

Ising (Conformal) Fields and Cluster Area Measures

Federico Camia ^{*†}

*Department of Mathematics, Vrije Universiteit Amsterdam
De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands*

Charles M. Newman ^{‡§}

*Courant Inst. of Mathematical Sciences, New York University
251 Mercer Street, New York, NY 10012, USA*

Abstract

We provide a representation for the scaling limit of the $d = 2$ critical Ising magnetization field as a (conformal) random field using SLE (Schramm-Loewner Evolution) clusters and associated renormalized area measures. The renormalized areas are from the scaling limit of the critical FK (Fortuin-Kasteleyn) clusters and the random field is a convergent sum of the area measures with random signs. Extensions to off-critical scaling limits, to $d = 3$ and to Potts models are also considered.

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1 Introduction

The Ising model in $d = 2$ dimensions is perhaps the most studied statistical mechanical model and has a special place in the theory of critical phenomena since the groundbreaking work of Onsager [27]. Its scaling limit at or near the critical point is recognized to give rise to Euclidean (quantum) field theories. In particular, the scaling limit of the lattice magnetization field should be a Euclidean random field and, at the critical point, the simplest reflection-positive conformal field theory Φ^0 [4, 13]. As such, there have been

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†E-mail: fede@few.vu.nl

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§E-mail: newman@courant.nyu.edu

a variety of representations in terms of free fermion fields [33] and explicit formulas for correlation functions (see, e.g., [25, 28] and references therein). In this paper, we provide a construction of Φ^0 in terms of random geometric objects associated with Schramm-Loewner Evolutions (SLEs) [31] (see also [12, 19, 22, 44]) and Conformal Loop Ensembles (CLEs) [35, 36, 43] — namely, a gas (or random process) of continuum loops and associated clusters and (renormalized) area measures.

Two such loop processes arise in the results announced by Smirnov [37–41] (see also the work of Riva and Cardy in [29] — in particular Sections 6 and 7 there) that the full scaling limit of critical Ising spin cluster boundaries (respectively, FK random cluster boundaries) is given by the (nested version of) CLE with parameter $\kappa = 3$ (resp., $\kappa = 16/3$). One can try to associate with each continuum cluster \mathcal{C}_j^* or external boundary loop \mathcal{L}_j^* in the scaling limit a finite area measure μ_j^* representing the rescaled number of sites in the corresponding lattice cluster (where $*$ is SP for the spin case and FK for the random cluster case). We can in fact do this for the FK case and expect it to also be valid for the spin case.

Although one might try to represent the Euclidean field Φ^0 using spin clusters by a sum $\sum_k \chi_k \mu_k^{SP}$, where the χ_k 's are $+1$ or -1 depending on whether \mathcal{C}_k^{SP} corresponds to a $+$ or $-$ spin cluster, this does not seem to work. Instead, we use the FK clusters, which leads to $\Phi^0 = \sum_j \eta_j \mu_j^{FK}$, where the η_j 's are independent random signs. The (countable) family $\{\mu_j^{FK}\}$ is a “point” process with each μ_j^{FK} a “point” and where distinct “points” should be orthogonal measures.

For a bounded $\Lambda \subset \mathbb{R}^2$ with nonempty interior, one expects that $\sum_j \mu_j^{FK}(\Lambda) = \infty$. This would follow from the scaling covariance expected for $\{\mu_j^{FK}\}$ and described at the end of this section. The same happens for the corresponding measures in independent percolation that count so-called “one-arm” sites, as follows from work of Garban, Pete and Schramm [16, 17]. Nevertheless, for any $\varepsilon > 0$ only finitely many μ_j^{FK} 's will have support that intersects Λ and has diameter greater than ε . Furthermore, with probability one, $\sum_j [\mu_j^{FK}(\Lambda)]^2 < \infty$ which leads to convergence (at least in L_2) of the sum with random signs $\sum_j \eta_j \mu_j^{FK}(\Lambda)$. We note that divergence of $\sum_j \mu_j^{FK}$ means that $\Phi^0 = \sum_j \eta_j \mu_j^{FK}$ is not a signed measure; i.e., even restricted to a bounded Λ , it is not the difference of two positive *finite* measures. For negative results of a similar sort, but in the context of Gaussian random fields, see [15].

