\sum_{i=1}^{n} x_i\right)^{2n+\beta}}{\Gamma(2n+\beta)} \theta^{2n+\beta-1} e^{-\left(\alpha+2\sum_{i=1}^{n} x_i\right)\theta}$$

$$\pi(\theta / x) = \frac{(\alpha + t)^{2n+\beta}}{\Gamma(2n+\beta)} \theta^{2n+\beta-1} e^{-(\alpha+t)\theta} \tag{4.10}$$

where $t = 2\sum_{i=1}^{n} x_i$

Estimator Under Squared Error Loss Function

By using squared error loss function $l(\hat{\theta},\theta) = c_1(\hat{\theta}-\theta)^2$ for some constant c_1 the risk function is given by

$$R(\hat{\theta},\theta) = E\left[l(\hat{\theta},\theta) \right]$$

$$= \int_0^\infty c_1(\hat{\theta}-\theta)^2 \frac{(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)} \theta^{2n+\beta-1} e^{-(\alpha+t)\theta} \, d\theta$$

$$= \frac{c_1(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)} \left[\hat{\theta}^2 \int_0^\infty \theta^{2n+\beta-1} e^{-(\alpha+t)\theta} \, d\theta + \int_0^\infty \theta^{2n+\beta+1} e^{-(\alpha+t)\theta} \, d\theta - 2\hat{\theta} \int_0^\infty \theta^{2n+\beta} e^{-(\alpha+t)\theta} \, d\theta \right]$$

$$= \frac{c_1(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)} \left[\hat{\theta}^2 \frac{\Gamma(2n+\beta)}{(\alpha+t)^{2n+\beta}} + \frac{\Gamma(2n+\beta+2)}{(\alpha+t)^{2n+\beta+2}} - 2\hat{\theta} \frac{\Gamma(2n+\beta+1)}{(\alpha+t)^{2n+\beta+1}} \right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 0$, we obtain the Bayes estimator as

$$\hat{\theta}_{Gs} = \frac{2n+\beta}{\alpha+t}, \quad \text{where } t = 2\sum_{i=1}^{n} x_i \tag{4.11}$$

Estimator Under Entropy Loss Function

Using entropy loss function $L(\delta) = a[\delta - \log(\delta) - 1]\,; a > 0,\ \delta = \frac{\hat{\theta}}{\theta}$, the risk function is given by

$$R(\hat{\theta},\theta) = \int_0^\infty a[\delta - \log(\delta) - 1] \frac{(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)} \theta^{2n+\beta-1} e^{-(\alpha+t)\theta} \, d\theta$$

$$=\frac{a(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\left[\hat{\theta}\frac{\Gamma(2n+\beta-1)}{(\alpha+t)^{2n+\beta-1}}-\log\hat{\theta}\frac{\Gamma(2n+\beta)}{(\alpha+t)^{2n+\beta}}+\frac{\Gamma'(2n+\beta)}{(\alpha+t)^{2n+\beta}}-\frac{\Gamma 2n+\beta}{(\alpha+t)^{2n+\beta}}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we obtain the Baye's estimator as

$$\hat{\theta}_{GE}=\frac{2n+\beta-1}{\alpha+t},\quad \text{where } t=2\sum_{i=1}^{n}x_i \tag{4.12}$$

Estimator Under LINEX Loss Function

Using LINEX loss function $l(\theta,\hat{\theta})=\exp\left\{b\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1$ for some constant b the risk function is given by

$$R(\hat{\theta},\theta)=\int_0^\infty\left(\exp\left\{b\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1\right)\frac{(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\theta^{2n+\beta-1}\ e^{-(\alpha+t)\theta}\ d\theta$$

$$=\frac{(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\left[e^{b\hat{\theta}}\frac{\Gamma(2n+\beta)}{(\alpha+b+t)^{2n+\beta}}-b\hat{\theta}\frac{\Gamma(2n+\beta)}{(\alpha+t)^{2n+\beta}}+b\frac{\Gamma(2n+\beta+1)}{(\alpha+t)^{2n+\beta+1}}-\frac{\Gamma(2n+\beta)}{(\alpha+t)^{2n+\beta}}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we obtain the Bayes estimator as

$$\hat{\theta}_{GL}=\frac{1}{b}\log\left(\frac{b+\alpha+t}{\alpha+t}\right)^{2n+\beta} \tag{4.13}$$

Application

The data set was initially stated by Badar Priest (1982) on failure stresses (inGpa) of 65 single carbon fibers of length 50mm respectively. The data set is given as follows: 1.339,1.434,1.549,1.574,1.589,1.613,1.746,1.753,1.764,1.807,1.812,1.84,1.852,1.852,1.862,1.864,1.931,1.952,1.974,2.019,2.051,2.055,2.058,2.088,2.125,2.162,2.171,2.172,2.18,2.194,2.211,2.27,2.272,2.28,2.299,2.308,2.335,2.349,2.356,2.386,2.39,2.41,2.43,2.458,2.471,2.497,2.514,2.558,2.577,2.593,2.601,2.604,2.62,2.633,2.67,2.682,2.699,2.705,2.735,2.785,3.02,3.042,3.116,3.174. This data set had used by Al-Mutairi (2013) and Uzma et al. (2017).

The posterior estimates and posterior risks are calculated, and result is presented in table 4.1 and table 4.2.

Table 4.1: *Posterior estimates and Posterior variances using Jeffery's Prior*

	$\hat{\theta}_S$	$\hat{\theta}_L$		$\hat{\theta}_E$
		b = 0.5	b =1.0	
Posterior Estimates	0.2231	0.0003	0.0007	0.4427
Posterior Risks	0.0513	7.4402	2.1083	5.6629

Table 4.2: *Posterior estimates and Posterior variances using Gamma Prior*

	$\hat{\theta}_S$	$\hat{\theta}_L$		$\hat{\theta}_E$
		b = 0.5	b =1.0	
Posterior Estimates	0.5266	0.05432	0.0574	0.6434
Posterior Risks	0.1430	7.4023	2.0032	5.9721

It is clear from Table 4.1and Table 4.2, on equating the Bayes posterior risk of dissimilar loss functions, it is observed that the squared error loss function has less Bayes posterior risk in both non informative and informative priors than other loss functions. According to the decision rule of less Bayes posterior risk we accomplish that squared error loss function is more preferable loss function.

Conclusion

We have predominantly considered the Bayes estimator of the parameter of Ailamujia distribution using Jeffrey's prior and gamma prior supposing three different loss functions. The Jeffrey's prior gives the prospect of covering wide continuum of priors to get Bayes estimates of the parameter. From the results, we observe that in most cases, Bayesian Estimator under Squared error Loss function has the smallest posterior risk values for both prior's i.e, Jeffrey's and gamma prior information.

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Chapter 5:

Estimating the Parameter of Weighted Ailamujia Distribution using Bayesian Approximation Techniques

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Additional information is available at the end of the chapter

Introduction

Ailamujia distribution (also known as ЭРланга distribution) was proposed by Lv et al [1] for applicability in various engineering fields. He studied various descriptive measures of the newly developed lifetime model which include mean, variance, median and maximum likelihood estimate. Pan et al [2] considered this distribution for estimating intervals and testing of hypothesis. Long [3] obtained the Bayes' estimates of ЭРланга Distribution under Type II censoring for missing data with three different priors. Li [4] estimated the parameters of Ailamujia model considering the three loss functions under a non informative prior. Uzma et al. [5] proposed the weighted Ailamujia Distribution and applied in reliability analysis. Assume X denotes the life span of a product following the Weighted Ailamujia distribution, its probability density function and cumulative density function is given respectively as follows:

$$f(x,\theta)=\frac{(2\theta)^{c+2}}{\Gamma(c+2)}x^{c+1}e^{-2\theta x}; \qquad x\geq 0,\theta\geq 0. \tag{5.1}$$

where θ the shape parameter and c are the weight parameter

$$F(x,\theta)=\frac{1}{\Gamma(c+2)}\gamma(c+2,2\theta x) \;\; ; \quad x\geq 0,\theta\geq 0. \tag{5.2}$$

The likelihood function and the corresponding log likelihood of (5.1) are given in the equations (5.3) and (5.4) respectively as:

$$L(x\mid\theta)\propto\theta^{n(c+2)}e^{-2\theta\sum_{i=1}^{n}x_i}. \tag{5.3}$$

$$\log L(x \mid \theta) = n(c+2)\log\theta - 2\theta\sum_{i=1}^{n} x_i. \tag{5.4}$$

The major focus of the current manuscript is to examine the performance of unknown shape parameter θ of Weighted Ailamujia distribution under a variety of priors using the two Bayesian approximation techniques.

Bayesian Approximation Techniques of Posterior Modes

Bayesian paradigm gives a comprehensive model for updating the prior information in view of the current knowledge. Those who like the elegance of Bayesian outlook study important properties of the posterior and predictive distributions. If the resulting distribution is in closed form and difficult to characterize it, analytical or numerical approximation methods are often used for accuracy with less computational complicacy. Many authors have reviewed the approximation methods including Sultan and Ahmad [6, 7] for Kumaraswamy distribution and generalized Power function distribution, Kawsar and Ahmad [8] for Inverse Exponential and Uzma and Ahmad [9,10] for Inverse Lomax and Dagum distributions.

Normal Approximation

In Bayesian approach, approximation techniques for large samples usually consider the normal approximation to the posterior distribution. If the posterior distribution is less skewed with sharp peak, the most convenient way is to approximate it by normal distribution. This posterior distribution is usually localized near the posterior mode and behaves normal under different conditions when the sample size is increased. The Normal approximation for the posterior distribution $P(\theta \mid x)$ centered at mode is given as

$$P(\theta \mid x) \sim N\left[\hat{\theta}, \left\{I(\hat{\theta})\right\}^{-1}\right] \tag{5.5}$$

where $I(\hat{\theta}) = -\dfrac{\partial^2}{\partial\theta^2}\log P(\theta \mid x)$ (5.6)

Under Jeffery's Prior $g_1(\theta) \propto \dfrac{1}{\theta}$, the posterior distribution is given by

$$P_1(\theta \mid x) \propto g_1(\theta) L(x \mid \theta) \tag{5.7}$$

$$\Rightarrow \quad P_1(\theta \mid x) \propto \theta^{n(c+2)-1} e^{-2\theta T}, \text{ where } T = \sum_{i=1}^{n} x_i. \tag{5.8}$$

As such $\hat{\theta} = \frac{\partial}{\partial\theta} \log P_1(\theta \mid x) = \frac{n(c+2)-1}{2T}$ and $\left[I(\hat{\theta})\right]^{-1} = \frac{[n(c+2)-1]}{(2T)^2}$.

Therefore, $P_1(\theta \mid x) \sim N\left[\frac{[n(c+2)-1]}{2T}, \frac{n(c+2)-1}{(2T)^2}\right]$. (5.9)

Similarly, under Gamma Prior $g_2(\theta) \propto e^{-b_1\theta}\theta^{a_1-1}$, the posterior distribution is given by $P_2(\theta \mid x) \propto \theta^{n(c+2)+b_1-1} e^{-\theta[2T+a_1]}$ (8) and can be approximated as

$$P_2(\theta \mid x) \sim N\left[\frac{n(c+2)+a_1-1}{2T+b_1}, \frac{n(c+2)+a_1-1}{(2T+b_1)^2}\right]. \tag{5.10}$$

Also, under the Erlang prior $g_3(\theta) \propto \theta^{a_2} e^{-\theta b_2}$, the posterior distribution is $\theta^{n(c+2)+a_2} e^{-\theta(2T+b_2)}$ (5.10) and can be approximated as

$$P_3(\theta \mid x) \sim N\left[\frac{n(c+2)+a_2}{2T+b_2}, \frac{n(c+2)+a_2}{(2T+b_2)^2}\right]. \tag{5.11}$$

T-K Approximation

Laplace's method proves to be an efficient procedure for solving the difficult integrals which arise in mathematics. For approximating the average values of functions of parameters and marginal densities, Laplace's method is generally used. It is widely applicable method in statistics for its simpler computations than the MCMC methods etc. Moreover, Laplace method provides better view of the problem. The different manuscripts in literature which describe the method include Lindley [11], Tierney and Kadane [12] and Leonard, Huss and Tsui [13]. Tierney and Kadane presented the Laplace method for computing $E[h(\theta) \mid x]$ as

$$E[h(\theta) \mid x] \cong \frac{\hat{\phi}^* \exp\left[-nh^*(\hat{\theta}^*)\right]}{\phi \exp\left[-nh(\hat{\theta})\right]} \tag{5.12}$$

where $-nh(\hat{\theta}) = \log P(\theta \mid x)$; $-nh*(\hat{\theta}^*) = \log P(\theta \mid x) + \log h(\theta)$.

$\hat{\phi}^2 = -\left[-nh''(\hat{\theta})\right]^{-1}$; $\phi^{*2} = -\left[-nh''^*(\hat{\theta}^*)\right]^{-1}$.

Thus, for Weighted Ailamujia distribution, T-K approximation for shape parameter θ under different priors is obtained as:

Under Jeffery's Prior $g_1(\theta) \propto \frac{1}{\theta}$, the posterior distribution is given by the equation (6)

We have $nh(\theta) = \log P_1(\theta \mid x) = [n(c+2)-1]\log\theta - 2\theta T.$

For which, $nh'(\theta) = \dfrac{n(c+1)-1}{\theta} - 2T \Rightarrow \hat{\theta} = \dfrac{n(c+2)-1}{2T}.$

Also, $nh''(\hat{\theta}) = \dfrac{-[n(c+2)-1]}{\theta^2} = \dfrac{-(2T)^2}{[n(c+2)-1]}.$

$$\therefore \hat{\phi}^2 = \frac{n(c+2)-1}{(2T)^2} \Rightarrow \hat{\phi} = \frac{\sqrt{n(c+2)-1}}{2T}.$$

Now, $nh^*(\theta^*) = -nh(\theta) + \log h(\theta)$, where $h(\theta) = \theta$

$$\Rightarrow \qquad nh^*(\theta^*) = n(c+2)\log\theta - 2\theta T.$$

Then, $nh'^*(\theta^*) = \dfrac{n(c+2)}{\theta} - 2T \Rightarrow \hat{\theta}^* = \dfrac{n(c+2)}{2T}.$

Further, $nh''^*(\hat{\theta}) = \dfrac{-(2T)^2}{n(c+2)} \Rightarrow \hat{\phi}^* = \dfrac{\sqrt{n(c+2)}}{2T}$

$$E[\theta \mid x] = \left(\frac{n(c+2)}{n(c+2)-1}\right)^{n(c+2)-\frac{1}{2}} \frac{n(c+2)}{2T}\exp(-1).$$

$$E(\theta^2 \mid x) = \frac{\hat{\phi}^{**}\left[nh^{**}(\hat{\theta})\right]}{\hat{\phi}\left[nh(\hat{\theta})\right]}, \text{here } h(\theta) = \theta^2. \tag{5.13}$$

As such, $E[\theta^2 \mid x] = \left(\dfrac{n(c+2)+1}{n(c+2)-1}\right)^{n(c+2)-\frac{1}{2}} \left[\dfrac{n(c+2)+1}{2T}\right]^2 \exp(-2).$

$$Var(\theta \mid x) = \left(\frac{n(c+2)+1}{n(c+2)-1}\right)^{n(c+2)-\frac{1}{2}} \left[\frac{n(c+2)+1}{2T}\right]^2 \exp(-2) - \left\{\left(\frac{n(c+2)}{n(c+2)-1}\right)^{n(c+2)-\frac{1}{2}} \frac{n(c+2)}{2T}\exp(-1)\right\}^2. \tag{5.14}$$

Also, under Gamma Prior, the posterior distribution is given by the equation (5.9) and the estimates are given by

$$E(\theta \mid x)=\left[\frac{n(c+2)+a_1}{n(c+2)+a_1-1}\right]^{n(c+2)+a_1-\frac{1}{2}}\left[\frac{n(c+2)+a_1}{2T+b_1}\right]\exp(-1). \tag{5.15}$$

$$E(\theta^2 \mid x)=\left[\frac{n(c+2)+a_1+1}{n(c+2)+a_1-1}\right]^{n(c+2)+a_1-\frac{1}{2}}\left[\frac{n(c+2)+a_1+1}{2T+b_1}\right]^2\exp(-2)$$

$$V(\theta \mid x)=\left[\frac{n(c+2)+a_1+1}{n(c+2)+a_1-1}\right]^{n(c+2)+a_1-\frac{1}{2}}\left[\frac{n(c+2)+a_1+1}{2T+b_1}\right]^2\exp(-2)-$$

$$\left\{\left[\frac{n(c+2)+a_1}{n(c+2)+a_1-1}\right]^{n(c+2)+a_1-\frac{1}{2}}\left[\frac{n(c+2)+a_1}{2T+b_1}\right]\exp(-1)\right\}^2. \tag{5.16}$$

Similarly, under Erlang Prior $g_4(\theta)\propto\theta^{a_2}e^{-\theta b_2}$, the posterior distribution is given by the equation (5.10) and we have

$$E(\theta \mid x)=\left[\frac{n(c+2)+a_2+1}{n(c+2)+a_2}\right]^{n(c+2)+a_2+\frac{1}{2}}\left[\frac{n(c+2)+a_2+1}{(2T+b_2)}\right]\exp(-1). \tag{5.17}$$

$$E(\theta^2 \mid x)=\left[\frac{n(c+2)+a_2+2}{n(c+2)+a_2}\right]^{n(c+2)+a_2+\frac{1}{2}}\left[\frac{n(c+2)+a_2+2}{2T+b_2}\right]^2\exp(-2).$$

$$Var(\theta \mid x)=\left[\frac{n(c+2)+a_2+2}{n(c+2)+a_2}\right]^{n(c+2)+a_2+\frac{1}{2}}\left[\frac{n(c+2)+a_2+2}{2T+b_2}\right]^2\exp(-2)-$$

$$\left\{\left[\frac{n(c+2)+a_2+1}{n(c+2)+a_2}\right]^{n(c+2)+a_2+\frac{1}{2}}\left[\frac{n(c+2)+a_2+1}{(2T+b_2)}\right]\exp(-1)\right\}^2. \tag{5.18}$$

Applications

For comparing the efficacy of different priors and the two approximation techniques for the weighted Ailamujia distribution, we have considered the three real life data sets related to engineering field.

