Grey Time-delayed Verhulst Model for Forecasting the Market Diffusion of New Products

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Abstract

Taking account of the time-delayed phenomenon in diffusion of new products, we propose the time-delayed Verhulst model and then establish grey time-delayed Verhulst model using the method of greying differential equations. The related parameter packets of the novel model is obtained under the rule of OLS. The results show that traditional grey Verhulst model is a special example of grey time-delayed Verhulst model which can reflect the time-delayed information effectively. An actual example of market diffusion shows the modeling accuracy is remarkably improved by using grey time-delayed Verhulst model.

Keywords: Grey System, Grey Verhulst model, Time-delayed, Market Diffusion

1. Introduction

When a new product entering to the related market, managers want to know the sales and consumer acceptance rate of the new product in the market, because this information is very important for them to make marketing plans, production plans, and even the development strategies of the company. We first introduce two classical models for market diffusion, analyze the pros and cons and then proposed the problem which will be solved in this paper.

Assume that $N_t$ is the cumulative number using a new product. According to Malthus theory, we know the incremental purchase number is proportional to $N_t$. Thus, we have the following model:

$$\frac{dN(t)}{dt} = rN(t)$$

(1)

where $r$ is the growth rate of purchase. If $r > 0$, $N(t)$ obeys the law of exponential growth.

However, the market demand for new products is limited. It is impossible for any product purchase growing as an exponential trend when reaching a certain penetration rate. Considering the restriction of market demand for new products, Malthus model must be improved.

Assume that $K$ is the total number of potential consumers, the purchase growth rate increases with the decrease of $N_t$. When $N(t) \to K$, the purchase growth rate trends to 0. Thus, we have the following improved model:

$$\frac{dN(t)}{dt} = r\left(1 - \frac{N(t)}{K}\right)N(t)$$

(2)

Let $a = -r$, $b = -\frac{r}{K}$, Eq.(2) can be written as

$$\frac{dN(t)}{dt} +aN(t) = b\left(N(t)\right)^2$$

(3)

Eq.(3) is Verhulst model proposed by German mathematician Verhulst in 1937. Raw data is required to be an approximate S-shape for or establishing traditional Verhulst model, or ineffective. Professor Deng greyed the traditional Verhulst model based on the concepts and principles of grey differential equation, and obtained the following grey Verhulst model [1]:

$$x^{(0)}(k) + az^{(1)}(k) = b\left(z^{(1)}(k)\right)^2$$

(4)

The grey Verhulst model, which is a first-order one-variable grey differential equation, is a time series model. The original data vary with time, as do their randomness characteristics. It is one of the basic models of Grey System forecasting and control. Rather than relying on its or iginal data
distribution, Grey modeling is based on the Accumulated Generating method (AGO) that describes the series using grey exponential law. Thus, this model can be solved by constructing a grey differential equation. The simulated values are then derived from the inverse accumulated generating operation (IAGO), as well as forecasted values. This model has the capability to forecast well without large data samples. Also, this modeling procedure is simple to use and has the advantage of making short-term forecasting with a small data set.

Because grey Verhulst model relies on first-order accumulated generating operation (1-AGO) on the original data, the traditional Verhulst model is extended for forecasting data with an approximate mono-peak trend. It can be concluded that the grey Verhulst model excels the traditional Verhulst model in the range of application. Thus, it has been widely used recently [2-7]. The improvement has been made on grey Verhulst model regarding the selection of initial conditions and parameter estimation[8, 9]. The nonlinear grey Bernoulli model (NGBM(1,1)) is a extended model of grey Verhulst model and the GM(1,1) model, and was successfully used in simulating and forecasting values of annual unemployment rates of 10 selected countries [10] and foreign exchange rates of Taiwan’s major trading partners [11]. The power exponent \( n \) in this model can effectively reflect the nonlinearity of real systems and flexibly determine the form of the model. Namely, when \( n = 2 \), NGBM (1,1) devolves to grey Verhulst model. Thus, if the power exponent \( n \) is confirmed with an appropriate method, the forecast will be more precise than that delivered by grey Verhulst model. This indicates that the NGBM(1,1) model has remarkably improved the simulation and forecasting accuracy of the grey Verhulst model. The NGBM(1,1) model has been widely employed in the simulation and forecasting of series having non-linear variations. Zhou [12] selected the proper value of \( n \) by utilizing a particle swarm optimization (PSO) algorithm and used the model for forecasting the power load of the Hubei electric power network. A genetic algorithm-based NGBM (1,1) is used to forecast the output of Taiwan’s integrated circuit industry [13]. CO₂ emissions, energy consumption, and economic growth forecasts by an optimized NGBM (1,1) are demonstrably more precise than those by ARIMA [14]. These improvements have shown better results in both simulation and forecasting.