In the next section, we set up notation for the Ising model on the square lattice and its FK representation and review how the scaling limit of FK cluster boundaries may be viewed as a process of noncrossing continuum loops \mathcal{L}_j^{FK} and associated continuum clusters \mathcal{C}_j^{FK} . We then show why the natural scaling for the Ising spin variables at criticality to obtain a Euclidean (random) field Φ^0 leads to natural rescaled area measures μ_j^{FK} supported on \mathcal{C}_j^{FK} and to the representation of Φ^0 in terms of those measures. We also discuss why area measures μ_k^{SP} for spin clusters are not appropriate for representing Φ^0 , by using an example taken from the infinite temperature Ising model on the triangular lattice, \mathbb{T} .

In Section 3, we use (see Proposition 3.1) and then discuss how to verify a decay property of the critical Ising two-point correlation or equivalently the FK connectivity

function. Another essential ingredient in our analysis is a bound (see Prop. 3.2) on the number of macroscopic FK clusters. Although we focus on critical Ising-FK percolation on \mathbb{Z}^2 , similar arguments can be applied to other lattices and to independent (and, in principle, Ising spin) percolation. The case of independent percolation is discussed at the end of Section 3. In Section 4, we review the general conclusions of our work and discuss extensions to off-critical (or as they are sometimes called, near-critical) scaling limits, either as temperature $T \rightarrow T_c$, the critical temperature, with magnetic field $h = 0$, or else as $h \rightarrow 0$ with $T = T_c$. Finally, we propose there that a cluster area measure representation should also be valid for the $d = 3$ Ising model and for the $d = 2$ q -state Potts model with $q = 3$ or 4.

Before concluding this section, we wish to emphasize that this paper is meant to serve as an introduction, readable by both mathematicians and physicists, to a *representation* for the Ising scaling limit field Φ^0 in terms of the limit rescaled area measures $\{\mu_j^{FK}\}$. We hope this will prove useful in providing a general conceptual framework for field-based scaling limits like Aizenman and Burchard [2] did for connectivity-based ones. Although detailed explanations and proofs are provided in this paper for certain issues, others are avoided. In particular, although the next two sections of the paper provide arguments for the existence of both Φ^0 and $\{\mu_j^{FK}\}$ as (subsequence) limits of the corresponding lattice quantities, they do not provide the tools needed to prove that the limits are unique. This will be done in a future paper in collaboration with C. Garban, along with a proof of related properties such as that Φ^0 and $\{\mu_j^{FK}\}$ have the expected conformal covariance including that for $\alpha > 0$, $\alpha^{1/8}\Phi^0(\alpha z)$ and $\{\alpha^{-15/8}\mu_j^{FK}(d(\alpha z))\}$ are equidistributed with $\Phi^0(z)$ and $\{\mu_j^{FK}(dz)\}$.

2 Ising (Euclidean) Field

We consider the standard Ising model on the square lattice \mathbb{Z}^2 with Hamiltonian

$$\mathbf{H} = - \sum_{\{x,y\}} S_x S_y - h \sum_x S_x,$$

where the first sum is over nearest neighbor pairs in \mathbb{Z}^2 (or bonds $b = \{x, y\}$), the spin variables S_x are (± 1) -valued and the external field h is in \mathbb{R} .

When there is a unique infinite volume Gibbs distribution for some value of h and inverse temperature $\beta = 1/T$, we denote by $\langle \cdot \rangle_{\beta, h}$ its expectations. There is a critical β_c such that nonuniqueness occurs only for $h = 0$ and $\beta > \beta_c$. In particular, the critical Gibbs measure is unique and in that case we use the notation $\langle \cdot \rangle_c = \langle \cdot \rangle_{\beta_c, 0}$. By translation invariance, the two-point correlation $\langle S_x S_y \rangle_{\beta, h}$ is a function only of $y - x$, which in the critical case we denote by $\tau_c(y - x)$.

We want to study the random field associated with the spins on the rescaled lattice $a\mathbb{Z}^2$ in the scaling limit $a \rightarrow 0$. More precisely, for test functions $f(z)$ of bounded support

on \mathbb{R}^2 , we can define for the critical model

$$\Phi^a(f) = \int_{\mathbb{R}^2} f(z)\Phi^a(z)dz = \int_{\mathbb{R}^2} f(z)[\Theta_a \sum_{x \in \mathbb{Z}^2} S_x \delta(z - ax)]dz = \Theta_a \sum_{z \in a\mathbb{Z}^2} f(z)S_{z/a}, \quad (1)$$

with an appropriate choice of the scale factor Θ_a . Since $\Phi^a(f)$ is a random variable with zero mean, it is natural to choose Θ_a so that $\langle [\Phi^a(f)]^2 \rangle_c$ is bounded away from 0 and ∞ as $a \rightarrow 0$. Choosing Θ_a so that this second moment is exactly one for f the indicator function of the unit square $[0, 1]^2$ yields