Data Set 5. 1: The first data set is provided in Murthy et al. [14] about time between failures for 30 repairable items. The data are listed as the following:

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45,0.70,1.06,1.46, 0.30,1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17

DataSet 5.2: The second data represent the strength data measured in GPA, for single carbon fibres and impregnated 1000 carbon fiber tows reported by Badar and Priest [15]. We will be considering the single fibres of 10 mm in gauge length with sample sizes 63.

0.101,0.322,0.403,0.428,0.457,0.550,0.561,0.596,0.597,0.645,0.654,0.674,0.718,0.722,0.725,0.723,0.775,0.814,0.816,0.818,0.824,0.859,0.875,0.938,0.940,1.056,1.117,1.128,1.137,1.137,1.177,1.196,1.230,1.325,1.339,1.345,1.420,1.423,1.435,1.443,1.464,1.472,1.494,1.532,1.546,1.577,1.608,1.635,1.693,1.701,1.737,1.754,1.762,1.828,2.052,2.071,2.086,2.171,2.224,2.227,2.425,2.595,2.220

Data set 5.3: The third data set is on the strengths of 1.5 cm glass fibres. The data was originally obtained by workers at the UK National Physical Laboratory and it has been used by Bourguignon *et al.* [16].

Table 5.1: *Posterior estimates for Normal Approximation*

	c	Jeffery's Prior	Gamma Prior			Erlang Prior		
			$a_1=b_1=0.5$	$a_1=b_1=1.0$	$a_1=b_1=2.0$	$a_2=b_2=0.5$	$a_2=b_2=1.0$	$a_2=b_2=2.0$
Data Set I	0.5	0.79948 (0.00863)	0.80055 (0.00860)	0.80162 (0.00856)	0.80372 (**0.00849**)	0.81130 (0.00871)	0.81231 (0.00868)	0.81429 (0.00861)
	1	0.96153 (0.01038)	0.96174 (0.01033)	0.96194 (0.01028)	0.96235 (**0.01017**)	0.97249 (0.01045)	0.97263 (0.01039)	0.97292 (0.01028)
	2	1.28565 (0.01388)	1.28411 (0.01379)	1.28259 (0.01370)	1.27961 (**0.01353**)	1.29486 (0.01391)	1.29328 (0.01382)	1.29018 (0.01364)

Data Set II	0.5	0.99914 (0.00637)	0.99914 (0.00635)	0.99914 (0.00633)	0.99915 (**0.00629**)	1.00551 (0.00639)	1.00549 (0.00637)	1.00545 (0.00633)
	1	1.20025 (0.00766)	1.19961 (0.00763)	1.19898 (0.00760)	1.19772 (**0.00755**)	1.20597 (0.00767)	1.20532 (0.00764)	1.20402 (0.00758)
	2	1.60246 (0.01023)	1.60054 (0.01018)	1.59864 (0.01014)	1.59486 **(0.01005)**	1.60690 (0.01022)	1.60498 0.01018)	1.60117 0.01009
Data Set III	0.5	0.82429 (0.00434)	0.82475 (0.00433)	0.82521 (0.00432)	0.82612 (**0.00430**)	0.83000 (0.00436)	0.83045 (0.00435)	0.83133 (0.00433)
	1	0.99020 (0.00521)	0.99022 (0.00520)	0.99025 (0.00518)	0.99030 (**0.00516**)	0.99548 (0.00522)	0.99549 (0.00521)	0.99551 (0.00518)
	2	1.32202 (0.00696)	1.32118 (0.00694)	1.32034 (0.00691)	1.31867 **(0.00687)**	1.32643 (0.00696)	1.32557 (0.00694)	1.32388 (0.00690)

Table 5.2: *Posterior estimates for TK approximation*

	C	Jeffery's Prior	Gamma Prior			Erlang Prior		
			$a_1=b_1=0.5$	$a_1=b_1=1.0$	$a_1=b_1=2.0$	$a_2=b_2=0.5$	$a_2=b_2=1.0$	$a_2=b_2=2.0$
Data Set I	0.5	0.81029 (0.00875)	0.81131 (0.00871)	0.81232 (0.00868)	0.81430 (**0.00861**)	0.81667 (0.01766)	0.81765 (0.01759)	0.81957 (0.01744)
	1	0.97235 (0.01050)	0.97250 (0.01045)	0.97264 (0.01039)	0.97293 (**0.01028**)	0.97785 (0.02113)	0.97797 (0.02102)	0.97821 (0.02080)
	2	1.29646 (0.01400)	1.29487 (0.01391)	1.29329 (0.01382)	1.29019 (**0.01364**)	1.30023 (0.02805)	1.29862 (0.02787)	1.29547 (0.02751)
Data Set II	0.5	1.005532 (0.00641)	1.00551 (0.00639)	1.00549 (0.00637)	1.00546 (**0.00633**)	1.00869 (0.01287)	1.00866 (0.01283)	1.00860 (0.01275)
	1	1.20663 (0.00770)	1.20598 (0.00767)	1.20532 (0.00764)	1.20403 (**0.00758**)	1.20915 (0.01543)	1.20849 (0.01537)	1.20718 (0.01525)
	2	1.60884 (0.01027)	1.60691 (0.01022)	1.60498 (0.01018)	1.60117 (**0.01009**)	1.61009 (0.02053)	1.60815 (0.020443)	1.60432 (0.020266)

Data Set III	0.5	0.82956 (0.00436)	0.83000 (0.00436)	0.83045 (0.00435)	0.83133 (**0.00433**)	0.83263 (0.00877)	0.83307 (0.00875)	0.83394 (0.00872)
	1	0.99547 (0.00524)	0.99548 (0.00522)	0.99549 (0.00521)	0.99551 (**0.00518)**	0.99810 (0.01051)	0.99811 (0.01048)	0.99812 (0.01043)
	2	1.32729 (0.00699)	1.32643 (0.00696)	1.32558 (0.00694)	1.32388 (**0.00690**)	1.32906 (0.01399)	1.32819 (0.01394)	1.32648 (0.01385)

Conclusion

From table 5.1 and 5.2, it is clearly evident that the posterior variance of gamma prior is less than the other priors under both the approximation techniques especially when the value of both the hyper parameters a_1 and b_1 is taken as 2. Further, it is also noted that the values of normal approximation are less than the T-K approximation for all the three data sets.

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Chapter 6:

Bayesian Inference for Exponential Rayleigh Distribution Using R Software

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Additional information is available at the end of the chapter

Introduction

Exponential-Rayleigh (ER) distribution is a newly proposed lifetime model introduced and discussed by Kawsar and Ahmad (2017). It is a versatile distribution and can take a variety of shapes such as positively skewed, reversed-J and tends to be symmetric. Exponential-Rayleigh distribution is a continuous distribution with wide range of applications in reliability fields and is used for modelling lifetime phenomena. The cdf and pdf of the ERD are given as

$$f(x) = \lambda \beta x e^{\frac{\beta}{2}x^2} e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)}; \qquad x, \lambda, \beta > 0 \ , \tag{6.1}$$

$$F(x) = 1 - e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)} \tag{6.2}$$

The main purpose of this chapter is to study the Bayesian approach for the parameter of ER distribution. There are numerous good sources which provide the detailed explanation of Bayesian approach while then, a number of authors have studied and obtained various probability distributions based on the Estimation of the Bayesian approach. Ahmed et al. (2007) discussed the exponential distribution (ED) from a Bayesian point of view. James Dow (2015) obtained the Bayesian Inference for the parameter of Weibull-Pareto distribution. Naqash et al. (2016) studied Bayesian Analysis of Generalized Exponential Distribution while as Kawsar and Ahmad (2017) considered the estimation for the parameter of Weibull- Rayleigh (WR) distribution. They obtained Baye's estimators for the parameter of WR distribution by using different Informative and Non-Informative priors under different symmetric and asymmetric loss functions. They also compared the classical method with Bayesian method by using mean square error through simulation study with varying sample sizes.

Parameter Estimation

Consider a random sample $x_1, x_2, x_3, \ldots x_n$ having density function of (6.1) and then the likelihood function of the given distribution is as follows:

$$L(x) = \lambda^n \beta^n \prod_{i=1}^{n} \left\{ x_i e^{\frac{\beta}{2} x_i^2} e^{-\lambda\left(e^{\frac{\beta}{2} x_i^2} - 1\right)} \right\}. \tag{6.3}$$

The corresponding log likelihood function of the equation (6.3) is given as under:

$$\log L(x) = n \log \lambda + n \log \beta + \sum_{i=1}^{n} \log x_i + \frac{\beta}{2} \sum_{i=1}^{n} x_i^2 - \lambda \sum_{i=1}^{n} \left(e^{\frac{\beta}{2} x_i^2} - 1 \right). \tag{6.4}$$

Differentiating (6.4) with respect to λ, when the parameter β is assumed to be known, then the MLE is obtained as

$$\frac{\partial}{\partial \lambda} \log L(x) = 0, \Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \left(e^{\frac{\beta}{2} x_i^2} - 1 \right)}. \tag{6.5}$$

Posterior Distribution and Baye's Estimators under Non-Informative Prior Using Different Loss Functions:

The extended Jeffrey's prior suggested by Al-Kutubi (2005) is given as

$$g_1(\lambda) = \frac{1}{\lambda^{2c_1}};\ c_1 \in R^+ \tag{6.6}$$

Combining the likelihood function (6.3) and the above prior distribution, then the posterior density of λ is derived as follows:

$$g_1(\lambda \mid x) = \frac{T_1^{n-2c_1+1}}{\Gamma(n-2c_1+1)} \lambda^{n-2c_1} e^{-\lambda T_1} \ ; \ \lambda > 0 . \tag{6.7}$$

Hence the posterior density of $g_1(\lambda \mid x) \sim G((n-2c_1+1), T_1)$, where $T_1 = \sum_{i=1}^{n} \left(e^{\frac{\beta}{2} x_i^2} - 1 \right)$

With the above prior, we use three different loss functions namely Al-Bayyati's loss function (ABLF), Entropy loss function (ELF) and LINEX loss function (LLF) to find Bayes estimates for the parameter of model (6.1).

Under ABLF the risk function is given by

$$R(\hat{\lambda},\lambda)=\int_0^\infty \lambda^{c_2}(\hat{\lambda}-\lambda)^2\frac{T_1^{n-2c_1+1}}{\Gamma(n-2c_1+1)}\lambda^{n-2c_1}\ e^{-\lambda T_1}d\lambda \tag{6.8}$$

On solving (6.8), we get

$$R(\hat{\lambda},\lambda)=\frac{1}{\Gamma n-2c_1+1)}\left[\frac{\hat{\lambda}^2\Gamma(n-2c_1+c_2+1)}{T_1^{c_2}}+\frac{\Gamma(n-2c_1+c_2+3)}{T_1^{c_2+2}}-\frac{2\hat{\lambda}\Gamma(n-2c_1+c_2+2)}{T_1^{c_2+1}}\right].$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the Bayes estimator:

$$\hat{\lambda}_{ABLF}=\frac{(n-2c_1+c_2+1)}{T_1}\text{; where } T_1=\sum_{i=1}^{n}\left(e^{\frac{\beta}{2}x_i^2}-1\right)\ . \tag{6.9}$$

Remark 6.1:

Replacing $c_2=0$ and $c_1=1/2$ in (6.9) we get the same Bayes estimator as obtained under SELF using the Jeffrey's prior, replace $c_2=0$ and $c_1=3/2$ we get the same Bayes estimator as obtained under SELF using Hartigan's prior and replace $c_2=0$ and $c_1=0$ we get the we get the same Bayes estimator as obtained under SELF using Uniform prior. Under ELF the risk function is given by

$$R(\hat{\lambda},\lambda)=b_1\int_0^\infty\left[\frac{\hat{\lambda}}{\lambda}-\log\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right]\frac{T_1^{n-2c_1+1}}{\Gamma(n-2c_1+1)}\lambda^{n-2c_1}\ e^{-\lambda T_1}\ d\lambda\ . \tag{6.10}$$

On solving (6.10), we get

$$R(\hat{\lambda},\lambda)=b_1\left[\hat{\lambda}\frac{T_1}{(n-2c_1)}-\log(\hat{\lambda})+\frac{\Gamma'(n-2c_1+1)}{\Gamma(n-2c_1+1)}-1\right].$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the Bayes estimator:

$$\hat{\lambda}_{ELF}=\frac{(n-2c_1)}{T_1}\text{; where } T_1=\sum_{i=1}^{n}\left(e^{\frac{\beta}{2}x_i^2}-1\right)\ . \tag{6.11}$$

Remark 6.2:

Replacing $c_1 = 1/2$ in (6.11) we get the same Bayes estimator as obtained under the Jeffrey's prior, replace $c_1 = 3/2$ we get the Hartigan's prior and replace $c_1 = 0$ we get the Uniform prior.

Under LLF the risk function is given by

$$R(\hat{\lambda},\lambda) = \int_0^\infty \left(\exp\left\{b_2\left(\hat{\lambda}-\lambda\right)\right\} - b_2\left(\hat{\lambda}-\lambda\right) - 1\right) \frac{T_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)} \lambda^{n-2c_1}\, e^{-\lambda T_1} d\lambda. \qquad (6.12)$$

On solving (6.12), we get

$$R(\hat{\lambda},\lambda) = \frac{T_1^{\,n-2c_1+1}}{\Gamma(n-2c_1+1)} \left[\begin{array}{l} e^{b_2\hat{\lambda}} \dfrac{\Gamma(n-2c_1+1)}{(b_2+T_1)^{n-2c_1+1}} - b_2\hat{\lambda}\dfrac{\Gamma(n-2c_1+1)}{T_1^{\,n-2c_1+1}} \\ \qquad\qquad + b_2 \dfrac{\Gamma(n-2c_1+2)}{T_1^{\,n-2c_1+2}} - \dfrac{\Gamma(n-2c_1+1)}{T_1^{\,n-2c_1+1}} \end{array} \right]$$

$$R(\hat{\lambda},\lambda) = e^{b_2\hat{\lambda}} \left(\frac{T_1}{b_2+T_1}\right)^{n-2c_1+1} - b_2\hat{\lambda} + b_2\frac{(n-2c_1+1)}{T_1} - 1.$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the Bayes estimator:

$$\hat{\lambda}_{LLF} = \frac{1}{b_2}\log\left(\frac{b_2+T_1}{T_1}\right)^{n-2c_1+1} \text{; where } T_1 = \sum_{i=1}^{n}\left(e^{\frac{\beta}{2}x_i^2} - 1\right). \qquad (6.13)$$

Remark 6.3:

If we put $c_1 = 1/2$ in (6.13) we get the same Bayes estimator as obtained under the Jeffrey's prior, If $c_1 = 3/2$ we get the Hartigan's prior and If $c_1 = 0$ we get the Uniform prior.

Posterior Distribution and Baye's Estimators under Informative Prior Using Different Loss Functions:

The gamma distribution is used as an informative prior with hyper parameters a *and* b, having the following p.d.f as:

$$g_2(\lambda) \propto \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}; \quad 0 < \lambda < \infty, a, b > 0. \qquad (6.14)$$

Combining the likelihood function (6.3) and the prior distribution (6.14), then the posterior density of λ is derived as follows:

$$g_2(\lambda \mid x) = \frac{T_2^{\,n+a}}{\Gamma(n+a)} \lambda^{n+a-1} \; e^{-\lambda T_2} \quad ; \quad \lambda > 0 \; . \tag{6.15}$$

Hence the posterior density of $g_2(\lambda \mid x) \sim G((n+a), T_2)$; where $T_2 = b + \sum_{i=1}^{n} \left(e^{\frac{\beta}{2} x_i^2} - 1 \right)$.

Remark 6.4:

For $a = b = 0$ in (6.15), the posterior distribution under the gamma prior reduces to posterior distribution under the Jeffrey's prior.

For $a = 1, b = 0$ in (6.15), the posterior distribution under the gamma prior reduces to posterior distribution under the Uniform prior.

Under ABLF the risk function is given by

$$R(\hat{\lambda}, \lambda) = \int_0^\infty \lambda^{c_2} (\hat{\lambda} - \lambda)^2 \frac{T_2^{\,n+a}}{\Gamma(n+a)} \lambda^{n+a-1} \; e^{-\lambda T_2} d\lambda \tag{6.16}$$

On solving (3.16), we get

$$R(\hat{\lambda}, \lambda) = \frac{1}{\Gamma n+a)} \left[\frac{\hat{\lambda}^2 \Gamma(n+c_2+a)}{T_2^{\,c_2}} + \frac{\Gamma(n+a+c_2+1)}{T_2^{\,c_2+2}} - \frac{2\hat{\lambda}\Gamma(n+a+c_2+1)}{T_2^{\,c_2+1}} \right].$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the Bayes estimator:

$$\hat{\lambda}_{ABLF} = \frac{(n+a+c_2)}{T_2} \text{; where } T_2 = b + \sum_{i=1}^{n} \left(e^{\frac{\beta}{2} x_i^2} - 1 \right) \; . \tag{6.17}$$

Remark 6.5:

Replacing $c_2 = 0$ in (3.17), we get the same Bayes estimator as obtained under the SELF.