The question now is: Does the purchase growth rate \( dN(t)/dt \) in Eq.(2) have any relationship with \( N(t) \) at other times besides \( t \)? As we know, the following factors may cause delays: 1. Information dissemination of new products takes time; 2. The process for consumers making decision to buy new products after receiving related information takes time. Therefore, we need to expand grey Verhulst model to a new model which can reflect time-delayed information.

2. The Grey Time-delayed Verhulst Model

2.1. Time-delayed Verhulst model

We first give time-delayed Verhulst model. Assume that the purchase growth rate of a new product has a relationship with the cumulative purchase number at the time interval \( [t - \tau_0 , t] \). Thus, Eq.(2) can be improved as follow:

\[
\frac{dN(t)}{dt} = r \left( 1 - \frac{N(t)}{K} \right) \int_{t-\tau_0}^{t} N(t - \tau) d\tau
\]

(5)

where \( \tau \) is the time-delayed variable, \( 0 \leq \tau \leq \tau_0 \).

It is difficult for us to solve the above complicated differential equation by classical theory. The method of greying differential equations [15] is a feasible choice to find the approximate solution in this situation.

2.2. Greying the time-delayed Verhulst model

According to the concepts and principles of grey differential equation [1] and the method of greying differential equations [15], we can establish grey time-delayed Verhulst model.

Let \( g_\omega \) be a mapping, \( S\{g_\omega\}, i = 1, 2, \cdots \) be a set of grey items in model, \( g_\omega \) be the grey mapping, \( S\{\omega_\omega\}, i = 1, 2, \cdots \) be a set of white items in model, \( \omega_\omega \) be the white mapping.
We call $g_\omega$ the greying mapping of the model, only if

1. $g_\omega : S(a_\omega) \to S(g_i)$;
2. $g_1 : dx/dt (white \ derivative) \to x^{(0)}(k) (grey \ derivative)$,
   $g_2 : x (background \ value) \to z^{(1)}(k) (grey \ background \ set)$
3. $(g_1, g_2)$ is the basic mapping of the white differential equation.

If we let $N(t) = x$ in Eq. (5), Time-delayed Verhulst model can be expressed as following:

$$\frac{dx}{dt} = r \left( 1 - \frac{x}{K} \right) \int_0^\tau x(t-\tau) d\tau \tag{6}$$

Greying Eq.(6), we have

$$x^{(0)}(k) = r \left( 1 - \frac{z^{(1)}(k)}{K} \right) \left( \sum_{\tau=0}^\tau z^{(1)}(k-\tau) \right) \tag{7}$$

where. $k = 1, 2, \ldots, n.$

Removing the brackets, we have

$$x^{(0)}(k) - r \sum_{\tau=0}^\tau z^{(1)}(k-\tau) = -\frac{r}{K} \sum_{\tau=0}^\tau z^{(1)}(k) z^{(1)}(k-\tau) \tag{8}$$

Let $a = -r, b = -\frac{r}{K}$, Eq.(6) can be expressed as follow:

$$x^{(0)}(k) + a \sum_{\tau=0}^\tau z^{(1)}(k-\tau) = b \sum_{\tau=0}^\tau z^{(1)}(k) z^{(1)}(k-\tau) \tag{9}$$

We call Eq.(9) gery time-delayed Verhulst model.

According to the de finition of bac kground o f grey model, we hav e $2 \leq k \leq n, 2 \leq k - \tau \leq n$ and $\tau \geq 0$, thus, $0 \leq \tau \leq n - 2$.

From the expression of the above model, we can see that the time-delayed items are included in gery time-delayed Verhulst model. When the time-delayed parameter $\tau_0 = 0$, Eq.(9) equals traditional gery Verhulst model:

$$x^{(0)}(k) + a \sum_{\tau=0}^\tau z^{(1)}(k-\tau) = b \sum_{\tau=0}^\tau z^{(1)}(k) z^{(1)}(k-\tau) \tag{10}$$

The whitenization equation of the grey Verhulst model is

$$\frac{dx^{(1)}}{dt} + ax^{(1)}(t) = b \left( x^{(1)}(t) \right)^2 \tag{11}$$

and the solution of Eq(11) is given by

$$x^{(1)}(t) = \frac{1}{e^{\omega t}} \left[ \frac{1}{x^{(0)}(0)} \frac{b}{a} \left( 1 - e^{-\omega t} \right) \right] \tag{12}$$

Thus, we have the following time response sequence

$$x^{(1)}(k+1) = \frac{ax^{(1)}(0)}{bx^{(0)}(0) + \left( a - bx^{(1)}(0) \right) e^{\omega \tau}} \tag{13}$$

The Verhulst model is mainly used to describe and to study processes with saturated states (or say sigmoid processes). For example, this model is often used in the prediction of human populations, biological growth, reproduction, economic life span of consumable products, etc. From the solution of the Verhulst equation, it can be seen that when $t \to \infty$, if $a > 0$, then $x^{(1)}(t) \to \infty$; if $a < 0$, then,
\( x^{(i)}(t) \to \frac{a}{b} \). That is, when \( t \) is sufficiently large, for any \( k > t \), \( x^{(i)}(k+1) \) and \( x^{(i)}(k) \) will be sufficiently close. At this time,
\[
x^{(i)}(k+1) = x^{(i)}(k+1) - x^{(i)}(k) > 0.
\]
So, the system approaches extinction.