$$\Theta_a^{-1} = \sqrt{\sum_{z, w \in \Lambda_{1,a}} \langle S_{z/a} S_{w/a} \rangle_c} = \sqrt{\sum_{x, y \in \Lambda_{1/a}} \tau_c(y - x)}, \quad (2)$$

where $\Lambda_{L,a} = [0, L]^2 \cap a\mathbb{Z}^2$ and $\Lambda_L = \Lambda_{L,1} = [0, L]^2 \cap \mathbb{Z}^2$.

One way to formulate the FK representation of the Ising model (for $h = 0$ and $\beta \leq \beta_c$) is that coexisting with the (± 1) -valued spin variables S_x on the sites x of \mathbb{Z}^2 are $\{0, 1\}$ -valued occupation variables n_b on the bonds $b = \{x, y\}$ of \mathbb{Z}^2 . The occupied or open ($n_b = 1$) FK bonds determine FK clusters, C_i , which are the sets of sites x in \mathbb{Z}^2 connected to each other by paths of open FK bonds. One can generate the S_x 's from the n_b 's by assigning independent symmetric ± 1 random signs η_i to the C_i 's and then setting $S_x = \eta_i$ for every x in C_i . If we write $x \xleftrightarrow{FK} y$ to denote that x and y are in the same FK cluster, it is immediate that the FK connectivity function at criticality is simply given by

$$P(x \xleftrightarrow{FK} y) = \langle S_x S_y \rangle_c = \tau_c(y - x).$$

Denoting by E_c expectation in the critical system, by \hat{C}_i^a the restriction of the cluster aC_i in $a\mathbb{Z}^2$ to $[0, 1]^2$, and by $|\hat{C}_i^a|$ the number of $(a\mathbb{Z}^2)$ -sites in \hat{C}_i^a , we have

$$\Theta_a^{-2} = \sum_{x, y \in \Lambda_{1/a}} \tau_c(y - x) = \sum_{x, y \in \Lambda_{1/a}} P(x \xleftrightarrow{FK} y) = E_c \left(\sum_i |\hat{C}_i^a|^2 \right). \quad (3)$$

By the definition of Θ_a we see that the rescaled areas $W_i^a = \Theta_a |\hat{C}_i^a|$ are uniformly square summable in the sense that $E_c \sum_i (W_i^a)^2 = 1$ for all a . We would like to argue that, at least along subsequences of a 's tending to zero, $\{W_i^a\}$ has a nontrivial limit in distribution. This is already partly clear — i.e., no W_i^a can diverge to $+\infty$. But what prevents them all from tending to zero as $a \rightarrow 0$? It turns out that this uses the following hypothesis about $\tau_c(y - x)$ (where $1/\sqrt{2}$ is the appropriate constant for the lattice \mathbb{Z}^2) — roughly speaking, that it decays like $\|y - x\|^{-2\theta}$ with $\theta < 1$, where $\|\cdot\|$ denotes Euclidean norm or that $\sum_{\|x\| \leq r} \tau_c(x)$ diverges as a power when $r \rightarrow \infty$. (It also uses that the crossing probability of an annulus is bounded away from one as $a \rightarrow 0$ — see (10) and Prop. 3.2.)

Hypothesis 2.1. *For some fixed $\theta < 1$, there are constants $K_1 > 0$ and $K_2 < \infty$ such that for any small $\varepsilon > 0$ and then for any $x \in \mathbb{Z}^2$ with large $\|x\|$,*

$$K_2 \tau_c(x_\varepsilon) \geq \tau_c(x) \geq K_1 \varepsilon^{2\theta} \tau_c(x_\varepsilon) \quad (4)$$

for any $x_\varepsilon \in \mathbb{Z}^2$ with $\|x_\varepsilon - \varepsilon x\| \leq 1/\sqrt{2}$.

As we will discuss, the clusters $\{C_i^a = aC_i\}$ on the rescaled lattice $a\mathbb{Z}^2$ will converge in the scaling limit to full plane continuum clusters $\{\mathcal{C}_j^{FK}\}$ in \mathbb{R}^2 . In that limit most of the lattice clusters disappear because they are not of macroscopic size. The importance of the lower bound on $\tau_c(x)$ in (4) is that it guarantees (see Prop. 3.1) that the rescaled areas of the microscopic clusters are negligible (at least in a square summable sense). That is, the contribution to $\sum_i (W_i^a)^2$ coming from clusters C_i^a whose intersection with the unit square has small macroscopic diameter is negligible. A corresponding statement is true for the clusters that contribute to the field $\Phi^a(f)$ for more general test functions f of bounded support. The significance of the upper bound on $\tau_c(x)$ in (4) is that it easily implies that $\langle [\Phi^a(f)]^2 \rangle_c$ is bounded away from 0 and ∞ as $a \rightarrow 0$.