Under ELF the risk function is given by

$$R(\hat{\lambda}, \lambda) = b_1 \int_0^\infty \left[\frac{\hat{\lambda}}{\lambda} - \log\left(\frac{\hat{\lambda}}{\lambda} \right) - 1 \right] \frac{T_2^{\,n+a}}{\Gamma(n+a)} \lambda^{n+a-1} \; e^{-\lambda T_2} d\lambda \, . \tag{6.18}$$

On solving (6.18), we get

$$R(\hat{\lambda},\lambda)=b_1\left[\hat{\lambda}\frac{T_2}{(n+a-1)}-\log(\hat{\lambda})+\frac{\Gamma'(n+a)}{\Gamma(n+a)}-1\right].$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the Bayes estimator:

$$\hat{\lambda}_{ELF}=\frac{(n+a-1)}{T_2}\quad;\text{where}\,T_2=b+\sum_{i=1}^{n}\left(e^{\frac{\beta}{2}x_i^2}-1\right)\ . \tag{6.19}$$

Under LLF the risk function is given by

$$R(\hat{\lambda},\lambda)=\int_0^\infty\left(\exp\left\{b_2\left(\hat{\lambda}-\lambda\right)\right\}-b_2\left(\hat{\lambda}-\lambda\right)-1\right)\frac{T_2^{\,n+a}}{\Gamma(n+a)}\lambda^{n+a-1}\ e^{-\lambda T_2}d\lambda \tag{6.20}$$

On solving (6.20), we get

$$R(\hat{\lambda},\lambda)=\frac{T_2^{\,n+a}}{\Gamma(n+a)}\left[\begin{array}{l}e^{b_2\hat{\lambda}}\dfrac{\Gamma(n+a)}{(b_2+T_2)^{n+a}}-b_2\hat{\lambda}\dfrac{\Gamma(n+a)}{T_2^{\,n+a}}\\ \qquad\qquad+b_2\dfrac{\Gamma(n+a+1)}{T_2^{\,n+a+1}}-\dfrac{\Gamma(n+a)}{T_2^{\,n+a}}\end{array}\right]$$

$$R(\hat{\lambda},\lambda)=e^{b_2\hat{\lambda}}\left(\frac{T_2}{b_2+T_2}\right)^{n+a}-b_2\hat{\lambda}+b_2\frac{(n+a)}{T_2}-1.$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the Bayes estimator:

$$\hat{\lambda}_{LLF}=\frac{1}{b_2}\log\left(\frac{b_2+T_2}{T_2}\right)^{n+a}\quad;\text{where}\,T_2=b+\sum_{i=1}^{n}\left(e^{\frac{\beta}{2}x_i^2}-1\right)\ . \tag{6.21}$$

Simulation Study

In the simulation study, three samples of sizes 25, 50 and 100 to signify small, medium, and large data sets have been generated from R software to examine the performance of Classical and Bayesian estimates for the parameter of Exponential-Rayleigh (ER) distribution under different priors using different loss functions. The data sets are obtained by using the inverse cdf method and the value of the parameters $\alpha\,\&\,\lambda$ are chosen as $\alpha=0.5$ and $\lambda=0.5,1.0\,\&\,1.5$. The values of Jeffrey's extension were $c_1=(0.4,1.4)$ and the values of hyper parameters were a= (0.4, 1.4) and b= (0.4, 1.4). The results are replicated 1000 times and the average results are presented in table 6.1 and table 6.2.

Table 6.1: *Baye's estimators and MSE (in parenthesis) under Extension of Jeffery's prior*

n	β	λ	c_1	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{ABLF}$		$\hat{\lambda}_{ELF}$	$\hat{\lambda}_{LLF}$	
					c_2=0.3	c_2=-0.3		b_2=0.4	b_2=-0.4
25	0.5	0.5	0.4	0.44148 (0.0887)	0.45031 **(0.01033)**	0.43972 (0.0114)	0.42736 (0.0131)	0.44346 (0.0111)	0.44659 (0.01072)
			1.4	0.44148 (0.0887)	0.41499 **(0.01445)**	0.40439 (0.0163)	0.39204 (0.0188)	0.40826 (0.0156)	0.41115 (0.01513)
		1.0	0.4	0.88297 (0.4342)	0.90063 **(0.04131)**	0.87944 (0.0459)	0.85471 (0.0525)	0.88380 (0.0449)	0.89638 (0.04217)
			1.4	0.88297 (0.4342)	0.82999 **(0.05784**)	0.80879 (0.0654)	0.78408 (0.0755)	0.81366 (0.0636)	0.82524 (0.06366)
		1.5	0.4	1.32445 (1.1814)	1.35094 **(0.09295)**	1.31916 (0.1034)	1.28207 (0.1182)	1.32110 (0.1027)	1.34939 (0.09341)
			1.4	1.32445 (1.1810)	1.24498 (**0.13015**)	1.21319 (0.1473)	1.176115 (0.1700)	1.21625 (0.1456)	1.24230 (0.13152)
50	0.5	0.5	0.4	0.4887 (0.0168)	0.49356 **(0.00484)**	0.48769 (0.0049)	0.48085 (0.0051)	0.48967 (0.0049)	0.49159 (0.00487)
			1.4	0.4887 (0.0168)	0.47401 (0.00528)	0.46815 (0.0056)	0.46131 (0.0061)	0.47016 (0.0054)	0.47200 (0.00539)
		1.0	0.4	0.97735 (0.0605)	0.98712 **(0.01935)**	0.97539 (0.0197)	0.96171 (0.0206)	0.97744 (0.0196)	0.98511 (0.01940)
			1.4	0.97735 (0.0605)	0.94803 (**0.02112)**	0.93629 (0.0224)	0.92262 (0.0244)	0.93849 (0.0221)	0.94587 (0.02135)
		1.5	0.4	1.4662 (0.1519)	1.48068 **(0.04353)**	1.46309 (0.0445)	1.44256 (0.0464)	1.46332 (0.0445)	1.48058 (0.04353)
			1.4	1.46602 (0.1519)	1.42204 **(0.04751)**	1.40445 (0.0505)	1.38392 (0.0549)	1.40502 (0.0504)	1.42159 (0.04758)
100	0.5	0.5	0.4	0.51758 (0.0083)	0.51344 **(0.00286)**	0.51706 (0.0029)	0.52016 (0.0030)	0.518075 (0.0030)	0.51915 (0.00305)
			1.4	0.51758 (0.0083)	0.50308 (**0.00264**)	0.50671 (0.0026)	0.50981 (0.0027)	0.50773 (0.0026)	0.50879 (0.00271)
		1.0	0.4	1.03515 (0.0329)	1.02687 **(0.01146)**	1.03412 0.01190	1.04033 (0.0123)	1.03508 0.01197	1.039375 (0.01229)
			1.4	1.03515 (0.0329)	1.00617 **(0.01056)**	1.01341 (0.0107)	1.01962 (0.0109)	1.01442 (0.0107)	1.01863 (0.01087)
		1.5	0.4	1.55273 (0.0730)	**0.51344** **(0.00286)**	1.55117 0.0267	1.56049 (0.0278)	1.55102 (0.0267)	1.56068 (0.02784)
			1.4	1.55273 (0.0730)	0.50308 (**0.00264**)	1.52012 (0.0240)	1.52944 (0.0245)	1.52006 (0.0240)	1.52953 (0.02455)

Table 6.2: Baye's estimators and MSE (in parenthesis) under Gamma prior

n	β	λ	$a=b$	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{ABLF}$		$\hat{\lambda}_{ELF}$	$\hat{\lambda}_{LLF}$	
					c_2=0.3	c_2=-0.3		b_2=0.4	b_2=-0.4
25	0.5	0.5	0.4	0.44148 (0.08879)	0.45066 **(0.0102)**	0.44014 (0.01139)	0.42786 (0.01301)	0.44384 (0.01096)	0.44697 (0.0106)
			1.4	0.44148 (0.08879)	0.46013 **(0.0094)**	0.44979 (0.01036)	0.43773 (0.01173)	0.45339 (0.01001)	0.45654 (0.0097)
		1.0	0.4	0.88297 (0.43422)	0.89505 **(0.0418)**	0.874151 (0.04665)	0.84977 (0.05338)	0.87849 (0.04557)	0.89082 (0.0427)
			1.4	0.88297 (0.43422)	0.89858 **(0.0401)**	0.87839 (0.04469)	0.85483 (0.05098)	0.88256 (0.04369)	0.89452 (0.0410)
		1.5	0.4	1.32445 (1.18140)	1.33328 **(0.0961)**	1.30215 (0.10750)	1.26584 (0.12319)	1.30423 (0.10669)	1.33158 (0.0967)
			1.4	1.32445 (1.18140)	1.31685 **(0.0977)**	1.28725 (0.10948)	1.25273 (0.12536)	1.28937 (0.10858)	1.31506 (0.0984)
50	0.5	0.5	0.4	0.48867 (0.01628)	0.49358 **(0.0048)**	0.48774 (0.00493)	0.48093 (0.00514)	0.48971 (0.00489)	0.49162 (0.0048)
			1.4	0.48867 (0.01628)	0.49847 **(0.0047)**	0.49268 (0.00483)	0.48593 (0.00498)	0.49462 (0.00481)	0.49653 (0.0047)
		1.0	0.4	0.97735 (0.06605)	0.98334 **(0.0192)**	0.97170 (0.01976)	0.95813 (0.02073)	0.97375 (0.01965)	0.98133 (0.0193)
			1.4	0.97735 (0.06605)	0.98366 **(0.0188)**	0.97224 (0.01937)	0.95892 (0.02029)	0.97425 (0.01927)	0.98169 (0.0189)
		1.5	0.4	1.46602 (0.15194)	1.46931 **(0.0432)**	1.45193 (0.04464)	1.43164 (0.04700)	1.45225 (0.04461)	1.46915 (0.0432)
			1.4	1.46602 (0.15194)	1.45609 **(0.0426)**	1.43919 (0.04447)	1.41948 (0.04725)	1.43955 (0.04443)	1.45586 (0.0427)
100	0.5	0.5	0.4	0.51758 (0.00838)	0.51341 **(0.0028)**	0.51702 (0.00297)	0.52012 (0.00308)	0.51804 (0.00300)	0.51911 (0.0030)
			1.4	0.51758 (0.00838)	0.51591 **(0.0029)**	0.51950 (0.00306)	0.52259 (0.00319)	0.52051 (0.00309)	0.52158 (0.0031)
		1.0	0.4	1.03515 (0.03295)	1.02469 **(0.0112)**	1.03191 (0.01169)	1.03809 (0.01212)	1.03288 (0.01175)	1.03715 (0.0120)
			1.4	1.03515 (0.03295)	1.02447 **(0.0111)**	1.03158 (0.01155)	1.03771 (0.01198)	1.03254 (0.01161)	1.03677 (0.0119)
		1.5	0.4	1.55273 (0.07305)	1.53388 **(0.0250)**	1.54468 (0.02591)	1.55394 (0.02682)	1.54455 (0.02589)	1.55412 (0.0268)
			1.4	1.55273 (0.07305)	1.52577 **(0.0240)**	1.53641 (0.02475)	1.54553 (0.02549)	1.53630 (0.02474)	1.54567 (0.0255)

From table 6.1 and table 6.2 we conclude that Al-Bayyati's loss function gives the minimum MSE as compared to the other loss functions and among the priors Gamma prior gives the less MSE than other assumed priors.

Conclusion

In this chapter, we have paralleled the Baye's estimates of the parameter of the Exponential-Rayleigh (ER) distribution under extension of Jeffrey's prior and gamma prior using different loss functions with that of maximum likelihood estimate. From the results, Al-Bayyati's loss function gives the minimum MSE as compared to the other loss functions and among the priors Gamma prior gives the less MSE than other assumed priors.

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Chapter 7:

Designing Accelerated Life Testing for Product Reliability Under Warranty Prospective

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Additional information is available at the end of the chapter

Introduction

In the current era, every manufacturer producing goods want to increase their sales by providing some incentives to customers in the form of warranty and guaranty. It is a written statement presented by Manufacturers to the customers, promising them to repair or replace the product they purchased, if necessary, within a specified period of time. It is additionally a way of advertising the standard of the merchandise and thereby boosting sales. A detailed review of assorted problems associated with product warranties will be found in Blischke and Murthy (1992a, 1992b, 1994) and Chien (2010).

One of the types of warranty policies is rebate warranty. Under this scheme, a customer (buyer) is refunded by some percent of money (sales price) if the product meets the defect during the warranty time spam. Batteries and tires are the items sold under this warranty scheme. Common forms include: lump sun, and pro-rata rebates. Other issues and discussions related to warranty policies can be found in Mitra and Patankar (1993), Murthy (1990), Murthy and Blischke (1992).

The manufacturers shall only provide these incentives when they have the faith on the products that their product has the ability to serve at least for the time period warranty is given. Therefore, it is essential for manufacturers to test the reliability and performance of the products before letting them serve in the market. This can be done by using accelerated life testing (ALT) on products, where products are put at higher stresses than normal to induce failure and predict their life under normal use conditions. ALT also helps Manufacturers to predict the various costs associated with the product under warranty policy. The main aim of ALT is to find the failure data of such products and systems by subjecting them to the higher levels of stresses. Hence accelerated life testing is needed to quickly provide the information about the life distribution of products.

For analysing ALT efficiently and to obtain performance data, the experimenter needs to determine the testing method, statistical model, form of the life data and a suitable statistical method. Analysing these measures properly, provides the best estimates of the product's life and performance under usual conditions.

There are researchers who combined accelerating life testing and warranty models. GuangbinYang, (2010), provided a method for describing the warranty cost, and its confidence interval. El-Dessoukey (2015) used accelerated life tests along with Exponentiated Pareto distribution to describe age replacement policy under warranty policy.

The article describes how to use accelerated life testing procedures for predicting the cost of age replacement of units or products under warranty policy. Under constant stress, the generalized exponential distribution is assumed to cover the lifetimes of the products. The chapter also describes the age replacement policy in the combination of pro-rata rebate warranty for non-repairable units.

Model Description and Test Method

ALT is a best used method for reliability and life prediction of systems or components. Designing the ALT plan, needs to determine the following:

- **i.** The statistical distribution of failure times of products.
- **ii.** Type of data used, complete or censored.
- **iii.** Type of censoring scheme.
- **iv.** The type of stress to be used.
- **v.** The stress level selected.
- **vi.** The percentage of test units allocated for each stress level.
- **vii.** The mathematical model describing the relation between life and stress (life-stress relationship).

This study is dealt with constant stress and type-I censored data under the assumption that the lifetimes of the units follows generalized exponential distribution. There are k levels of high stresses V_j, $j = 1, 2, ..., K$ and assume that V_u is the normal use condition satisfying $V_u < V_1 < V_2 <.... < V_k$. At each stress level, there are n_j units put on test and the experiment terminates once the number of failures r_j among these n_j units are observed.

For the detailed review of constant stress ALT one may refer to Abdel-Ghaly et al. (1998), El-Dessouky (2001), Attia et al. (2011), and Attia et al. (2013). The studies have shown that the two-parameter Generalised Exponential distribution can selectively used in place of two-parameter gamma and two-parameter Weibull distributions for analysing many lifetime data

(see, Gupta and Kundu 1999). The probability density function (pdf) of a generalised Exponential distribution is given by

$$f(t_{ij},\alpha_j,\beta)=\frac{\alpha_j}{\beta}e^{-\frac{t_{ij}}{\beta}}\left(1-e^{-\frac{t_{ij}}{\beta}}\right)^{\alpha_j-1};\ \alpha_j>0,\beta>0,t_{ij}>0 \tag{7.1}$$

where, the shape papameter $\alpha_j>0,$ and the scale parameter $\beta>0,$ hence denoted by $GE(\alpha,\beta)$. The CDF of generalised Exponential distribution is

$$F(t_{ij},\alpha_j,\lambda)=\left(1-e^{-\frac{t_{ij}}{\beta}}\right)^{\alpha_j};\ \alpha_j>0,\beta>0,t_{ij}>0 \tag{7.2}$$

The survival function is given by

$$S(t_{ij},\alpha_j,\lambda)=1-\left(1-e^{-\frac{t_{ij}}{\beta}}\right)^{\alpha_j} \tag{7.3}$$

The failure rate or hazard rate is given by

$$h(t_{ij},\alpha_j,\lambda)=\frac{\frac{\alpha_j}{\beta}e^{-\frac{t_{ij}}{\beta}}\left(1-e^{-\frac{t_{ij}}{\beta}}\right)^{\alpha_j-1}}{1-\left(1-e^{-\frac{t_{ij}}{\beta}}\right)^{\alpha_j}}$$

The studies have shown that GE distribution very well represents the life of modern products. The hazard function behaves like as of gamma distribution, see Ahmad (2010).

It is also assumed that the stress $V_j,\ j=1,2,...,k$ only affects the shape parameter of the Generalised exponential model, α_j through a life stress model called power rule model given by:

$$\alpha_j=CV_j^{-p};\ \ C>0,\ p>0,\ \ j=1,2,...,k \tag{7.4}$$

where C is the proportionality constant, and p is the power of the applied stress, are the two model parameters.

The Estimation Procedure

The likelihood function of an observation t (failure time of an item) is developed at stress level V_j. Since at each stress level V_j, there are n_j units put under test. Here, the total population experimental units are N= $\sum_{i=1}^{k} n_j$. Applying type-I censoring at each stress level, it can be seen that the once the censoring time "t_0" is reached the experiment automatically terminates. Assume that $r_j \left(\leq n_j\right)$ failures are observed at the jth stress level prior to the termination of the test and $\left(n_j - r_j\right)$ units still survived. Therefore, likelihood function becomes:

$$L\left(\alpha_j, C, \beta\right) = \prod_{j=1}^{k} \frac{n_j}{\left(n_j - r_j\right)!} \left[\prod_{i=1}^{r_j} f\left(t_{ij}; \alpha_j, C, \beta\right)\right] \left[1 - F\left(t_0\right)\right]^{n_j - r_j} \tag{7.5}$$

Where t_0 is the time of cessation of the test.