When resolving practical problems, we often face processes with sigmoid sequences of raw data. In this case, we can take the sequences of the original data as \( x^{(i)} \) and the 1-IAGO sequence as \( x^{(0)} \) to establish a Verhulst model to simulate \( x^{(i)} \) directly.

### 2.3. Parameters identification of grey time-delayed Verhulst model

**Theorem 1.** Let \( x^{(0)}(k) \) be the market diffusion sequence of a new product

\[
x^{(i)}(k) = AGO x^{(0)}(k),
\]

\[
z^{(i)}(k) = MEAN x^{(i)}(k) = 0.5\left(x^{(i)}(k) + x^{(i)}(k-1)\right).
\]

Grey time-delayed Verhulst model of \( x^{(i)}(k) \) is:

\[
x^{(i)}(k) + a \sum_{\tau=0}^{\tau_0} z^{(i)}(k-\tau) = b \sum_{\tau=0}^{\tau_0} z^{(i)}(k) z^{(i)}(k-\tau)
\]

(14)

So the primary parameter package \( P_{IB} \) of grey time-delayed Verhulst model is:

\[
P_{IB} = (a, b)
\]

(15)

\[
a = \frac{GE - CH}{FG - C^2}
\]

(16)

\[
b = \frac{FH - CE}{FG - C^2}
\]

(17)

The secondary parameter package \( P_{IBB} \) of grey time-delayed Verhulst model is:

\[
P_{IBB} = (C, E, F, G, H)
\]

(18)

where

\[
C = \frac{1}{M} \sum_{k=1}^{n} x^{(i)}(k) (M - z^{(i)}(k))^2,
\]

\[
E = \sum_{k=1}^{n} \left( M - z^{(i)}(k) \right) \cdot x^{(i)}(k),
\]

\[
F = \sum_{k=1}^{n} \left( M - z^{(i)}(k) \right)^2,
\]

\[
G = \frac{1}{M} \sum_{k=1}^{n} \left[ z^{(i)}(k) \left( M - z^{(i)}(k) \right) \right]^2,
\]

\[
H = \frac{1}{M} \sum_{k=1}^{n} \left( k - z^{(i)}(k) \right) x^{(i)}(k).
\]

**Proof.** Let \( k = 2, 3, \cdots, n \) in grey time-delayed Verhulst model. For \( 2 \leq k - \tau \leq n \), \( 0 \leq \tau \leq \tau_0 \), thus, \( \tau_0 + 2 \leq k \leq n - \tau_0 \).

We have
\[ x^{(t)}(r_0 + 2) + a \sum_{t=0}^{f_0} x^{(t)}(r_0 - t + 2) = b \sum_{t=0}^{f_0} z^{(t)}(r_0 + 2) z^{(t)}(r_0 - t + 2), \]
\[ x^{(t)}(r_0 + 3) + a \sum_{t=0}^{f_0} x^{(t)}(r_0 - t + 3) = b \sum_{t=0}^{f_0} z^{(t)}(r_0 + 3) z^{(t)}(r_0 - t + 3), \]
\[ \vdots \]
\[ x^{(t)}(k) + a \sum_{t=0}^{f_0} x^{(t)}(k - t) = b \sum_{t=0}^{f_0} z^{(t)}(k) z^{(t)}(k - t), \]
\[ \vdots \]
\[ x^{(t)}(n - r_0) + a \sum_{t=0}^{f_0} x^{(t)}(n - r_0 - t) = b \sum_{t=0}^{f_0} z^{(t)}(n - r_0) z^{(t)}(n - r_0 - t). \]

So
\[ y_N = BP_{1B} \]
where
\[ B = \begin{bmatrix}
\sum_{t=0}^{f_0} z^{(t)}(r_0 - t + 2) & \sum_{t=0}^{f_0} z^{(t)}(r_0 + 2) z^{(t)}(r_0 - t + 2) \\
\sum_{t=0}^{f_0} z^{(t)}(r_0 - t + 3) & \sum_{t=0}^{f_0} z^{(t)}(r_0 + 3) z^{(t)}(r_0 - t + 3) \\
\vdots & \vdots \\
\sum_{t=0}^{f_0} z^{(t)}(n - r_0) & \sum_{t=0}^{f_0} z^{(t)}(n - r_0) z^{(t)}(n - r_0 - t)
\end{bmatrix}, \quad y_N = \begin{bmatrix}
\begin{bmatrix} x^{(t)}(r_0 + 2) \\
x^{(t)}(r_0 + 3) \\
\vdots \\
x^{(t)}(n - r_0)
\end{bmatrix} \\
\begin{bmatrix} x^{(t)}(r_0 + 3) \\
x^{(t)}(r_0 + 3) \\
\vdots \\
x^{(t)}(n - r_0)
\end{bmatrix}
\end{bmatrix}, \quad P_{1B} = \begin{bmatrix} a \\
\end{bmatrix}. \]