In a series of papers, the authors constructed [9] a certain process of loops in the plane and proved [10] (see also [11]) convergence to it in the scaling limit of the collection of boundaries of all (macroscopic) clusters for critical independent site percolation on the triangular lattice. In the limit there is no self-crossing or crossing of different loops but there is self-touching and touching between different loops. Moreover, the loops are locally SLE_6 curves.

Similar results for the 2D critical Ising model on the square lattice have been announced by Smirnov [37–41]. There one considers either the boundaries between plus and minus spin clusters [41], or the loops in the medial lattice that separate FK from dual FK clusters [40] (see Figure 1). We will focus on those loops which separate FK clusters in the original \mathbb{Z}^2 lattice on their *inside* from dual FK clusters in the dual lattice on their *outside*. In the scaling limit of spin cluster boundaries one would obtain simple loops that do not touch each other and locally are SLE_3 type curves. In the case of FK cluster boundaries, there would instead be self-touching and touching between different loops (but no crossing), like in the percolation case. Now, however, the loops would locally be $\text{SLE}_{16/3}$ type curves.

In the FK case, each loop L_i^a that we consider on the medial lattice of $a\mathbb{Z}^2$ is the outer boundary of a rescaled FK cluster C_i^a . The inner boundary of C_i^a is made of “daughter” loops $L_{i,k}^a$ corresponding to the “holes” in C_i^a . In the scaling limit $a \rightarrow 0$, one can analogously identify a continuum cluster \mathcal{C}_j^{FK} as the closed set left after removing from \mathbb{R}^2 the (open) exterior of the loop \mathcal{L}_j^{FK} and the (open) interiors of its daughter loops $\mathcal{L}_{j,n}^{FK}$ (with interiors and exteriors defined using winding numbers). We remark that because the scaling limit is only a limit in distribution and no special effort was made to coordinate indexing for clusters in the lattice and in the continuum, we use different letters, i and j , for the two indices. We denote by $\{\mu_j^{FK}\}$ the finite measures supported on $\{\mathcal{C}_j^{FK}\}$ corresponding to the limit of the rescaled areas $\{W_i^a\}$ as $a \rightarrow 0$, in the sense, e.g., that $\{\mu_j^{FK}(\Lambda_1)\}$ is the scaling limit of the rescaled areas $\{W_i^a\}$. The existence and nontriviality of $\{\mu_j^{FK}(\Lambda_1)\}$ (or of $\{\mu_j^{FK}(f) = \int f(z)\mu_j^{FK}(dz)\}$ for more general test functions $f(z)$ of bounded support) will follow from Hypothesis 2.1 (see Prop. 3.1) and Prop. 3.2, as noted above. The collection $\{\mu_j^{FK}\}$ ought to be a functional of $\{\mathcal{L}_j^{FK}\}$ as has recently been proved in the independent percolation context by Garban, Pete and Schramm [16, 17].

Letting $\{\eta_j\}$ denote i.i.d. symmetric (± 1) -valued variables, one obtains the following