Using InL to denote the natural logarithm of $L\left(\alpha_j, C, \beta\right)$, then we have

$$InL = K + \sum_{j=1}^{k} r_j \left(InC - In\beta\right) - p\sum_{j=1}^{k} r_j InV_j - \sum_{j=1}^{k}\sum_{i=1}^{r_j} \frac{t_{ij}}{\beta} + \sum_{j=1}^{k}\sum_{i=1}^{r_j} \left(CV_j^{-p} - 1\right) In\left(W\left(t_{ij}\right)\right) + \sum_{j=1}^{k} \left(n_j - r_j\right) In\left[1 - \left(W\left(t_0\right)\right)^{CV_j^{-p}}\right] \tag{7.6}$$

where K is a constant, $W\left(t_{ij}\right) = 1 - \exp\left(\frac{-t_{ij}}{\beta}\right) \& W\left(t_0\right) = 1 - \exp\left(\frac{-t_0}{\beta}\right)$

The first derivative of the logarithm of the likelihood function in equation (7.6) with respect to $\beta, C \,\&\, p$ are obtained as:

$$\frac{\partial InL}{\partial \beta} = -\sum_{j=1}^{k} \frac{r_j}{\beta} + \sum_{j=1}^{k}\sum_{i=1}^{r_j} \frac{t_{ij}}{\beta^2} - \sum_{j=1}^{k}\sum_{i=1}^{r_j} \frac{\left(CV_j^{-p} - 1\right)}{W\left(t_{ij}\right)} \frac{t_{ij}}{\beta^2} \exp\left(\frac{-t_{ij}}{\beta}\right) + \sum_{j=1}^{k} \frac{\left(n_j - r_j\right)}{\varphi\left(t_0\right)} \frac{t_0 CV_j^{-p}}{\beta^2} \left(W\left(t_0\right)\right)^{CV_j^{-p} - 1} \exp\left(-\frac{t_0}{\beta}\right) \tag{7.7}$$

$$\frac{\partial InL}{\partial C}=\sum_{j=1}^{k}\frac{r_j}{C}+\sum_{j=1}^{k}\sum_{i=1}^{r_j}V_j^{-p}InW\left(t_{ij}\right)-\sum_{j=1}^{k}\frac{\left(n_j-r_j\right)}{\varphi\left(t_0\right)}V_j^{-p}\left(W\left(t_0\right)\right)^{CV_j^{-p}}In\left(W\left(t_0\right)\right) \tag{7.8}$$

$$\frac{\partial InL}{\partial p}=\sum_{j=1}^{k}r_jInV_j-\sum_{j=1}^{k}\sum_{i=1}^{r_j}CV_j^{-p}InV_jIn\left(W\left(t_{ij}\right)\right)+\sum_{j=1}^{k}\frac{\left(n_j-r_j\right)}{\varphi\left(t_0\right)}CV_j^{-p}InV_jInW\left(t_0\right)\left(W\left(t_0\right)\right)^{CV_j^{-p}} \tag{7.9}$$

Where, $\phi\left(t_0\right)=1-\left(W\left(t_0\right)\right)^{CV_j^{-p}}$

The ML estimates of β, C and p are obtained by equating the above equations to zero. Also, the variance-covariance matrix is obtained using the fisher information matrix of the form:

$$I=\begin{bmatrix}\frac{-\partial^2 InL}{\partial\beta^2} & \frac{-\partial^2 InL}{\partial C\partial\beta} & \frac{-\partial^2 InL}{\partial p\partial\beta}\\ \frac{-\partial^2 InL}{\partial\beta\partial C} & \frac{-\partial^2 InL}{\partial C^2} & \frac{-\partial^2 InL}{\partial P\partial C}\\ \frac{-\partial^2 InL}{\partial\beta\partial p} & \frac{-\partial^2 InL}{\partial C\partial p} & \frac{-\partial^2 InL}{\partial p^2}\end{bmatrix} \tag{7.10}$$

Where,

$$\frac{\partial^2 InL}{\partial\beta^2}=\sum_{j=1}^{k}\frac{r_j}{\beta^2}-2\sum_{j=1}^{k}\sum_{i=1}^{r_j}\frac{t_{ij}}{\beta^3}-\sum_{j=1}^{k}\sum_{i=1}^{r_j}\left[\left(CV_j^{-p}-1\right)te^{\frac{t_{ij}}{\beta}}\frac{2\beta e^{\frac{t_{ij}}{\beta}}-2\beta-t_{ij}}{\beta^4\left(e^{\frac{t_{ij}}{\beta}}-1\right)^2}\right]+\sum_{j=1}^{k}\left(n_j-r_j\right)t_0CV_j^{-p}$$

$$\left[\frac{\left(W\left(t_0\right)\right)^{CV_j^{-p}}\left\{\left(\left(2V_j^p\beta^3-t_0V_j^p\beta^2\right)\left(W\left(t_0\right)\right)^{CV_j^{-p}}+t_0V_j^p\right)e^{\frac{t_0}{p}}-2V_j^p\beta^3\left(W\left(t_0\right)\right)^{CV_j^{-p}}-Ct_0\right\}}{V_j^p\left\{\beta^2\left(W\left(t_0\right)\right)CV_j^{-p}-1\right\}^2\left(e^{\frac{t}{\beta}}-1\right)^2}\right] \tag{7.11}$$

$$\frac{\partial^2 InL}{\partial C^2}=-\sum_{j=1}^{k}\frac{r_j}{C^2}-\sum_{j=1}^{k}\left(n_j-r_j\right)V_j^{-p}\left(W\left(t_0\right)\right)^{CV_j^{-p}}\left\{In\left(W\left(t_0\right)\right)\right\}^2\left\{\frac{\varphi\left(t_0\right)V_j^{-p}+\left(W\left(t_0\right)\right)^{CV_j^{-p}}V_j^{-p}}{\left(\varphi\left(t_0\right)\right)^2}\right\} \tag{7.12}$$

$$\frac{\partial^2 InL}{\partial p^2}=\sum_{j=1}^{k}\sum_{i=1}^{r_j}CV_j^{-p}\left(InV_j\right)^2InW\left(t_{ij}\right)+$$

$$\sum_{j=1}^{k}\left(n_j-r_j\right)InV_jCV_j^{-2p}InW\left(t_0\right)\left[\frac{\left(W\left(t_0\right)\right)^{CV_j^{-p}}InV_j\left\{\phi\left(t_0\right)V_j^{-p}+CInW\left(t_0\right)\right\}}{\left(\phi\left(t_0\right)\right)^2}\right] \tag{7.13}$$

$$\frac{\partial^2 InL}{\partial\beta\partial C} = -\sum_{j=1}^{k}\sum_{i=1}^{r_j}\frac{t_{ij}}{\beta^2}e^{-\frac{t_{ij}}{\beta}} + \sum_{j=1}^{k}(n_j - r_j)t_0(W(t_0))^{CV_j^{-p}}\left(\frac{CInW(t_0)+V_j^p}{\phi(t_0)\beta^2\left(e^{\frac{t_0}{\beta}}-1\right)}\right) \tag{7.14}$$

$$\frac{\partial^2 InL}{\partial p\partial C} = -\sum_{j=1}^{k}\sum_{i=1}^{r_j}V_j^{-p}InW(t_{ij})InV_j - \sum_{j=1}^{k}(n_j - r_j)InW(t_0)\left[(W(t_0))^{CV_j^{-p}}\left(\frac{CInW(t_0)-\phi(t_0)}{\phi(t_0)V_j^{2p}}\right)\right] \tag{7.15}$$

$$\frac{\partial^2 InL}{\partial\beta\partial p} = \sum_{j=1}^{k}\sum_{i=1}^{r_j}\frac{t_{ij}}{\beta^2}\frac{InV_j}{W(t_{ij})}e^{\frac{-t_{ij}}{\beta^2}} + \sum_{j=1}^{k}t_0\frac{(n_j - r_j)}{V_j^p\beta^2}CV_j^{-p}InV_j(W(t_0))^{CV_j^{-p}}\left[\frac{CInW(t_0)+V_j^p\phi(t_0)}{(\phi(t_0))^2(e^{t_0/\beta}-1)}\right] \tag{7.16}$$

Thus, the approximate $(1-\lambda)100\%$ confidence intervals for $\alpha, C \,\&\, p$ are given by:

$$\hat{\beta} \pm Z_{\lambda/2}\sqrt{\text{var}(\hat{\beta})}, \qquad \hat{C} \pm Z_{\lambda/2}\sqrt{\text{var}(\hat{C})}, \qquad \hat{p} \pm Z_{\lambda/2}\sqrt{\text{var}(\hat{p})},$$

Where $Z_{\lambda/2}$ is the $100(1-\lambda/2)$ percentile of a standard normal variate.

Simulation Procedure

Numerical studies are carried out to check the performances of the ML estimates. The invariance property of MLE is used to estimate the MLEs of shape Parameter β_j through;

$$\alpha_j = CV_j^{-p}; \;\; C > 0, p > 0, \;\; j = 1, 2, ..., k$$

The detailed steps are given below

1) The total of thousand random samples of sizes 50, 100, 150 and 200 are generated from Generalised Exponential distribution.
2) Three different levels of stress, k=3, are chosen as below. $(V_1 = 1, V_2 = 1.5, V_3 = 2), n_j = \frac{n}{3} \,\&\, r_j = 60\% n_j$.
3) For sample sizes, type-I censored samples are used to estimate the parameters using Newton-Raphson method.
4) The RABs and MSE are tabulated for all sets of (β_0, C_0, p_0).
5) Using the invariance property of MLEs, we calculate the MLEs of the shape parameter α_u at usual stress level $V_u = 0.5$. We also calculate the reliability function for different values of $\beta, C, p \,\&\, t_0$,

$$\hat{R}_u(t_0) = 1 - \left(1 - e^{-\frac{t}{\beta_0}}\right)^{\alpha_0}$$

Also, at each at mission time $(t_0 = 1.5, 1.8 \ \& \ 2.2)$, the MLEs of the reliability function are predicted for all sets of parameters.

Simulation results are summarised in Tables (7.1), (7.2) and (7.3). Tables (7.1) and (7.2) give the estimators, RABs and MSEs. Tables (7.3) give the estimated shape parameter under $V_u = 0.5$. And the reliability function is predicted under $V_u = 0.5$.

Table 7.1: *The Estimates, Relative Bias and MSE of the parameters* $(\beta, C, P, \alpha_1, \alpha_2, \alpha_3)$ *under type-II censoring*

n	Parameters	$(\beta_0 = 0.25, C_0 = 1.5, p_0 = 1)$			$(\beta_0 = 1, C_0 = 1.5, p_0 = 1)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
50	β	0.229	0.084	0.078	0.930	0.070	0.076
	C	1.411	0.059	0.044	1.421	0.052	0.063
	P	0.930	0.070	0.062	0.925	0.075	0.071
	α_1	1.411	0.059	0.044	1.421	0.052	0.063
	α_2	0.967	0.057	0.042	0.976	0.050	0.062
	α_3	0.740	0.056	0.042	0.748	0.049	0.059
100	β	0.234	0.064	0.064	0.947	0.053	0.056
	C	1.429	0.047	0.035	1.436	0.042	0.048
	P	0.933	0.067	0.071	1.098	0.098	0.077
	α_1	1.429	0.047	0.035	1.436	0.042	0.048
	α_2	0.955	0.045	0.034	0.920	0.040	0.046
	α_3	0.748	0.044	0.033	0.670	0.039	0.045
150	β	0.239	0.044	0.042	0.951	0.049	0.051
	C	1.449	0.034	0.034	1.561	0.040	0.041
	P	0.946	0.054	0.046	1.052	0.052	0.046
	α_1	1.449	0.034	0.034	1.561	0.040	0.041
	α_2	0.987	0.033	0.033	1.018	0.039	0.040
	α_3	0.752	0.032	0.032	0.752	0.038	0.039

200	β	0.252	0.008	0.010	0.976	0.024	0.030
	C	1.480	0.013	0.003	1.467	0.022	0.023
	P	1.113	0.113	0.010	0.955	0.045	0.031
	α_1	1.480	0.013	0.003	1.467	0.022	0.023
	α_2	0.942	0.012	0.002	0.996	0.021	0.023
	α_3	0.684	0.012	0.003	0.756	0.021	0.022

Table 7.2: *The Estimates, Relative Bias and MSE of the parameters* $(\beta, C, P, \alpha_1, \alpha_2, \alpha_3)$ *under type-II censoring*

N	parameters	$(\beta_0 = 0.25, C_0 = 1, p_0 = 1)$			$(\beta_0 = 1, C_0 = 1, p_0 = 1.5)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
50	β	0.231	0.076	0.081	0.928	0.072	0.067
	C	1.103	0.103	0.047	1.071	0.071	0.067
	P	0.929	0.071	0.058	1.421	0.052	0.063
	α_1	1.103	0.103	0.047	1.071	0.071	0.067
	α_2	0.756	0.074	0.046	0.601	0.069	0.065
	α_3	0.579	0.072	0.045	0.400	0.068	0.064
100	β	0.237	0.052	0.057	0.939	0.061	0.055
	C	1.098	0.098	0.042	1.056	0.044	0.050
	P	0.937	0.063	0.075	1.436	0.042	0.065
	α_1	1.098	0.098	0.042	1.056	0.044	0.050
	α_2	0.750	0.095	0.040	0.590	0.043	0.049
	α_3	0.573	0.093	0.039	0.390	0.043	0.048
150	β	0.239	0.044	0.045	0.948	0.052	0.042
	C	1.049	0.049	0.033	1.041	0.041	0.036
	P	0.958	0.042	0.040	1.561	0.040	0.056
	α_1	1.049	0.049	0.033	1.041	0.041	0.036
	α_2	0.711	0.048	0.032	0.552	0.040	0.035
	α_3	0.539	0.047	0.032	0.352	0.039	0.034

200	β	0.252	0.008	0.017	0.981	0.019	0.036
	C	1.060	0.060	0.023	1.035	0.025	0.031
	P	1.013	0.013	0.021	1.467	0.022	0.025
	α_1	1.060	0.060	0.023	1.035	0.025	0.031
	α_2	0.702	0.060	0.022	0.570	0.024	0.030
	α_3	0.525	0.059	0.022	0.374	0.024	0.030

Table 7.3: *The estimated shape parameter and Reliability function at normal stress level taking n=200.*

β_0	C_0	P_0	α_0	t_0	$R_u(t_0)$
0.25	1.5	1	3.201165	0.2 0.4 0.6	0.8518 0.5141 0.2624
1	1.5	1	2.843896	0.2 0.4 0.6	0.9922 0.9573 0.8959
0.25	1	1	2.139189	0.2 0.4 0.6	0.7208 0.3826 0.1840
1	1	1.5	2.861221	0.2 0.4 0.6	0.9924 0.9581 0.8974

The Replacement Policy with the prospective of Pro-Rata Rebate Warranty Scheme

This warranty policy is applicable on the non-repairable products. The product is replaced upon failure or at a certain time age (τ), which among the two occurs first. Upon failure at $t \le \tau$, a failure replacement is performed with $Cd > 0$ (downtime cost) and $Cp > 0$ (purchasing cost). The customer is refunded a proportion of sales price Cp if the defect/failure occurs in the warranty period (w). The rebate function is given by:

$$R(t) = \begin{cases} Cp\left(1 - \dfrac{t}{w}\right) & 0 \le t \le w \\ 0 & t > w \end{cases} \qquad (7.17)$$

There is some literature available on age-replacement policy, e.g. Chien and Chen (2007a), Chien and Chen (2007b), Huang et al. (2008), Chien (2010), Chien at al. (2014), Na and Sheng

(2014), have used the different warranty policies and observed their effects under both producer and consumer perspective. The current study dealt with estimating the expected total cost and expected cost rate for age replacement of units under warranty policy. The pro-rata rebate warranty policy has also been taken into consideration. It is assumed that there is no salvage value for the preventively replaced product. The preventive replacement is carried out with cost Cp at the product age τ.

Therefore, the total cost incurred in a renewal cycle is:

$$C(d)=\begin{cases} Cd+Cp-R(t) & 0\le t\le w \\ Cd+Cp & w<t<\tau \\ Cp & t\ge\tau \end{cases} \tag{7.18}$$

According to Chein (2010) and Chein et.al (2014), the expected total cost function under this policy is:

$$E(C(t))=CdF(\tau)+Cp\frac{\int_0^w \bar{F}(u)du}{w} \tag{7.19}$$

And, the expected cost rate is

$$E(CR(t))=\frac{E(C(t))}{\int_0^\tau \bar{F}(u)du} \tag{7.20}$$

Where $\int_0^\tau \bar{F}(u)du$ is the expected cycle time which is denoted by $E(T(\tau))$.

Under Generalised Exponential distribution:

We have

$$F(u)=\left(1-e^{-\frac{u}{\beta}}\right)^{\alpha};\ u>0,\alpha,\beta>0 \tag{7.21}$$

Therefore z

Also,

$$\int_0^\tau \bar{F}(u)du=\tau-\int_0^\tau\left(1-e^{-\frac{u}{\beta}}\right)^{\alpha}du \tag{7.23}$$

Substituting equations (7.22) and (7.23) into equations (7.19) and (7.20), respectively, we obtain the expected total cost and expected cost rate for the non-repairable product. It can be seen that the function defined in the above equations does not have the elementary integral. Therefore, a numerical approximation can be obtained by substituting the values of all the parameters involved, except for the variable of integration.

Now for an application example, if the item is replaced with downtime cost $Cd = 50$ and purchasing cost $Cp = 1000$. The expected total cost, expected cost rate, and expected cycle time are estimated. Also, the estimated values of the parameters of generalised exponential distribution α and β are obtained under normal conditions as shown in table 7.4.

Table 7.4: *The expected total cost, the expected cycle time and the expected cost rate for age-replacement under warranty policy on Generalised Exponential distribution.*

B	A	w	τ	E(C(τ))	E(T(τ))	CR(τ)
2	0.2	5	7	930.1232	5.7921	230.885
3	0.2	5	7	944.1763	6.4909	215.4057
4	0.2	5	7	955.8282	6.9264	199.7983
5	0.2	5	7	973.2882	7.3589	174.5714
5	0.3	5	7	882.6932	6.0825	183.0282
5	0.4	5	7	830.1848	5.2488	204.6751
5	0.4	6	7	903.7913	6.2867	208.0381
5	0.4	7	7	947.5114	7.5112	218.3333
5	0.4	8	8	988.8976	8.0751	203.3333
5	0.4	8	9	1012.512	8.4758	200.4356
5	0.4	8	10	1050.812	8.8752	216.6667

Results and Conclusion

In the table (7.1) and (7.2), it can observe that the modules of the difference between the true value of the parameter and its estimator converges to zero, hence consistent. From table (7.3), it can be noticed that the reliability function decreases as the mission time t_0 increases. It is

obvious that whenever a product is tested for a long period of time, its reliability decreases because of the wear out in the product. The studies show that the preventive replacement will be strongly affected under PRRW. Particularly, when the product is proven to failures, and adding the PRRW will extend the optimal replacement age closer to the warranty period.

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Chapter 8:

Gamma Rayleigh Distribution: Properties and Application

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Additional information is available at the end of the chapter

Introduction

In recent years generating new distributions to analyse different life time data has received considerable attention. A number of methods are available in literature that can be used to generalise the existing models to make them more flexible. Among various diverse generalizing methods available, the generalization of our interest is T-X family of distribution by Alzetreeh *et al* (2012). . Let $r(t)$ be the PDF of a non-negative continuous random variable T defined on $[0, \infty)$, and let $F(x)$ denote the CDF of a random variable X. Then the CDF for the T-X family of distributions for random variable X is

$$G(x) = \int_0^{-\log(1-F(x))} r(t)\,dt \quad . \tag{8.1}$$

And the corresponding PDF is given by

$$g(x) = \frac{f(x)}{1-F(x)} r\{-\log(1-F(x))\} \quad . \tag{8.2}$$

Let T follow gamma distribution with PDF

$$r(t) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} e^{-\frac{t}{\beta}} t^{\alpha-1} \quad ; \alpha, \beta > 0, t \geq 0 \ . \tag{8.3}$$

Using (8.3) in (8.1) we obtain the CDF of Gamma-X family of distribution given as

$$G(x) = \frac{\gamma(\alpha, -\log(1-F(x))/\beta)}{\Gamma(\alpha)} \quad . \tag{8.4}$$

Where $\gamma(\alpha, x) = \int_0^x z^{\alpha-1} e^{-z} dz$ is the incomplete gamma function.