Under the rule of OLS, we have
\[ P_{1B} = (B^T B)^{-1} B^T y_N. \]

Thus,
\[ P_{1B} = \begin{bmatrix} a \\
\end{bmatrix}. \]

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3. An actual example of market diffusion

In August, 2002, the opening of "blog-China" (http://www.BLOGchina.com) marks blog is rising in China. As time goes by, people have more in-depth understanding of blog, and more and more people began to use the blog. 2005 is the "The first year of blog popularity", in this year, the number of blog users reached to 900 million. In 2006, the blog is driving on the fast lane and develops steadily, at the end of the year, the number of Chinese blog (referring to an effective blog space) has more than 20 million. In this section, the advantage of the grey time-delayed Verhulst model over the traditional one is demonstrated by the actual example of China’s blog diffusion in [16-18]. The total number of China’s valid blog space from 2002 to 2007 is in Table 1.

<table>
<thead>
<tr>
<th>Year</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blog scale</td>
<td>8</td>
<td>30</td>
<td>150</td>
<td>900</td>
<td>2080</td>
<td>4200</td>
</tr>
</tbody>
</table>


From the table 1, we can get the following sequences:

\[ x^{(0)} = (8, 22, 120, 750, 1180, 2120) \]
\[ x^{(1)} = (8, 30, 150, 900, 2080, 4200) \]
\[ z^{(1)} = (\ldots, 19, 90, 525, 1490, 3140) \]

Let \( T_0 = 1 \). According to Theorem 1, we have

\[ P_{1b} = (a, b)^T = (-0.8342501, -0.00012138)^T \]

The grey time-delayed Verhulst model is

\[ x^{(0)}(k) - 0.8342501(z^{(1)}(k) + z^{(1)}(k - 1)) = -0.00012138(z^{(1)}(k))^2 + z^{(1)}(k)z^{(1)}(k - 1) \]

Take \( e(k)\% \) as the relative absolute error, \( e_{avg} \% \) as the average relative absolute error, \( p_{avg} \% \) as the average precision. The modeling results of China’s valid blog space are showed in the table 2.

<table>
<thead>
<tr>
<th>Year</th>
<th>Actual values</th>
<th>Traditional Grey Verhulst Model</th>
<th>Grey Time-delayed Verhulst Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \hat{x}(k) )</td>
<td>( e(k)% )</td>
</tr>
<tr>
<td>2003</td>
<td>22</td>
<td>19.57</td>
<td>-11.02</td>
</tr>
<tr>
<td>2004</td>
<td>120</td>
<td>47.71</td>
<td>-60.24</td>
</tr>
<tr>
<td>2005</td>
<td>750</td>
<td>115.12</td>
<td>-84.67</td>
</tr>
<tr>
<td>2006</td>
<td>1180</td>
<td>271.34</td>
<td>-77.01</td>
</tr>
<tr>
<td>2007</td>
<td>2120</td>
<td>607.03</td>
<td>-71.37</td>
</tr>
</tbody>
</table>

\[ e_{avg} \% = -- 60.82 \]
\[ p_{avg} \% = -- 39.18 \]
The numbers of blog

Fig. 1. Curves of actual data, traditional grey Verhulst model and grey time-delayed model

It can be seen from Table 2 and Fig. 1, if we directly apply traditional grey Verhulst model to forecast the number of Chinese blogs, the modeling error is big and the predicted values are much smaller than the actual values. The main reason for this is that the traditional model ignores the time-delayed information in the diffusion of new products. Because grey time-delayed Verhulst model takes account of the time-delayed information in modeling, thus it can obtain a much higher accuracy. The average relative absolute error is 18.66% and the average precision is 81.34%. Thus, the modeling result of grey time-delayed Verhulst model is satisfactory.

4. Conclusions

As the information dissemination of new products and the decision making process of consumers taking necessary time, we should add the time-delayed factors to the grey forecasting model to achieve satisfactory accuracy. This paper expands the traditional grey Verhulst model to a novel form containing time-delayed information. Comparatively, grey time-delayed Verhulst model prevails in the availability of acquiring higher modeling accuracy.

5. Acknowledgement

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6. References