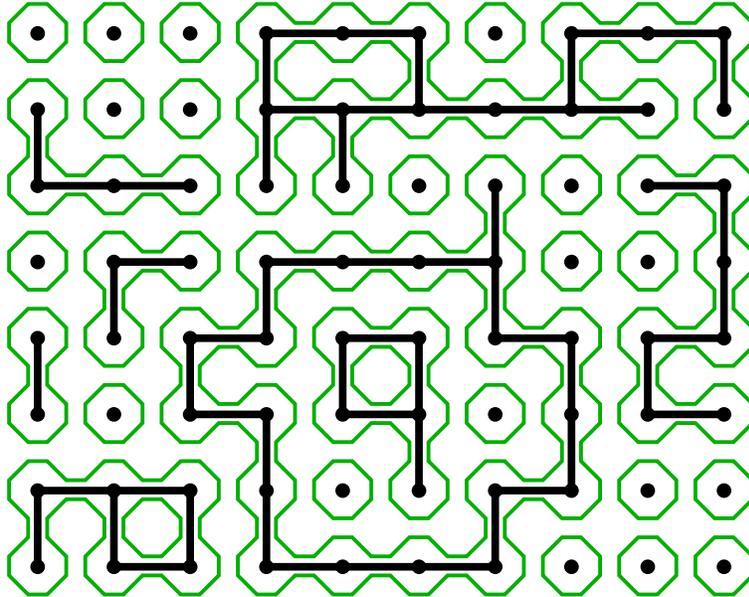


Figure 1: Example of an FK bond configuration in a rectangular region and the associated loops in the medial lattice. Black dots represent sites of \mathbb{Z}^2 , black horizontal and vertical edges represent open FK bonds and the lighter (green) loops are on the medial lattice. We focus on those loops which have \mathbb{Z}^2 -sites immediately on their *inside*.

representation of the Euclidean field Φ^0 : for test functions $f(z)$ of bounded support,

$$\Phi^0(f) = \sum_j \eta_j \mu_j^{FK}(f) = \int_{\mathbb{R}^2} f(z) \sum_j \eta_j \mu_j^{FK}(dz). \quad (5)$$

To be more precise, the sums in (5) should first be restricted to clusters with diameter greater than ε and then convergence (in L_2) as the cutoff $\varepsilon \rightarrow 0$ will follow from the square summability discussed earlier.

As noted in the Introduction, one might be tempted to represent the Euclidean field using spin clusters and hence SLE₃ type loops. If we use $\{\mu_k^{SP+}\}$ and $\{\mu_{k'}^{SP-}\}$ to denote the limits of appropriately rescaled areas of plus and minus spin clusters, respectively, then on a formal level, by decomposing the righthand side of (1) into the contribution from plus and minus clusters, one might expect that Φ^0 of (5) would also be given by $\sum \mu_k^{SP+} - \sum \mu_{k'}^{SP-}$ (with some resummation needed to handle the difference of two presumably divergent series) as an alternative to $\sum \eta_j \mu_j^{FK}$. This appears not to be so, as can be understood by considering the simple situation of the Ising model on the triangular lattice \mathbb{T} at $\beta = 0$.

The latter is noncritical as an Ising model and the correct Euclidean field obtained by using the noncritical FK clusters (which are just isolated sites since $\beta = 0$) and the $\beta = 0$ version of (2), is two-dimensional Gaussian white noise. But if one considers the Ising spin clusters, this is critical independent site percolation on \mathbb{T} and the formal

expression $\sum \mu_k^{SP+} - \sum \mu_{k'}^{SP-}$, besides the resummation issue, seems unrelated to white noise. Indeed, applying and then removing a cutoff, as explained after (5), in this case would probably not lead to the physically correct limit.

3 Area Measure

In the previous section we gave a representation of the Ising Euclidean spin field in terms of rescaled counting measures that give the “areas” of macroscopic Ising-FK clusters. In this section we first explain how to use Hypothesis 2.1 to get the existence of nontrivial limits in distribution of these area measures, at least along subsequences of a 's tending to zero. We then explain how to verify Hypothesis 2.1 first for critical Ising-FK percolation on \mathbb{Z}^2 and then for critical independent site or bond percolation on \mathbb{T} or \mathbb{Z}^2 . Using the notation introduced in Section 2 and denoting by $\text{diam}(\hat{C}_i^a)$ the Euclidean diameter of \hat{C}_i^a , we have the following proposition.

Proposition 3.1. *Hypothesis 2.1 implies that*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{a \rightarrow 0} \Theta_a^2 E_c \left(\sum_{i: \text{diam}(\hat{C}_i^a) \leq \varepsilon} |\hat{C}_i^a|^2 \right) = 0.$$

The usefulness of Prop. 3.1 is based on the additional result that for every fixed ε , in the scaling limit there will only be finitely many FK clusters with diameter larger than ε that intersect $[0, 1]^2$; this important feature of the scaling limit will be discussed below — see Prop. 3.2. Once one has Prop. 3.2, it then follows from Prop. 3.1 that the collection $\{\Theta_a |\hat{C}_i^a|\}$ has nontrivial subsequential limits; i.e., it is not possible that all $\Theta_a |\hat{C}_i^a|$'s scale to zero as $a \rightarrow 0$. Said more physically, Prop. 3.1 implies that there is a negligible contribution to the magnetization from FK clusters whose linear size is small on a macroscopic lengthscale while Prop. 3.2 says that there are only finitely many larger clusters touching any bounded region. Together, they lead to the representation (5).