The corresponding PDF of Gamma-X family is given by

$$G(x)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\ f(x)\{-\log(1-F(x))\}^{\alpha-1}(1-F(x))^{\frac{1}{\beta}-1} \tag{8.5}$$

A number of distributions have been developed using gamma-X generalization. A few among them are: Gamma-Pareto Distribution and Its Applications by Alzaatreh *et al.* (2014), the gamma-normal distribution: Properties and applications by Alzaatreh *et al.* (2012) etc.

In this context we propose an extension of Rayleigh distribution known as *Gamma- Rayleigh distribution* (GRD for short) using gamma-X family of distribution in order to make the distribution more flexible to real life data. The outline of this paper is as follows: in section 8.2, the PDF and CDF of proposed distribution i.e., GRD is derived. Various statistical properties of the distribution such as moments, moment generating function, mode etc. are discussed in section 8.3. The reliability measures of the distribution are discussed in section 8.4. The expressions for different information measures of the distribution is obtained in section 8.5. In section 8.6, expressions for mean deviation and median are derived. The parameter estimation of the parameters of the distribution is discussed in section 8.7. In section 8.8 the application of the proposed model is debated using real life examples and finally some conclusions and discussions are given at the end.

Derivation of GRD

The cumulative distribution function (CDF) of Rayleigh distribution is given by

$$F(x)=1-e^{-\frac{x^2}{2\theta^2}} \tag{8.6}$$

The probability density function (PDF) of Rayleigh distribution is given by

$$f(x)=\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\ ;\ \theta>0, x>0 \tag{8.7}$$

Using (8.6) in (8.4) we get, the CDF of GRD given by

$$G(x)=\frac{\gamma\left(\alpha,\frac{x^2}{2\theta^2\beta}\right)}{\Gamma(\alpha)} \tag{8.8}$$

and the corresponding PDF of GRD is given by

$$g(x)=\frac{x}{\theta^2\Gamma(\alpha)\beta^{\alpha}}e^{-x^2/2\beta\theta^2}\left(\frac{x^2}{2\theta^2}\right)^{\alpha-1};\alpha,\beta,\theta>0, x>0 \tag{8.9}$$

If we put $\alpha=1$ and $\beta=1$ in (8.9) we get the PDF of Rayleigh distribution.

Statistical Properties and Reliability Measures

In this section, the basic statistical properties of the proposed distribution are investigated.

Moments

The rth moment about origin can be obtained as

$$\mu_r' = \int_0^\infty x^r \, g(x)\, dx$$

$$= \int_0^\infty x^r \frac{x}{\theta^2 \Gamma(\alpha) \beta^\alpha} e^{-x^2/2\beta\theta^2} \left(\frac{x^2}{2\theta^2} \right)^{\alpha-1} dx$$

$$= \frac{(2\theta^2\beta)^{r/2} \Gamma\left(\alpha + \frac{r}{2}\right)}{\Gamma(\alpha)} \tag{8.10}$$

Where $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz$ is the gamma function.

Putting r=1, 2, 3, 4 in (8.10) we can obtain first four moments about origin.

Mean and Variance of GRD

The mean and variance of the GRD is given as

$$\text{Mean} = \frac{(2\theta^2\beta)^{1/2} \Gamma\left(\alpha + \frac{1}{2}\right)}{\Gamma(\alpha)} \quad \text{and variance} = \frac{(2\theta^2\beta)}{\Gamma(\alpha)} \left[\Gamma(\alpha+1) - \frac{\left(\Gamma\left(\alpha + \frac{1}{2}\right)\right)^2}{\Gamma(\alpha)} \right]$$

Moment Generating function

The moment generating function of the GRD can be derived as

$$M_X(t) = E\left(e^{tx}\right)$$

$$M_X(t) = \int_0^\infty e^{tx} g(x) dx$$

$$M_X(t) = \int_0^\infty (1 + tx + \frac{(tx)^2}{2!} + \cdots) f(x)dx$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty x^r g(x)dx$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

Using (10) in above equation, we get

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{(2\theta^2\beta)^{r/2}\Gamma\left(\alpha + \frac{r}{2}\right)}{\Gamma(\alpha)}$$

Mode

The mode of the GRD can be obtained as

$$\frac{\partial}{\partial x} \log g(x) = 0$$

$$\Rightarrow \frac{\partial}{\partial x} \log\left[\frac{x}{\theta^2\Gamma(\alpha)\beta^\alpha} e^{-x^2/2\beta\theta^2} \left(\frac{x^2}{2\theta^2}\right)^{\alpha-1}\right] = 0$$

$$\Rightarrow x = \theta(\beta(2\alpha - 1))^{\frac{1}{2}}$$

Harmonic mean

The harmonic mean of the GRD can be obtained as

$$H..M = \int_0^\infty \frac{1}{x} g(x)\, dx$$

$$= \int_0^\infty \frac{1}{x} \frac{x}{\theta^2\Gamma(\alpha)\beta^\alpha} e^{-x^2/2\beta\theta^2} \left(\frac{x^2}{2\theta^2}\right)^{\alpha-1} dx$$

$$= \frac{\Gamma\left(\alpha - \frac{1}{2}\right)}{(2\theta^2\beta)^{1/2}\Gamma(\alpha)} \; ; \alpha > \frac{1}{2}$$

Reliability analysis of GRD

Survival and Failure Rate Functions

The survival function, hazard rate function and reverse hazard rate function associated with the GRD is given by Eqn. (8.11), (8.12) and (8.13) respectively

$$S(x) = \frac{\Gamma\left(\alpha, x^2 / 2\theta^2\beta\right)}{\Gamma(\alpha)} \tag{8.11}$$

$$h(x) = \frac{x}{\theta^2\beta^\alpha\Gamma\left(\alpha, \frac{x^2}{2\theta^2\beta}\right)} e^{-x^2/2\beta\theta^2}\left(\frac{x^2}{2\theta^2}\right)^{\alpha-1} \tag{8.12}$$

$$\tau(x) = \frac{x}{\theta^2\beta^\alpha\gamma\left(\alpha, \frac{x^2}{2\theta^2\beta}\right)} e^{-x^2/2\beta\theta^2}\left(\frac{x^2}{2\theta^2}\right)^{\alpha-1} \tag{8.13}$$

Mean Residual Time and Mean Waiting Time

The mean residual time is given by

$$\mu(t) = E(T - t \mid T > 0) = \frac{1}{S(t)}\left(E(t) - \int_0^t xg(x)dx\right) - t$$

The mean residual time of GRD is given as

$$\mu(t) = (2\theta^2\beta)^{1/2} \frac{\Gamma\left(\alpha + \frac{1}{2}, \frac{t^2}{2\beta\theta^2}\right)}{\Gamma\left(\alpha, \frac{t^2}{2\beta\theta^2}\right)} - t$$

Also, the mean waiting time which is the waiting time elapsed since the failure of an item given that that this failure has happened in the interval [0, t] is given by

$$\bar{\mu}(t) = t - \frac{1}{F(t)}\int_0^t xg(x)dx$$

The mean waiting time of GRD is given as

$$\bar{\mu}(t) = t - \frac{(2\theta^2\beta)^{1/2}\gamma\left(\alpha + \frac{1}{2}, \frac{t^2}{2\beta\theta^2}\right)}{\gamma\left(\alpha, \frac{t^2}{2\beta\theta^2}\right)}$$

The graphs of PDF, CDF and hazard function for different value of the parameter are given below

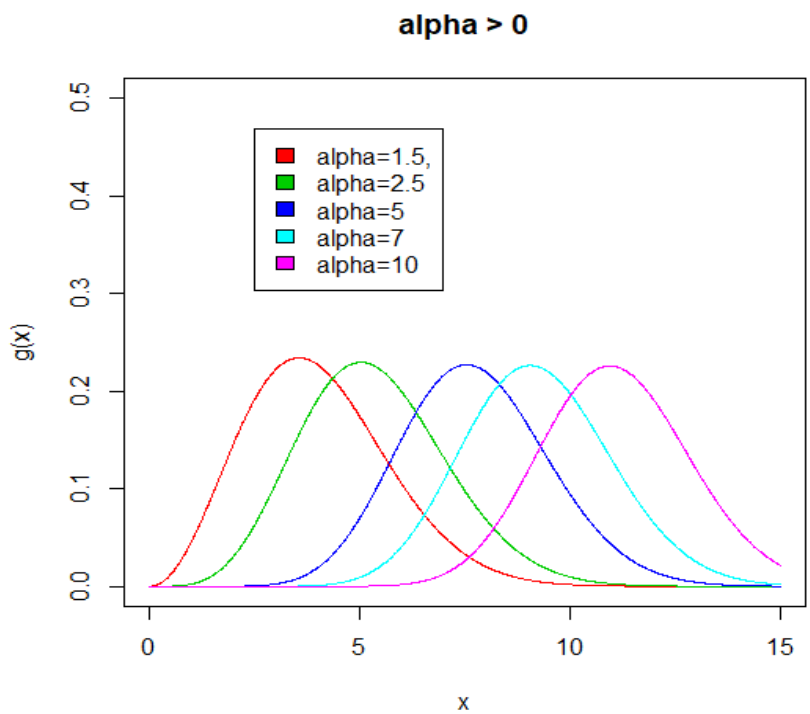


Figure 8.1: *Graph of density function*

Fig 8.1 represents Graphs of Probability density function of Gamma Rayleigh distribution for different values of parameter α when β *and* θ are fixed. Fig 8.2 represents Graphs of Probability density function of Gamma Rayleigh distribution for different values of parameter β when α *and* θ are fixed. Fig 8.3 represents Graphs of Probability density function of Gamma Rayleigh distribution for different values of parameter θ when β *and* α are fixed. Fig 8.4 represents Graphs of Probability density function of Gamma Rayleigh distribution for different values of parameter α , β *and* θ . Fig 8.5 represents Graphs of hazard function of Gamma Rayleigh distribution for different values of parameter α , β *and* θ .

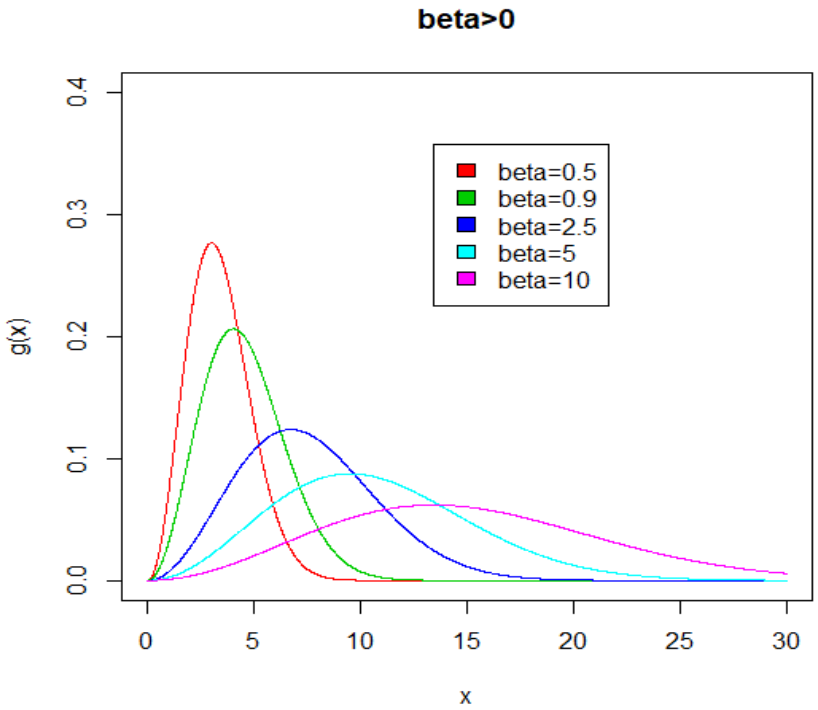


Figure 8.2: *Graph of density function*

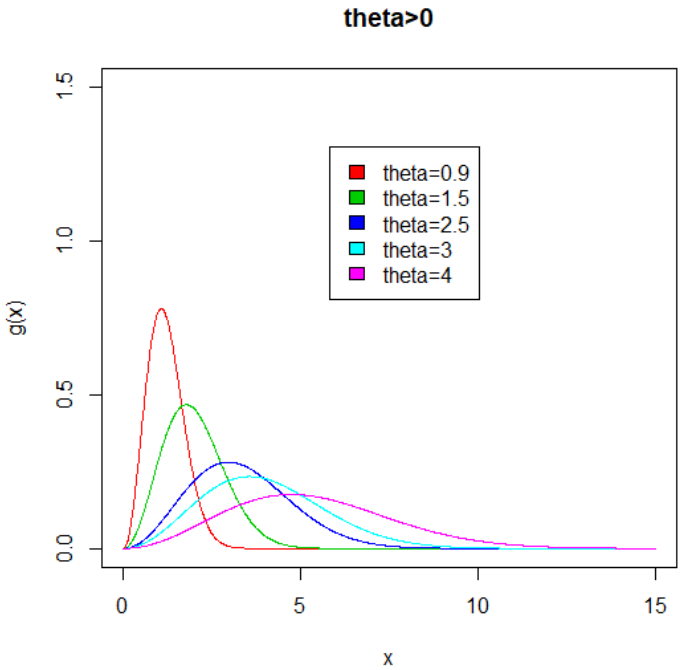


Figure 8.3: *Graph of density function*

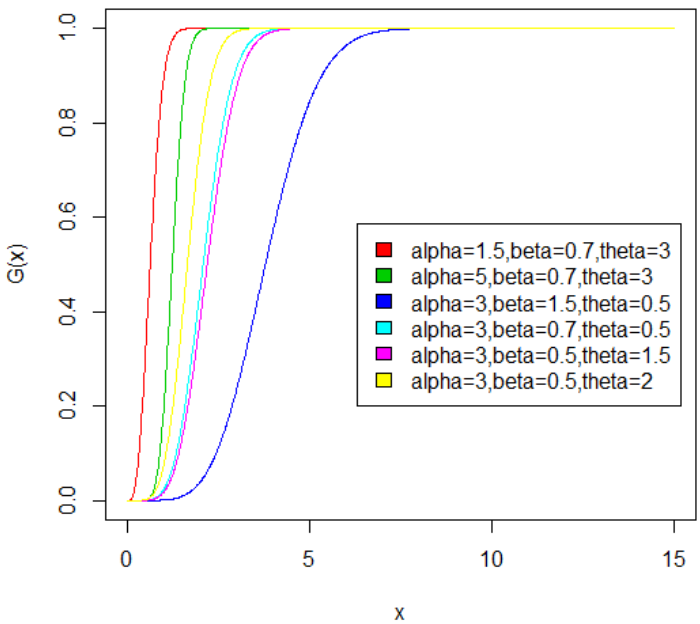


Figure 8.4: *Graph of density function*

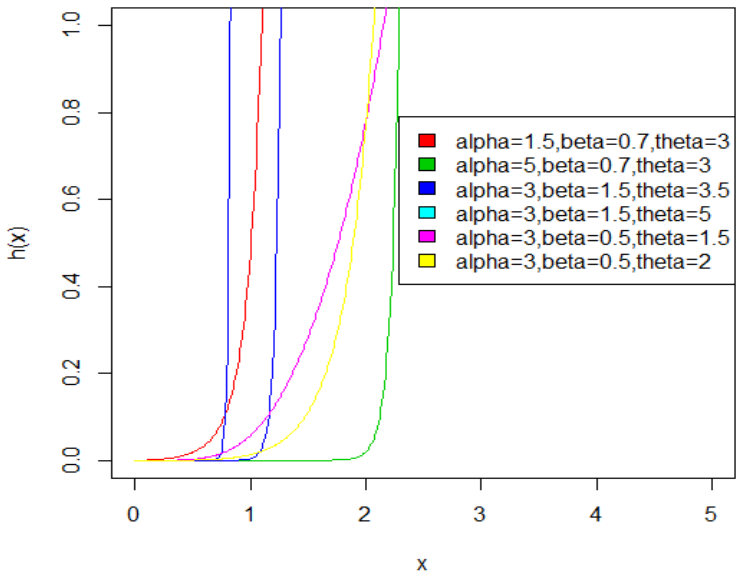


Figure 8.5: *Graph of density function*

Information Measures

Renyi Entropy

The Renyi entropy is denoted by $I_R(\rho)$ and is defined as:

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_{-\infty}^{\infty} f(x)^{\rho}\,dx\right\};\ \rho > 0, \rho \neq 1$$

Therefore, the Renyi entropy for GRD is given as

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\left(\frac{2}{\theta^2\beta}\right)^{\frac{(\rho-1)}{2}} \frac{\Gamma\left(\rho\left(\alpha-\frac{1}{2}\right)+\frac{1}{2}\right)}{(\Gamma(\alpha))^{\rho}\,\rho^{\left(\rho\left(\alpha-\frac{1}{2}\right)+\frac{1}{2}\right)}}\right\};\ \alpha > \frac{1}{2}$$

Shannon Entropy

The Shannon entropy is defined as

$$\eta_X = E[-\log f(x)]$$

The Shannon entropy for gamma –x family is given as

$$\eta_X = -E\left[\log f(F^{-1}(1-e^{-T}))\right] + \alpha(1-\beta) + \log\beta + \log\Gamma(\alpha) + (1-\alpha)\psi(\alpha) \qquad (8.14)$$

Where ψ the digamma function and T the gamma random variable with parameters α and β (see Alzaatreh, *et al.* [3] for proof details).