Proof of Proposition 3.1. Using Hypothesis 2.1, we can compare $\sum_{z' \in \Lambda_{\varepsilon' r}} \tau_c(z')$ for small ε' as $r \rightarrow \infty$ to $\sum_{z \in \Lambda_r} \tau_c(z)$ by using the second inequality of (4) to compare each $\tau_c(z')$ to the $\tau_c(z)$'s with $\varepsilon' z$ in the unit length square centered on z' (so that we may take z' as $z_{\varepsilon'}$). Since there are approximately $(1/\varepsilon')^2$ such z sites, we have that

$$\liminf_{r \rightarrow \infty} \frac{\sum_{z \in \Lambda_r} \tau_c(z)}{(1/\varepsilon')^2 (\varepsilon')^{2\theta} \sum_{z' \in \Lambda_{\varepsilon' r}} \tau_c(z')} \geq K_1.$$

Using this lower bound (with $r = 1/2a$ and $\varepsilon' = 2\varepsilon$) and (3), we have that

$$\begin{aligned}
\limsup_{a \rightarrow 0} \Theta_a^2 E_c \left(\sum_{i: \text{diam}(\hat{C}_i^a) \leq \varepsilon} |\hat{C}_i^a|^2 \right) &\leq \limsup_{a \rightarrow 0} \frac{\sum_{x, y \in \Lambda_{1/a}, \|x-y\| \leq \varepsilon/a} \tau_c(y-x)}{\sum_{x, y \in \Lambda_{1/a}} \tau_c(y-x)} \\
&\leq \limsup_{a \rightarrow 0} \frac{K'(1/a)^2 \sum_{z' \in \Lambda_{\varepsilon/a}} \tau_c(z')}{K''(1/a)^2 \sum_{z \in \Lambda_{1/2a}} \tau_c(z)} \\
&= K'''' \varepsilon^{2(1-\theta)}. \tag{6}
\end{aligned}$$

The proposition follows from the observation that the last expression in (6) tends to zero as $\varepsilon \rightarrow 0$ since $\theta < 1$. \square

The next two lemmas will be used to verify Hypothesis 2.1. Let $B_x(r)$ denote $\{y \in \mathbb{Z}^2 : \|y-x\| \leq r\}$, and denote its \mathbb{Z}^2 -boundary by $\partial B_x(r)$. If the subscript is omitted, we refer to the disc centered at the origin 0. We denote by $P_{\partial B(r)}^W$ (W for wired) the critical FK measure inside $B(r)$ with wired (i.e., everything connected) boundary condition on $\partial B(r)$. The next lemma is based on the FKG inequalities.

Lemma 3.1.

$$\tau_c(y-x) \leq P_{\partial B(\|x-y\|/3)}^W(0 \xleftrightarrow{FK} \partial B(\|x-y\|/3)) P(0 \xleftrightarrow{FK} \partial B(\|x-y\|/3)).$$

Proof of Lemma 3.1.

$$\begin{aligned}
\tau_c(y-x) &\leq P(x \xleftrightarrow{FK} \partial B_x(\|x-y\|/3) \text{ and } y \xleftrightarrow{FK} \partial B_y(\|x-y\|/3)) \\
&\leq P(x \xleftrightarrow{FK} \partial B_x(\|x-y\|/3) \mid y \xleftrightarrow{FK} \partial B_y(\|x-y\|/3)) \\
&\quad P(y \xleftrightarrow{FK} \partial B_y(\|x-y\|/3)) \\
&\leq P_{\partial B(\|x-y\|/3)}^W(0 \xleftrightarrow{FK} \partial B(\|x-y\|/3)) P(0 \xleftrightarrow{FK} \partial B(\|x-y\|/3)),
\end{aligned}$$

where in the last step we have used FKG. \square

The next lemma uses RSW bounds [30, 34], namely, that the probability $p_{FK}^a(x; r_1, r_2)$ that there is an open FK $a\mathbb{Z}^2$ -circuit in an (r_1, r_2) -annulus centered at x is bounded away from zero and one as $a \rightarrow 0$ by constants that depend only on r_1/r_2 . In fact, we only need a lower bound; i.e.,

$$\text{for any } x \in \mathbb{R}^2 \text{ and some } 0 < r_1 < r_2 < \infty, \liminf_{a \rightarrow 0} p_{FK}^a(x; r_1, r_2) > 0. \tag{7}$$

This is not immediate in the Ising case, since there is not currently a direct proof of RSW for critical FK percolation (as opposed to the independent percolation case). However, as we explain after the proof of the lemma, RSW follows from announced results about the scaling limit of spin cluster boundaries [41], combined with the Brownian loop soup representation of CLE_3 [24, 36, 43]; also, the lower bound (7) for some r_1, r_2 implies both upper and lower bounds for all r_1, r_2 .

Lemma 3.2. *Assuming (7), there exists a constant $K > 0$ such that*

$$\tau_c(y - x) \geq KP_{\partial B(\|x-y\|/3)}^W(0 \xleftrightarrow{FK} \partial B(\|x-y\|/3))^2.$$

Before giving the proof, we state an immediate consequence of this lemma, the preceding one and the fact that $P_{\partial B(r)}^W(0 \xleftrightarrow{FK} \partial B(r)) \geq P(0 \xleftrightarrow{FK} \partial B(r))$.

Corollary 3.1. *Assuming (7), $P(0 \xleftrightarrow{FK} \partial B(\|x-y\|/3))$ and $P_{\partial B(\|x-y\|/3)}^W(0 \xleftrightarrow{FK} \partial B(\|x-y\|/3))$ are comparable (up to constants) as $\|x-y\| \rightarrow \infty$.*

Proof of Lemma 3.2. Let $A(x, r)$ denote the intersection of \mathbb{Z}^2 and the annulus with outer radius r and inner radius $r/2$ centered at x and let $\text{circ}_{FK}(A(x, r))$ denote the event that there is an open FK circuit in $B_x(r)$ surrounding $B_x(r/2)$. Let $F(x, r)$ be the event that $\text{circ}_{FK}(A(x, r))$ occurs and the outermost open FK circuit contained in $B_x(r)$ and surrounding $B_x(r/2)$ is connected to x by an open FK path. We have

$$\begin{aligned} \tau_c(y - x) &\geq P(F(x, \|x-y\|/3) \cap F(y, \|x-y\|/3)) \\ &\quad \cap \partial B_x(\|x-y\|/6) \xleftrightarrow{FK} \partial B_y(\|x-y\|/6)) \\ &\geq K'' P(F(0, \|x-y\|/3))^2, \end{aligned} \tag{8}$$

where the second inequality follows from FKG and the constant $K'' > 0$ follows from RSW. We then note that

$$\begin{aligned} P(F(0, r)) &= \sum_{\text{circuits } \gamma} P(\gamma \text{ is the outermost open circuit in } A(0, r) \text{ and } 0 \xleftrightarrow{FK} \gamma) \\ &= \sum_{\text{circuits } \gamma} P(0 \xleftrightarrow{FK} \gamma \mid \gamma \text{ is the outermost open circuit in } A(0, r)) \\ &\quad P(\gamma \text{ is the outermost open circuit in } A(0, r)) \\ &\geq \sum_{\text{circuits } \gamma} P_{\partial B(r)}^W(0 \xleftrightarrow{FK} \gamma) P(\gamma \text{ is the outermost open circuit in } A(0, r)) \\ &\geq P_{\partial B(r)}^W(0 \xleftrightarrow{FK} \partial B(r)) \\ &\quad \sum_{\text{circuits } \gamma} P(\gamma \text{ is the outermost open circuit in } A(0, r)) \\ &= P_{\partial B(r)}^W(0 \xleftrightarrow{FK} \partial B(r)) P(\exists \text{ an open circuit in } A(0, r)). \end{aligned}$$

Inserting this bound into (8) concludes the proof. \square

RSW for FK percolation on \mathbb{Z}^2 . In order to get RSW, we assume (from [41]) that the “full” scaling limit of the Ising model converges to (the nested version of) CLE_3 . We can then use the representation of CLE_3 in terms of the Brownian loop soup [24, 36, 43], assuming that $\kappa = 3$ corresponds to a density of the Brownian loop soup below its critical density (which should correspond to $\kappa = 4$). A single Brownian loop has positive

probability of “surrounding” a disc of fixed radius r_1 centered at the origin. Let γ be such a loop and consider its loop-cluster, built recursively from the (countably many) Brownian loops by saying that any two loops which touch (and thus cross, with probability one) are in the same loop-cluster. Given that the density of the Brownian loop soup is assumed to be below the critical density, the loop-cluster of γ is contained with probability one inside a sufficiently large disc. Thus, for some $r_2 > r_1$, there is strictly positive probability that the (r_1, r_2) -annulus centered at the origin contains a CLE_3 circuit. Back on the lattice $a\mathbb{Z}^2$, this gives a positive probability, bounded away from zero as $a \rightarrow 0$, that the external boundary of an Ising spin cluster provides such a circuit. But this in turn implies the same for a closed (dual) FK circuit, and hence by self-duality at the critical point, the same for an open FK circuit.