We have

$$-E(\log f(F^{-1}(1-e^{T}))) = -\frac{1}{2} - \frac{1}{2}(\psi(\alpha) + \log\beta) + \log\theta + \alpha\beta \qquad (8.15)$$

Using (8.15) in (8.14) we get the Shannon entropy of GRD as given below

$$\eta_X = \frac{1}{2}\log\beta + \log\theta + \alpha + \log\Gamma(\alpha) - \frac{1}{2} + (\frac{1}{2} - \alpha)\psi(\alpha)$$

Mean Deviation About Mean and Median

The mean deviation about mean of the GRD can be obtained as

$$D(\mu) = \int_0^{\mu}(\mu - x)g(x)\,dx + \int_{\mu}^{\infty}(x-\mu)g(x)\,dx$$

$$D(\mu) = 2\int_0^{\mu} (\mu - x) g(x)\, dx$$

$$D(\mu) = \frac{2\mu\gamma\left(\alpha, \frac{\mu^2}{2\beta\theta^2}\right)}{\Gamma(\alpha)} - \frac{2^{3/2}(\theta^2\beta)^{1/2}\gamma\left(\alpha + \frac{1}{2}, \frac{\mu^2}{2\beta\theta^2}\right)}{\Gamma(\alpha)}$$

The mean deviation about median of the GRD can be obtained as

$$D(M) = \int_0^{M} (M - x) g(x)\, dx + \int_M^{\infty} (x - M) g(x)\, dx$$

$$D(M) = \mu - 2\int_0^{M} x\, g(x)\, dx$$

$$D(M) = \mu - \frac{2^{3/2}(\theta^2\beta)^{1/2}\gamma\left(\alpha + \frac{1}{2}, \frac{M^2}{2\beta\theta^2}\right)}{\Gamma(\alpha)}$$

Parameter Estimation

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from the GRD then the log likelihood function is given by

$$\log L = \sum_{i=1}^{n} \log x_i - 2n\log\theta - n\log\Gamma(\alpha) - n\alpha\log\beta - \sum_{i=1}^{n}\left(\frac{x_i^2}{2\theta^2\beta}\right) + (\alpha - 1)\sum_{i=1}^{n}\log\left(\frac{x_i^2}{2\theta^2}\right)$$

(8.16)

Taking the derivative of the natural logarithm of the likelihood function (8.16) w.r.t α, β and θ respectively and equation to zero we get the following three equations:

$$\frac{\partial}{\partial\alpha}\log L = -n\psi(\alpha) - n\log\beta + \sum_{i=1}^{n}\log\left(\frac{x_i^2}{2\theta^2}\right) = 0 \tag{8.17}$$

Where $\psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$

$$\frac{\partial}{\partial\beta}\log L = -\frac{n\alpha}{\beta} + \sum_{i=1}^{n}\left(\frac{x_i^2}{2\theta^2\beta^2}\right) = 0 \tag{8.18}$$

$$\frac{\partial}{\partial\theta}\log L = -\frac{2n\alpha}{\theta} + \sum_{i=1}^{n}\left(\frac{x_i^2}{\theta^3\beta}\right) = 0 \tag{8.19}$$

The MLE's of the parameters α, β and θ can be obtained by solving system of equations (8.17), (8.18) and (8.19). Methods such as Newton –Raphson technique can be used to solve these non-linear equations.

Application

In this section, three real life data sets are used to demonstrate the usefulness of GRD. The analysis is performed by using R Software. The distribution that are being used for comparison purpose with the proposed model are

1. Rayleigh distribution (RD) given by Rayleigh (1980) with PDF

$$g(x,\theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \quad ;\theta > 0, x > 0$$

2. Weibull- Rayleigh distribution (WRD) given by Ahmad *et al.* (2017). with PDF

$$g(x,\alpha,\beta,\theta) = \frac{\alpha x^2}{\beta\theta^2}\left(\frac{x^2}{2\beta\theta^2}\right)^{\alpha-1} e^{-\left(\frac{x^2}{2\beta\theta^2}\right)^{\alpha}} ;\alpha,\beta,\theta > 0, x > 0$$

Data set 8.1: This data set were used by Birnbaum and Saunders (1969) and correspond to the fatigue time of 101 6061-T6 aluminium coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps).

Table 8.1: *MLE's estimates, AIC, BIC, AICC for the fitted models to the Data set*

Distribution	**MLE**			**-Log L**	**AIC**	**BIC**	**AICC**
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$				
GRD	11.652	8.459	9.7927	423.75	853.51	861.21	853.71
RD			97.2239	504.21	1010.4	1012.9	1010.6
WRD	3.2083	20.602	22.575	433.88	873.77	881.46	873.97

Data set 8.2: The data are the exceedances of flood peaks (in m3/s) of the Wheaton River near Car cross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place. This data were analysed by Choulakian and Stephens (2001) and are given below

1.7, 2.2 ,14.4 ,1.1 ,0.4, 20.6, 5.3 ,0.7, 13.0, 12.0, 9.3 ,1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1 ,2.5, 14.4, 1.7 37.6 0.6 2.2, 39.0 ,0.3 ,15.0 ,11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5 ,2.5, 27.0, 1.9, 2.8.

Table 8.2: *MLE's estimates, AIC, BIC, AICC for the fitted models to the Data set*

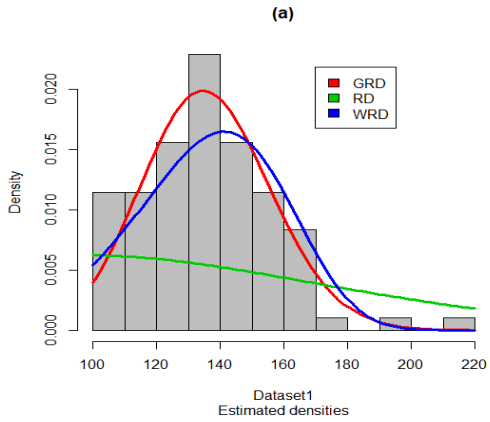
Distribution	MLE			-Log L	AIC	BIC	AICC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$				
GRD	0.324	6.408	8.4712	251.276	508.553	515.383	508.75
RD			12.207	607.675	609.952	609.952	607.87
WRD	0.450	4.091	4.066	251.498	508.997	515.827	509.19

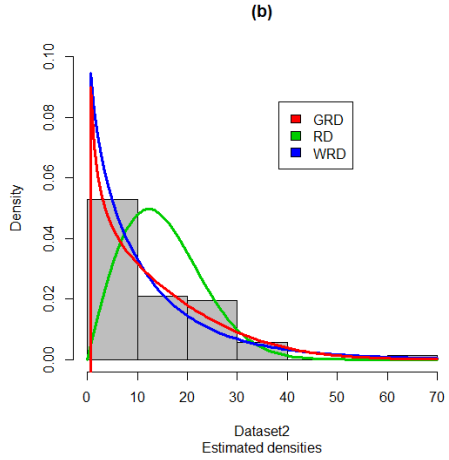
Data set 8.3: The data set represents the lifetime data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975).

1.1,1.4,1.3,1.7,1.9,1.8,1.6,2.2,1.7,2.7,4.1,1.8,1.5,1.2,1.4,3,1.7,2.3,1.6,2.

Table 8.3: *MLE's estimates, AIC, BIC, AICC for the fitted models to the Data set*

Distribution	MLE			-Log L	AIC	BIC	AICC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$				
GRD	2.3428	0.0528	4.0596	19.170	44.340	47.327	44.540
RD			1.4284	22.478	46.957	47.953	47.157
WRD	1.3935	0.2219	3.1969	20.586	47.172	50.160	47.372





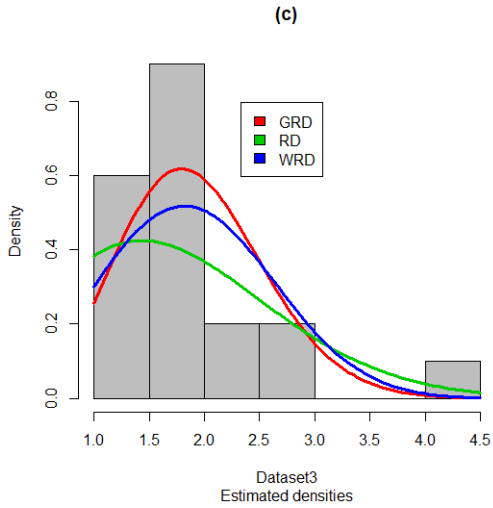


Figure 6: *(a) Plots of the estimated PDF of GRD and other competitive models for Data set 1.*

(b) Plots of the estimated PDF of GRD and other competitive models for Data set 2.

(c) Plots of the estimated PDF of GRD and other competitive models for Data set 3.

Discussion

Since the model with the least value of AIC, AICC, BIC are considered to be best fit, it can be seen from Table 8.1, Table 8.2 and Table 8.3 that GRD has the least value of AIC, AICC and BIC for all the data sets. Hence GRD fits the given data sets quite well as compared other models used for comparison. The histograms of the three data sets and the estimated PDF's of the proposed and competitive models are displayed in Figure 6(a), Figure 6(b) and Figure 6(c).

Conclusion

In this paper we have successfully defined a three parameter Gamma Rayleigh distribution based on T-X family of distribution introduced by Alzetreeh *et al* (2012). Some of the structural

properties including moments, mgf, and harmonic mean are studied. Also the entropy estimation of proposed Distribution is carried out. The parameters involved in the distribution are estimated by maximum likelihood method. The flexibility of this model is illustrated by means of three real life data sets and it is evident that the Gamma Rayleigh distribution provides better fit than Inverse Rayleigh distribution.

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Chapter 9:

A New Optimal Orthogonal Additive Randomized Response Model Based on Moments Ratios of Scrambling Variable

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Additional information is available at the end of the chapter

Introduction

The Randomized response (RR) technique was first presented by Warner (1965) mainly to cut down the probability of (i) reduced response rate and (ii) inflated response bias experienced in direct or open survey relating to sensitive issues. Some recent involvement to randomized response sampling is given by Fox and Tracy (1986), Singh and Mathur (2004, 2005), Gjestvang and Singh (2006), Singh and Tarray (2013, 2014, 2015, 2016) and Tarray and Singh (2016, 2017, 2018) . We below give the description of the model due to Singh (2010):

Singh (2010) Additive Model

Let there be k scrambling variables denoted by S_j , j = 1,2,…,k whose mean θ_j (i.e. $E(S_j) = \theta_j$) and variance γ_j^2 (i.e. $V(S_j) = \gamma_j^2$) are known. In Singh's (2010) proposed optimal new orthogonal additive model named as (POONAM), each respondent selected in the sample is requested to rotate a spinner, as shown in Fig. 9.1, in which the proportion of the k shaded areas, say P_1, P_2, … P_k are orthogonal to the means of the k scrambling variables, say $\theta_1, \theta_2, ..., \theta_k$ such that:

$$\sum_{j=1}^{k} P_j \theta_j = 0 \tag{9.1}$$

and

$$\sum_{j=1}^{k} P_j = 1 \tag{9.2}$$

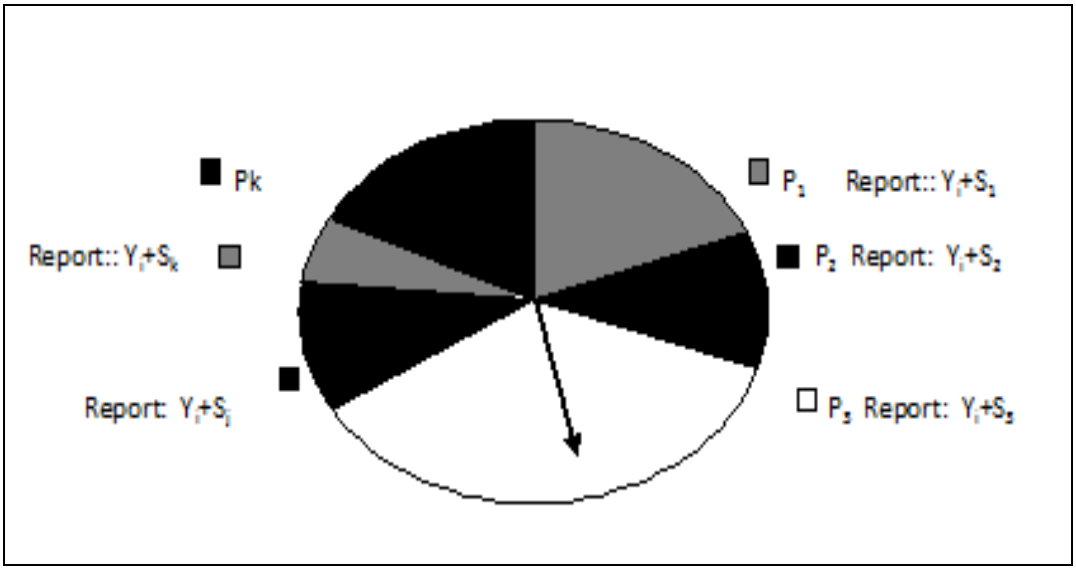

Figure 9.1: *Spinner for POONAM (Singh (2010))*

Now if the pointer stops in the jth shaded area, then the ith respondent with real value of the sensitive variable, say Y_i, is requested report the scrambled response Z_i as:

$$Z_i = Y_i + S_j \tag{9.3}$$

Assuming that the sample of size n is drawn from the population using simple random sampling with replacement (SRSWR). Singh (2010) suggested an unbiased estimator of the population mean μ_Y as

$$\hat{\mu}_Y = \frac{1}{n}\sum_{j=1}^{n} Z_j \tag{9.4}$$

The variance of $\hat{\mu}_Y$ is given by

$$V(\hat{\mu}_Y) = \frac{1}{n}\left[\sigma_y^2 + \sum_{j=1}^{k} P_j(\theta_j^2 + \gamma_j^2)\right] \tag{9.5}$$

The proposed procedure

It is to be noted that the mean θ_j and variance γ_j^2 of the jth scrambling variable S_j (j=1,2,…,k) are known. Author has to propose a new additive model based on standardized scrambling variable $S_j^* = \left(\frac{S_j^2}{\theta_j(1+C_j^2)}\right)$, j = 1,2,…,k .

As demonstrated in Fig. 9.2, in which the proportion of the k shaded areas, say P_1, P_2, … P_k are orthogonal to the means of the k scrambling variables $(S_j^*, \forall\ j = 1,2,...,k)$, say $\theta_1, \theta_2, ..., \theta_k$ such that:

$$\sum_{j=1}^{k} P_j \theta_j = 0 \tag{9.6}$$

and $$\sum_{j=1}^{k} P_j = 1 \tag{9.7}$$

Now if the pointer stops in the jth shaded area, then the ith respondent with real value of the sensitive variable, say Y_i, is requested report the scrambled response Z_i^* as:

$$Z_i^* = Y_i + S_j^*$$

(9.8)

we prove the following theorems.

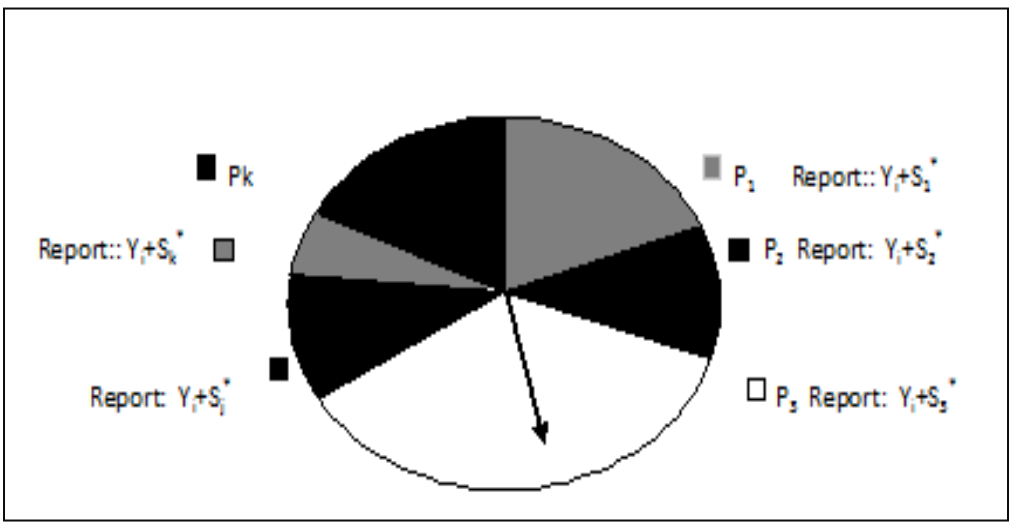

Figure 9.2: *Spinner for proposed procedure.*

Theorem 9.1

$$\hat{\mu}_{ST} = \frac{1}{n}\sum_{i=1}^{n} Z_i^* \tag{9.9}$$

Proof

Let E_1 and E_2 denote the expectations, then we have

$$E(\hat{\mu}_{ST}) = E_1 E_2\left[\frac{1}{n}\sum_{i=1}^{n} Z_i^*\right]$$

$$= E_1\left[\frac{1}{n}\sum_{i=1}^{n} E_2(Z_i^*)\right]$$

$$= E_1\left[\frac{1}{n}\sum_{i=1}^{n}\left\{Y_i\sum_{j=1}^{k} P_j + \sum_{j=1}^{k} P_j E_2(S_j^*)\right\}\right]$$

$$= E_1\left[\frac{1}{n}\sum_{j=1}^{n} Y_i\right] = \mu_Y \quad \text{, since } \sum_{j=1}^{k} P_j = 1 \text{ and } E_2(S_j^*) = \theta_j,$$

which proves the theorem.

Theorem 9.2

$$V(\hat{\mu}_{ST}) = \frac{1}{n}\left[\sigma_y^2 + \sum_{j=1}^{k}\frac{P_j A_j}{(1+C_j^2)^2}\right] \tag{9.10}$$

where

$$A_j = \left[\beta_2(S_j)C_j^4 + 4C_j^3 G_1(S_j) + 6C_j^2 + 1\right],$$

$\beta_2(S_j) = \dfrac{\mu_4(S_j)}{\gamma_j^4}$, $G_1(S_j) = \dfrac{\mu_3(S_j)}{\gamma_j^3}$ is the Fisher's measure of skewness, $\mu_3(S_j)$ and $\mu_4(S_j)$ are third and fourth central moments of the scrambling variable S_j.

Proof

$$V(\hat{\mu}_Y) = E_1 V_2(\hat{\mu}_Y) + V_1 E_2(\hat{\mu}_Y)$$

$$= E_1\left[V_2\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^*\right)\right] + V_1\left[E_2\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^*\right)\right]$$

$$= E_1\left[\frac{1}{n^2}\sum_{i=1}^{n} V_2(Z_i^*)\right] + V_1\left[\frac{1}{n}\sum_{i=1}^{n} E_2(Z_i^*)\right]$$

$$= \frac{1}{n}\sum_{j=1}^{k}\frac{P_j A_j}{(1+C_j^2)^2} + \frac{\sigma_y^2}{n}$$

$$= \frac{1}{n}\left[\sigma_y^2 + \sum_{j=1}^{k}\frac{P_j A_j}{(1+C_j^2)^2}\right]$$

Note that:

$$V_2(Z_i^*) = E_2\left(Z_i^{*2}\right) - \left(E_2\left(Z_i^*\right)\right)^2$$

$$E_2(Z_i^{*2}) = E_2\left(Y_i + S_j^*\right)^2 = E_2\left[Y_i^2 + S_j^{*2} + 2Y_i S_j^*\right]$$

$$= Y_i^2\sum_{j=1}^{k}P_j + \sum_{j=1}^{k}P_j E_2(S_j^{*2}) + 2Y_i\sum_{j=1}^{k}P_j E_2(S_j^*)$$

$$= Y_i^2 + \sum_{j=1}^{k}P_j E_2(S_j^{*2}) + 2Y_i\sum_{j=1}^{k}P_j\theta_j$$

$$= Y_i^2 + \sum_{j=1}^{k}P_j E_2(S_j^{*2}),$$

Since $\sum_{j=1}^{k} P_j\theta_j = 0,$

and $E_2(S_j^{*2}) = E_2\left\{\frac{S_j^4}{\theta_j^2(1+C_j^2)^2}\right\}$

$$= \frac{1}{\theta_j^2(1+C_j^2)^2}E_2(S_j^4)$$

$$= \frac{1}{\theta_j^2(1+C_j^2)^2}E_2\left\{(S_j-\theta_j)^4 + 6\theta_j^2(S_j-\theta_j)^2 + 4\theta_j(S_j-\theta_j)^3 + 4\theta_j^2(S_j-\theta_j)\right\}$$

$$= \frac{1}{\theta_j^2(1+C_j^2)^2}\left\{\mu_4(S_j) + 4\theta_j\mu_3(S_j) + 6\theta_j^2\gamma_j^2 + \theta_j^4\right\}$$

$$= \mu_4(S_j) + 4\theta_j\mu_3(S_j) + 6\theta_j^2\gamma_j^2 + \theta_j^4$$

$$= \frac{\theta_j^4 A_j}{\theta_j^2(1+C_j^2)^2} = \frac{\theta_j^2 A_j}{(1+C_j^2)^2}$$

Thus $E_2(Z_i^{*2}) = Y_i^2 + \sum_{j=1}^{k} \frac{P_j A_j}{(1+C_j^2)^2}$

Therefore $V_2(Z_i^*) = Y_i^2 + \sum_{j=1}^{k} \frac{P_j A_j}{(1+C_j^2)^2} - Y_i^2$

$$= \sum_{j=1}^{k} \frac{P_j A_j}{(1+C_j^2)^2}$$

Efficiency Comparison

From (9.5) and (9.4), we have

$$V(\hat{\mu}_{ST}) < V(\hat{\mu}_Y) \quad \text{if}$$

i.e. if $\frac{1}{n}\left[\sigma_y^2 + \sum_{j=1}^{k} \frac{P_j A_j}{(1+C_j^2)^2}\right] < \frac{1}{n}\left[\sigma_y^2 + \sum_{j=1}^{k} P_j\theta_j^2(1+C_j^2)\right]$

i.e if $\sum_{j=1}^{k} \frac{P_j A_j}{(1+C_j^2)^2} < \sum_{j=1}^{k} P_j\theta_j^2(1+C_j^2)$

i.e if $\sum_{j=1}^{k} P_j\left[\frac{A_j}{(1+C_j^2)^2} - \theta_j^2(1+C_j^2)\right] < 0$

i.e. if $\frac{A_j}{(1+C_j^2)^2} < \theta_j^2(1+C_j^2) \qquad \forall\ j = 1,2,\ldots,k$,

i.e. if $A_j < \theta_j^2(1+C_j^2)^3 \qquad \forall\ j = 1,2,\ldots,k$,

i.e. if $\theta_j^2 > \frac{A_j^2}{(1+C_j^2)^3} \qquad \forall\ j = 1,2,\ldots,k$. (9.11)

In case the scrambling variable S_j follows a normal distribution

(i.e. $S_j \sim N(\theta_j, \gamma_j^2)$, $j = 1,2,\ldots,k$), then A_j reduces to:

$$A_j^* = \left\{1 + 3C_j^2(2 + C_j^2)\right\} \tag{9.12}$$

Thus the condition (9.1) reduces to:

$$\theta_j^2 > \frac{(1 + 6C_j^2 + 3C_j^4)}{(1 + C_j^2)^3} \tag{9.13}$$

The condition (9.3) clearly indicates that $\left\{\theta_j^2 > \frac{(1 + 6C_j^2 + 3C_j^4)}{(1 + C_j^2)^3}, \forall\ j = 1,2,\ldots,k\right\}$ then the proposed model is always better.

Further , suppose $S_j \sim N(\theta_j, \gamma_j^2)$, $\forall\ j = 1,2,\ldots,k$, $\theta = 0$ and $\theta_j = 0\ \forall\ j = 1,2,\ldots,k$, then the variance expression in (9.5) and (9.4) respectively reduce to:

$$V(\hat{\mu}_Y) = \frac{1}{n}\left[\sigma_y^2 + \sum_{j=1}^{k} P_j \gamma_j^2\right] \tag{9.14}$$

and

$$V(\hat{\mu}_{ST}) = \frac{1}{n}\left[\sigma_y^2 + 3\right] \tag{9.15}$$

From (9.4) and (9.5) we have

$$V(\hat{\mu}_{ST}) - V(\hat{\mu}_Y) = \frac{1}{n}\left(\sum_{j=1}^{k} P_j \gamma_j^2 - 3\right)$$

$$= \frac{1}{n}\sum_{j=1}^{k} P_j\left(\gamma_j^2 - 3\right) \tag{9.16}$$

which is always positive if

$$\left(\gamma_j^2 - 3\right) > 0 \qquad \forall\ j = 1,2,\ldots,k$$

i.e. if $\gamma_j^2 > 3 \qquad \forall\ j = 1,2,\ldots,k$ (9.17)

Thus when $S_j \sim N(0, \gamma_j^2)$, $\forall\ j = 1,2,\ldots,k$, $\hat{\mu}_{ST}$ is more efficient as long as the condition (9.7) is satisfied.

In case S_j follows a normal distribution (i.e. $S_j \sim N(\theta_j, \gamma_j^2)$, $\forall\ j = 1,2,\ldots,k$), PRE of $\hat{\mu}_{ST}$ with $\hat{\mu}_Y$ by using the formula:

$$PRE(\hat{\mu}_{ST}, \hat{\mu}_Y) = \frac{\left[\sigma_y^2 + \sum_{j=1}^{k} P_j \left\{(\theta_j^2 + \gamma_j^2)\right\}\right]}{\left[\sigma_y^2 + \sum_{j=1}^{k} \frac{P_j A_j^*}{(1+C_j^2)^2}\right]} \times 100 \tag{9.18}$$

where A_j^* is given in (9.2).

Suppose γ=40, γ_1= 30, γ_2= 40, γ_3 = 20, γ_4= 10, P_1=0.02, P_2=0.05, P_3=0.06, P_4=0.87 with k = 4 . $\sigma_y^2, \theta_1, \theta_2, \theta_3$ and θ_4 as listed in Table 9.1.

Table 9.1: $PRE(\hat{\mu}_{ST}, \hat{\mu}_Y)$

σ_Y^2	θ_1	θ_2	θ_3	θ_4	PRE
	300	200	100	-25.20	18523.16
	800	700	600	-100.00	242264.29
	1300	1200	1100	-174.70	732808.85
25	1800	1700	1600	-249.40	1490172.55
	300	200	100	-25.20	4130.07
	800	700	600	-100.00	53073.44
	1300	1200	1100	-174.70	160380.06
125	1800	1700	1600	-249.40	326053.37
	300	200	100	-25.20	2362.49
	800	700	600	-100.00	29839.47
	1300	1200	1100	-174.70	90081.79
225	1800	1700	1600	-249.40	183091.37
	300	200	100	-25.20	1672.71
325	800	700	600	-100.00	20772.56

	1300	1200	1100	-174.70	62648.32
	1800	1700	1600	-249.40	127301.32
	300	200	100	-25.20	1305.25
	800	700	600	-100.00	15942.52
	1300	1200	1100	-174.70	48034.22
425	1800	1700	1600	-249.40	97581.38
	300	200	100	-25.20	1076.99
	800	700	600	-100.00	12942.05
	1300	1200	1100	-174.70	38955.77
525	1800	1700	1600	-249.40	79119.00
	300	200	100	-25.20	921.41
	800	700	600	-100.00	10897.13
	1300	1200	1100	-174.70	32768.55
625	1800	1700	1600	-249.40	66536.36
	300	200	100	-25.20	808.58
	800	700	600	-100.00	9414.01
	1300	1200	1100	-174.70	28281.11
725	1800	1700	1600	-249.40	57410.48
	300	200	100	-25.20	723.01
	800	700	600	-100.00	8289.13
	1300	1200	1100	-174.70	24877.59
825	1800	1700	1600	-249.40	50488.93

From Table 9.1 $PRE(\hat{\mu}_{ST}, \hat{\mu}_Y)$ are greater than 100. It shows $\hat{\mu}_{ST}$ is more efficient than $\hat{\mu}_Y$ with substantial gain. Thus, the estimator $\hat{\mu}_{ST}$ over $\hat{\mu}_Y$ is recommended.

Table 9.2: *PRE of* $\hat{\mu}_{ST}$ *ove* $\hat{\mu}_Y$.

σ_y^2	25	125	225	325	425	525	625	725	825
PRE	835.71	260.94	190.35	162.80	148.13	139.02	132.80	128.30	124.88

The minimum values from 9.2 is observed as 124.88 and maximum 835.71 with a median of 148.13.

Table 9.2 PRE remains higher if the value of σ_y^2 is small. In that case the value of σ_y^2 will be around 0.5 to 5.0 (see Singh (2010), p. 67). It is observed that the PRE value decreases from 5985.71 to 2675.00 as the value of σ_y^2 increases from 0.5 to 5.0 .

Case k = 2 and the $PRE(\hat{\mu}_{ST}, \hat{\mu}_Y)$ for different parameters. Results are shown in Table 9.3.

Thus, the estimator $\hat{\mu}_{ST}$ over the estimator $\hat{\mu}_Y$ is recommended.

Table 9.3: *PRE of the estimator* $\hat{\mu}_{ST}$ *over the estimator* $\hat{\mu}_Y$ *with k =2.*

P_1	θ_1	θ_2	σ_Y^2	PRE
			25	1514232.14
			125	331316.41
			225	186046.05
			325	129355.18
			425	99155.37
			525	80394.89
			625	67609.08
			725	58335.85
0.2	1300	-325.0	825	51302.54
			25	219089.29
			125	48003.91
			225	26993.42
			325	18794.21
0.4	300	-200.0	425	14426.40

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			525	11713.07
			625	9863.85
			725	8522.66
			825	7505.43
			25	1528536.91
			125	334445.57
			225	187802.78
			325	130576.32
			425	100091.20
			525	81153.47
			625	68246.87
			725	58886.03
0.4	800	-533.3	825	51786.27
			25	1289517.86
			125	282160.16
			225	158449.56
			325	110172.26
			425	84454.44
			525	68478.22
			625	57589.97
			725	49692.99
0.8	300	-1200.0	825	43703.50

Conclusion

This paper elucidates amelioration over the Singh's (2010) randomized response model. We have advocated the optimal orthogonal additive randomized response model. The proposed model is found to be more resourceful both theoretically as well as numerically than the additive randomized response model studied by Singh (2010). Thus, the suggested RR procedure is therefore indorsed for its use in practice as an alternative to Singh's (2010) model.

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Chapter 10:

Bayesian Approximation Techniques for Gompertz Distribution

Humaira Sultan

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Additional information is available at the end of the chapter

Introduction:

The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial tables. Gompertz probability distribution has many useful applications in areas of technology, medical, biological, and natural sciences. The Gompertz distribution was introduced by Gompertz (1825), and many authors have contributed to the statistical methodology and characterization of this distribution. Ismail (2010) discussed Bayes estimation for unknown parameters of Gompertz distribution and acceleration factors under partially accelerated life tests with Type-I censoring. Based on progressive first-failure censoring plans. Soliman et al. (2012) studied Bayes and frequentist estimators for two-parameter Gompertz distribution. Feroze and Aslam (2013) obtained point and interval estimates for the parameters of the two-component mixture of the Gompertz model based on Bayes Method along with posterior predictions for the future value from model. Sarabia et al. (2014) exploded several properties of the Gompertz distribution when lifetime or other kinds of data available fully observed. Prakash (2016) discussed about the Bayes prediction bound length under different censoring plans and statistical inference based on a random scheme under progressive Type-II censored data for Gompertz model. Reyad et al. (2016) introduced a comparative study for the E-Bayesian criteria with three various Bayesian approaches; Bayesian, hierarchical Bayesian and empirical Bayesian.

The probability density function of Gompertz distribution is given by

$$f(x) = \delta e^{x} e^{-\delta(e^{x}-1)} \quad ; \; x > 0; \delta > 0 \tag{10.1}$$

The likelihood function for (10.1) is given by

$$L(x) = \delta^n \prod_{i=1}^{n} e^x e^{-\delta(e^x - 1)}$$

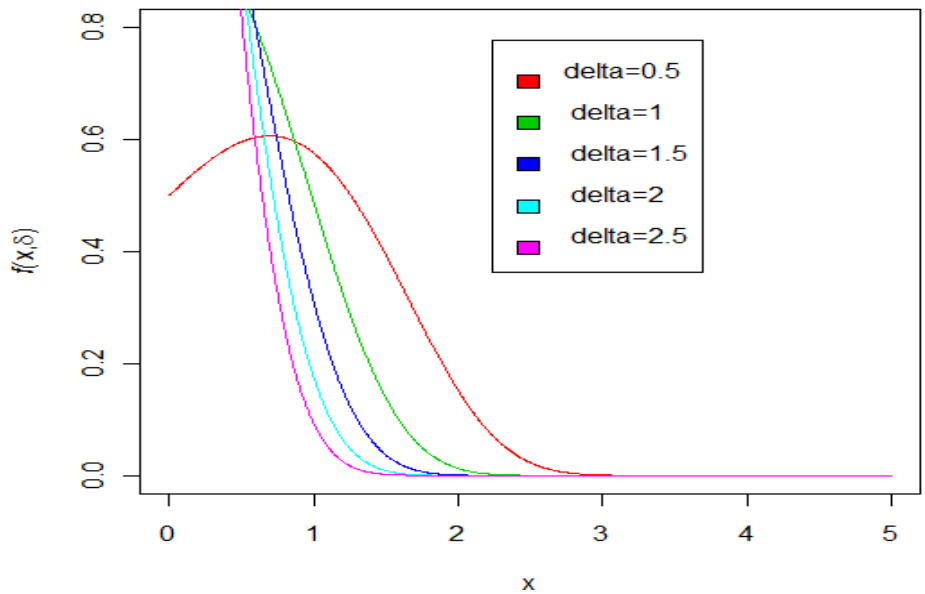

Figure 10.1: *represents probability density function of Gompertz distribution under different values of parameters.*

The Bayesian analysis is theoretically simple and probabilistically elegant. When posterior distribution is expressible in terms of complex analytical function and requires thorough calculation because of its numerical implementations, an approximate and large sample behavior of posterior distribution is studied. This is significant for two reasons: (a) asymptotic results provide valuable first order approximations when actual samples are relatively large, and (b) objective Bayesian methods obviously depend on the asymptotic properties of the assumed model. Thus, our current reading focuses to obtain the estimates of shape parameter of Gompertz distribution using two Bayesian approximation techniques i.e. normal approximation, T-K approximation.

Bayes Estimate of Shape Parameter of Gompertz Distribution using Normal Approximation:

If the posterior distribution $\Psi(\delta \mid x)$ is unimodal and roughly symmetric, it is convenient to approximate it by a normal distribution centered at the mode, yielding the approximation

$$\Psi(\delta \mid x) \sim \hat{N}\left(\hat{\delta}, \left[I(\hat{\delta})\right]^{-1}\right)$$

where $I(\hat{\delta}) = -\dfrac{\partial^2 \log P(\delta \mid y)}{\partial \delta^2}$ (10.2)

If the mode, $\hat{\delta}$ is in the interior parameter space, then $I(\delta)$ is positive; if $\hat{\delta}$ is a vector parameter, then $I(\delta)$ is a matrix. Some good sources on the topic is provided by Sultan et al. (2015).

In our study the normal approximations of Gompertz distribution under different priors is obtained as under:

Under extension of Jeffrey's prior $\varphi(\delta) \propto \left(\dfrac{1}{\delta}\right)^m ; m \in R^+$, the posterior distribution for δ is as

$$\Psi(\delta \mid x) \propto \delta^{n-m} e^{\sum_{i=1}^{n} x_i} e^{-\delta \sum_{i=1}^{n} (e^{x_i}-1)} \tag{10.3}$$

from which the posterior mode is obtained as $\hat{\delta} = \dfrac{n-m}{\sum_{i=1}^{n} (e^{x_i}-1)}$

and $\left[I(\hat{\delta})\right]^{-1} = \dfrac{n-m}{\left[\sum_{i=1}^{n} (e^{x_i}-1)\right]^2}$

Thus, the posterior distribution can be approximated as

$$\Psi(\delta \mid x) \sim N\left(\frac{n-m}{\sum_{i=1}^{n} (e^{x_i}-1)} ; \frac{n-m}{\left[\sum_{i=1}^{n} (e^{x_i}-1)\right]^2}\right)$$

Under the Inverse Levy prior $\varphi(\delta) \propto \delta^{-1/2} e^{-\frac{\delta c}{2}} ; c > 0; \delta > 0$, where c is the known hyper parameter, the posterior distribution for δ is as

$$\Psi(\delta \mid x) \propto \delta^{n-1/2} e^{\sum_{i=1}^{n} x_i} e^{-\delta\left[c/2 + \sum_{i=1}^{n} (e^{x_i}-1)\right]}$$

from which the posterior mode is obtained as $\hat{\delta} = \dfrac{n-1/2}{c/2 + \sum_{i=1}^{n} (e^{x_i}-1)}$

and $\left[I(\hat{\delta})\right]^{-1} = \dfrac{n-1/2}{\left[c/2 + \sum_{i=1}^{n} (e^{x_i}-1)\right]^2}$

Thus, the posterior distribution can be approximated as

$$\Psi(\delta \mid x) \sim N\left(\frac{n-1/2}{c/2+\sum_{i=1}^{n}(e^{x_i}-1)} ; \frac{n-1/2}{\left[c/2+\sum_{i=1}^{n}(e^{x_i}-1)\right]^2} \right)$$

Under gamma prior $\varphi(\delta) \propto \delta^{a-1} e^{-b\lambda}$; $a, b > 0; \delta > 0$ where a, b are the known hyper parameters, the posterior distribution for δ is approximated as

$$\Psi(\delta \mid x) \sim N\left(\frac{n+a-1}{\sum_{i=1}^{n}(e^{x_i}-1)+b} ; \frac{n+a-1}{\left[\sum_{i=1}^{n}(e^{x_i}-1)+b\right]^2} \right)$$

Bayes Estimate of Shape Parameter of Gompertz Distribution using T-K (Laplace) Approximation:

Tierney and Kadane (1986) gave Laplace method to evaluate $E(h(\delta) \mid x)$ as

$$E(h(\delta) \mid x) \cong \frac{\varpi^*}{\varpi} \exp\left\{ n\,h(\hat{\delta}^*) - n\,h(\hat{\delta}) \right\} \tag{10.4}$$

where $n\,h''(\hat{\delta}) = \ln \Psi(\delta \mid x)$; $n\,h''^*(\hat{\delta}^*) = \ln \Psi(\theta \mid x) + \ln h(\delta)$;

$$\hat{\varpi}^2 = -\left[n\,h''(\hat{\delta})\right]^{-1} ; \; \hat{\varpi}^{*2} = -\left[n\,h''^*(\hat{\delta}^*)\right]^{-1}$$

Under extension of Jeffrey's prior $\varphi(\delta) \propto \left(\frac{1}{\delta}\right)^m ; m \in R^+, m \in R^+$ the posterior distribution for δ is given in (10.3)

$$n\,h(\delta) = (n-m)\ln\delta - \delta \sum_{i=1}^{n}(e^{x_i}-1)$$

which implies $\hat{\delta} = \dfrac{n-m}{\sum_{i=1}^{n}(e^{x_i}-1)}$ that maximizes $n\,h(\lambda)$ since $n\,h''(\delta) = -\dfrac{n-m}{\delta^2} < 0$

Similarly $n\,h^*(\delta^*) = n\,h(\delta) + \ln h(\delta) = (n-m+1)\ln\delta - \delta \sum_{i=1}^{n}(e^{x_i}-1)$

From which $\hat{\delta}^* = \dfrac{n-m+1}{\sum_{i=1}^{n}(e^{x_i}-1)}$ that maximizes $nh^*(\delta^*)$ since

$$nh^{*''}(\delta^*) = -\frac{n-m+1}{\delta^2} < 0$$

Thus the maximum of $nh(\delta)$ and $nh^*(\delta^*)$ are given by

$$nh(\hat{\delta}) = \ln\left(\frac{n-m}{\sum_{i=1}^{n}(e^{x_i}-1)}\right)^{n-m} - (n-m) \qquad \&$$

$$nh^*(\hat{\delta}^*) = \ln\left(\frac{n-m+1}{\sum_{i=1}^{n}(e^{x_i}-1)}\right)^{n-m+1} - (n-m+1)$$

respectively.

The estimates of variances are given by

$$\varpi = \left.\frac{\partial^2 nh(\delta)}{\partial\delta^2}\right|_{\delta=\hat{\delta}} = \frac{(n-m)^{1/2}}{\sum_{i=1}^{n}(e^{x_i}-1)}$$

$$\& \qquad \varpi^* = \left.\frac{\partial^2 nh(\delta^*)}{\partial\delta^{*2}}\right|_{\delta^*=\hat{\delta}^*} = \frac{(n-m+1)^{1/2}}{\sum_{i=1}^{n}(e^{x_i}-1)}$$

So we have $E(\delta\,|\,x) \cong \dfrac{\varpi^*}{\varpi}\exp\left\{nh(\hat{\delta}^*) - nh(\hat{\delta})\right\}$

$$= \left(\frac{n-m+1}{\sum_{i=1}^{n}(e^{x_i}-1)}\right)\left(\frac{n-m+1}{n-m}\right)^{n-m+1/2} e^{-1}$$

Note that the relative error to exact the posterior mean $\dfrac{n-m+1}{\sum_{i=1}^{n}(e^{x_i}-1)}$) is

$$\left(\frac{n-m+1}{n-m}\right)^{n-m+1/2} e^{-1}.$$

To determine the second moment, assume $h(\delta) = \delta^2$

$$\therefore \qquad n\,h^{*}(\delta^{*}) = (n-m+2)\ln\delta - \delta\sum_{i=1}^{n}(e^{x_i}-1)$$

From which $E(\delta^2 \mid x) = \left(\dfrac{n-m+2}{\sum_{i=1}^{n}(e^{x_i}-1)}\right)^2 \left(\dfrac{n-m+2}{n-m}\right)^{n-m+1/2} e^{-2}$

Thus, variance is given by

$$\left(\frac{n-m+2}{\sum_{i=1}^{n}(e^{x_i}-1)}\right)^2 \left(\frac{n-m+2}{n-m}\right)^{n-m+1/2} e^{-2} - \left[\left(\frac{n-m+1}{\sum_{i=1}^{n}(e^{x_i}-1)}\right)\left(\frac{n-m+1}{n-m}\right)^{n-m+1/2} e^{-1}\right]^2$$

Under Gamma prior $\varphi(\delta) \propto \delta^{a-1}e^{-b\lambda}$; $a,b>0;\delta>0$

$$E(\delta \mid x) = \left(\frac{n+a-1}{b+\sum_{i=1}^{n}(e^{x_i}-1)}\right)\left(\frac{n+a}{n+a-1}\right)^{n+a+1/2} e^{-1}$$

where the relative error exact to the posterior mean $\left(\dfrac{n+a-1}{b+\sum_{i=1}^{n}(e^{x_i}-1)}\right)$ is $\left(\dfrac{n+a}{n+a-1}\right)^{n+a+1/2} e^{-1}$.

Further $E(\delta^2 \mid x) = \left(\dfrac{(n+a+1)(n+a-1)}{\left[b+\sum_{i=1}^{n}(e^{x_i}-1)\right]^2}\right)\left(\dfrac{n+a}{n+a-1}\right)^{n+a+1/2} e^{-2}$

Thus, variance is given by

$$\left(\frac{(n+a+1)(n+a-1)}{\left[b+\sum_{i=1}^{n}(e^{x_i}-1)\right]^2}\right)\left(\frac{n+a}{n+a-1}\right)^{n+a+1/2} e^{-2} - \left[\left(\frac{n+a-1}{b+\sum_{i=1}^{n}(e^{x_i}-1)}\right)\left(\frac{n+a}{n+a-1}\right)^{n+a+1/2} e^{-1}\right]^2$$

Under inverse levy prior $\varphi(\delta) \propto \delta^{-1/2} e^{-\frac{\delta c}{2}}; c > 0; \delta > 0$

$$E(\delta \mid x) = \left(\frac{n + 1/2}{c/2 + \sum_{i=1}^{n} (e^{x_i} - 1)} \right) \left(\frac{n + 1/2}{n - 1/2} \right)^{n} e^{-1}$$

where the relative error exact to the posterior mean $\left(\frac{n + 1/2}{c/2 + \sum_{i=1}^{n} (e^{x_i} - 1)} \right)$ is $\left(\frac{n + 1/2}{n - 1/2} \right)^{n} e^{-1}$

.

Further $E(\delta^2 \mid x) = \left(\frac{n + 3/2}{c/2 + \sum_{i=1}^{n} (e^{x_i} - 1)} \right)^2 \left(\frac{n + 3/2}{n - 1/2} \right)^{n} e^{-2}$

Thus, variance is given by

$$\left(\frac{n + 3/2}{c/2 + \sum_{i=1}^{n} (e^{x_i} - 1)} \right)^2 \left(\frac{n + 3/2}{n - 1/2} \right)^{n} e^{-2} - \left[\left(\frac{n + 1/2}{c/2 + \sum_{i=1}^{n} (e^{x_i} - 1)} \right) \left(\frac{n + 1/2}{n - 1/2} \right)^{n} e^{-1} \right]^2$$

Real life example 10.1

To examine the applicability of the results, real life data sets are analyzed. The data represents the survival times of 121 patients with breast cancer obtained from a large hospital which is widely reported in some literatures like Ramos et al. (2013)).

0.3,0.3,4.0,5.0,5.6,6.2,6.3,6.6,6.8,7.4,7.5,8.4,8.4,10.3,11.0,11.8,12.2,12.3,13.5,14.4,14.4,14.8,15.5,15.7,16.2,16.3,16.5,16.8,17.2,17.3,17.5,17.9,19.8,20.4,20.9,21.0,21.0,21.1,23.0,23.4,23.6,24.0,24.0,27.9,28.2,29.1,30.0,31.0,31.0,32.0,35.0,35.0,37.0,37.0,37.0,38.0,38.0,38.0,39.0,39.0,40.0,40.0,40.0,41.0,41.0,41.0,42.0,43.0,43.0,43.0,44.0,45.0,45.0,46.0,46.0,47.0,48.0,49.0,51.0,51.0,51.0,52.0,54.0,55.0,56.0,57.0,58.0,59.0,60.0,60.0,60.0,61.0,62.0,65.0,65.0,67.0,67.0,68.0,69.0,78.0,80.0,83.0,88.0,89.0,90.0,93.0,96.0,103.0,105.0,109.0,109.0,111.0,115.0,117.0,125.0,126.0,127.0,129.0,129.0,139.0,154.0.

Real life example 10.2

Consider the data of survival times of 45 gastric cancer patients given chemotherapy and radiation treatment (Bekker et al. 2000).

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.586, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.033

The Bayes estimates and posterior standard error (given in parenthesis) for both the examples under normal and T-K (Laplace) approximation based on non- informative and informative priors have been presented in table 10.1, 10.2

Table 10.1: *Posterior estimates and posterior standard error (in parenthesis) under normal and T-K (Laplace) approximations*

	Jeffrey's prior			**Gamma prior**			**Inverse levy prior**		
	m=0.5	**m=1**	**m=1.5**	**a=b=1**	**a=b=2**	**a=b=3**	**c=1**	**c=2**	**c=3**
$\hat{\delta}_{NA}$	0.0925 (0.0172)	0.0909 (0.0171)	0.0893 (0.0167)	0.0937 (0.0171)	0.0966 (0.0173)	0.0994 (0.0175)	0.09221 (0.0170)	0.0919 (0.0167)	0.0916 (0.0166)
$\hat{\delta}_{LA}$	0.0956 (0.0173)	0.0941 (0.0172)	0.0925 (0.0170)	0.0969 (0.0185)	0.0997 (0.0182)	0.1025 (0.0180)	0.0954 (0.0171)	0.0953 (0.0169)	0.0951 (0.0168)

Table 10.2: *Posterior estimates and posterior standard error (in parenthesis) under normal and T-K (Laplace) approximations:*

	Jeffrey's prior			**Gamma prior**			**Inverse levy prior**		
	m=0.5	**m=1**	**m=1.5**	**a=b=1**	**a=b=2**	**a=b=3**	**c=1**	**c=2**	**c=3**
$\hat{\delta}_{NA}$	0.1336 (0.0309)	0.1321 (0.0299)	0.1306 (0.0198)	0.1347 (0.02008)	0.1372 (0.0202)	0.1398 (0.0204)	0.1332 (0.0200)	0.1328 (0.0199)	0.1324 (0.0188)
$\hat{\delta}_{LA}$	0.1366 (0.02025)	0.1351 (0.02014)	0.1336 (0.02002)	0.1377 (0.02031)	0.1402 (0.02046)	0.1428 (0.02061)	0.1364 (0.02022)	0.1362 (0.02019)	0.1360 (0.02016)

Simulation study

In our simulation study we have generated a sample of sizes n=20, 50, 75 to see the result of small, medium, and large samples on the estimators. The results are simulated 5000 times and the average of the results has been presented in the tables 10.3, 10.4. To inspect the performance of Bayesian estimates for shape parameter of Gompertz distribution under different approximation techniques, estimates are obtainable along with posterior standard error given in parenthesis in the below tables.

Table10.3: *Posterior estimates and posterior standard deviation (in parenthesis) under normal approximation*

n	δ	Jeffrey's prior			Gamma prior			Inverse levy prior		
		m=0.5	m=1.0	m=1.5	a=b=1	a=b=2	a=b=3	c=1	c=2	c=3
20	0.9	0.9271 (0.2099)	0.9032 (0.2072)	0.8795 (0.2044)	0.9076 (0.2029)	0.9116 (0.1989)	0.9153 (0.1951)	0.8849 (0.2004)	0.8465 (0.1917)	0.8113 (0.1837)
	1.5	1.3021 (0.2948)	1.2687 (0.2910)	1.2353 (0.2872)	1.2518 (0.2799)	1.2370 (0.2699)	1.2238 (0.2609)	1.2205 (0.2764)	1.1486 (0.2601)	1.0847 (0.2456)
	2.5	1.8949 (0.4291)	1.8463 (0.4235)	1.7977 (0.4179)	1.7713 (0.3961)	1.7086 (0.3728)	1.6553 (0.3529)	1.7271 (0.3911)	1.5865 (0.3592)	1.4672 (0.3322)
50	0.9	1.1378 (0.1617)	1.1263 (0.1609)	1.1148 (0.1601)	1.1234 (0.1588)	1.1207 (0.1569)	1.1181 (0.1551)	1.1122 (0.1581)	1.0878 (0.1546)	1.0644 (0.1512)
	1.5	1.1956 (0.1699)	1.1835 (0.1690)	1.1541 (0.1682)	1.1792 (0.1667)	1.1751 (0.1645)	1.1711 (0.1624)	1.1674 (0.1659)	1.1405 (0.1621)	1.1148 (0.1584)
	2.5	1.5784 (0.2243)	1.5625 (0.2232)	1.5466 (0.2221)	1.5451 (0.2185)	1.5288 (0.2141)	1.5134 (0.2098)	1.5297 (0.2174)	1.4838 (0.2109)	1.4406 (0.2047)
75	0.9	0.8118 (0.0941)	0.8009 (0.0934)	0.8009 (0.0934)	0.8084 (0.0933)	0.8104 (0.0929)	0.8124 (0.0925)	0.8030 (0.0930)	0.7944 (0.0920)	0.7861 (0.0911)
	1.5	1.1887 (0.1377)	1.1807 (0.1372)	1.1727 (0.1367)	1.1779 (0.1360)	1.1751 (0.1347)	1.1724 (0.1336)	1.1701 (0.1355)	1.1519 (0.1334)	1.1344 (0.1314)
	2.5	1.5473 (0.1792)	1.5369 (0.1786)	1.5265 (0.1781)	1.5259 (0.1762)	1.5155 (0.1738)	1.5054 (0.1715)	1.5158 (0.1756)	1.4855 (0.1721)	1.4565 (0.1687)

Table10.4: *Posterior estimates and posterior standard error (in parenthesis) under T-K approximation*

n	δ	Jeffrey's prior			Gamma prior			Inverse levy prior		
		m=0.5	**m=1**	**m=1.5**	**a=b=1**	**a=b=2**	**a=b=3**	**c=1**	**c=2**	**c=3**
20	**0.9**	0.9748 (0.2152)	0.9510 (0.2126)	0.9272 (0.2099)	0.9532 (0.2079)	0.9552 (0.2036)	0.9571 (0.1995)	0.9521 (0.2102)	0.9305 (0.2054)	0.9099 (0.2009)
	1.5	1.3691 (0.3023)	1.3357 (0.2986)	1.3024 (0.2948)	1.3147 (0.2868)	1.2961 (0.2763)	1.2797 (0.2667)	1.3249 (0.2925)	1.2834 (0.2834)	1.2445 (0.2748)
	2.5	1.9925 (0.4399)	1.9439 (0.4345)	1.8953 (0.4291)	1.8603 (0.4058)	1.7903 (0.3816)	1.7308 (0.3608)	1.9002 (0.4195)	1.8160 (0.4010)	1.7390 (0.3840)
50	**0.9**	1.1608 (0.1633)	1.1493 (0.1625)	1.1378 (0.1617)	1.1427 (0.1786)	1.1427 (0.1584)	1.1397 (0.1565)	1.1476 (0.1614)	1.1347 (0.1596)	1.1221 (0.1579)
	1.5	1.2198 (0.1716)	1.2077 (0.1707)	1.1956 (0.1699)	1.2028 (0.1684)	1.1981 (0.1661)	1.1937 (0.1639)	1.2052 (0.1696)	1.1911 (0.1676)	1.1771 (0.1656)
	2.5	1.6104 (0.2266)	1.5944 (0.2254)	1.5785 (0.2243)	1.5761 (0.2206)	1.5588 (0.2161)	1.5425 (0.2118)	1.5851 (0.2230)	1.5606 (0.2196)	1.5369 (0.2162)
75	**0.9**	0.8227 (0.0946)	0.8172 (0.0943)	0.8118 (0.0941)	0.8192 (0.0939)	0.8211 (0.0935)	0.8230 (0.0931)	0.8182 (0.0941)	0.8138 (0.0936)	0.8094 (0.0931)
	1.5	1.2046 (0.1386)	1.1967 (0.1381)	1.1887 (0.1377)	1.1936 (0.1369)	1.1906 (0.1356)	1.1877 (0.1344)	1.1951 (0.1375)	1.1857 (0.1364)	1.1765 (0.1354)
	2.5	1.5680 (0.1804)	1.5577 (0.1798)	1.5473 (0.1792)	1.5463 (0.1773)	1.5354 (0.1749)	1.5250 (0.1726)	1.5519 (0.1786)	1.5361 (0.1767)	1.5207 (0.1750)

Conclusion

We presented approximate to Bayesian integrals of Gompertz distribution depending upon numerical integration and simulation study and showed how to study posterior distribution by means of simulation study. From the findings of above tables (1, 2, 3, 4) it has been found that the large sample distribution could be improved when prior is taken into account. In all cases (simulated data as well as real life data) normal approximation, T-K approximation, Bayesian estimates under informative priors are better than those under non-informative priors especially the Inverse levy distribution proves to be efficient with minimum posterior standard deviation. Further we accomplish that the posterior standard deviation based on different priors tends to decrease with the increase in sample size. It indicates that the estimators attained are consistent. It can also be detected that the performance of Bayes estimates under informative priors (inverse levy) is better than non-informative prior.

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