To conclude the discussion of RSW, we note that it is not difficult to show that one can use FKG to obtain from open circuits contained in overlapping (r_1, r_2) -annuli a “necklace” structure that provides open crossings of rectangles of arbitrary aspect ratio (see [6] for more details about such arguments). The rectangle crossings can then be used, once again with the help of FKG, to obtain circuits inside arbitrary annuli with probability bounded away from zero. By self-duality one also has closed (dual) crossings of rectangles and these can be used to bound the probability of open circuits away from one.

Proposition 3.2. *For $z \in \mathbb{R}^2$, let $N^a(z, r_1, r_2)$ denote the number of distinct clusters C_i^a that include sites in both $\{y \in a\mathbb{Z}^2 : \|y - z\| < r_1\}$ and $\{y \in a\mathbb{Z}^2 : \|y - z\| > r_2\}$. Assuming (7), for any $0 < r_1 < r_2 < \infty$, there exists $\lambda \in (0, 1)$ such that for all $z \in \mathbb{R}^2$ and all small $a > 0$ and any $k = 1, 2, \dots$,*

$$P(N^a(z, r_1, r_2) \geq k) \leq \lambda^k. \tag{9}$$

It follows that for any bounded $\Lambda \subset \mathbb{R}^2$ and $\varepsilon > 0$, the number of distinct clusters C_i^a of diameter $> \varepsilon$ touching Λ is bounded in probability as $a \rightarrow 0$.

Proof of Proposition 3.2. The proof is by induction on k . For $k = 1$, the result follows from RSW since $N^a(z, r_1, r_2) \geq 1$ is equivalent to the *absence* of a closed (dual) circuit in the (r_1, r_2) -annulus about z , which by self-duality at the critical point has the same probability as absence of an open circuit, which in turn is bounded away from one as $a \rightarrow 0$. Now suppose $N^a(z, r_1, r_2) \geq k - 1$. Then one may do an exploration of the C_i^a 's that touch $\{y \in a\mathbb{Z}^2 : \|y - z\| < r_1\}$ until $k - 1$ are found that reach $\{y \in a\mathbb{Z}^2 : \|y - z\| > r_2\}$, making sure that all cluster explorations have been fully completed without obtaining information about the outside of the clusters. At that point, the complement D of some random finite $D^c \subset a\mathbb{Z}^2$ remains to be explored and the (conditional) FK distribution in D is $P_{\partial D}^F$ with a *free* boundary condition on the boundary (or boundaries) between D and D^c . By RSW, the $P_{\partial D}^F$ -probability of an open crossing in D of the (r_1, r_2) -annulus is

bounded above by the original $P(N^a(z, r_1, r_2) \geq 1)$. Thus we have

$$\begin{aligned}
P(N^a(z, r_1, r_2) \geq k) &= P(N^a(z, r_1, r_2) \geq k - 1) \\
&\quad P(N^a(z, r_1, r_2) \geq k \mid N^a(z, r_1, r_2) \geq k - 1) \\
&= P(N^a(z, r_1, r_2) \geq k - 1) E(P_{\partial D}^F(N^a(z, r_1, r_2) \geq 1)) \\
&\leq P(N^a(z, r_1, r_2) \geq k - 1) P(N^a(z, r_1, r_2) \geq 1) \\
&\leq \lambda^k.
\end{aligned}$$

The last claim of the proposition follows from (9) because one may choose $O([\text{diam}(\Lambda)/\varepsilon]^2)$ points z_ℓ in \mathbb{R}^2 so that any C_i^a of diameter $> \varepsilon$ touching Λ will be counted in $N^a(z_\ell, \varepsilon/4, \varepsilon/2)$ for at least one z_ℓ . \square

We next explain how to verify Hypothesis 2.1 for critical Ising-FK percolation and for independent percolation; we do not have a verification for Ising spin percolation, although we expect it to be true in that case also. It may be of interest to note that for critical independent percolation, one can obtain a representation like (5), but with the $\text{SLE}_{16/3}$ -based measures μ_j^{FK} replaced by SLE_6 -based ones μ_j^{IN} , for the scaling limit of the lattice “divide and color” model [18]. (Here and below we use the letters IN to distinguish independent from FK percolation.) The original divide and color model, and the one most analogous to the FK representation of the ($h = 0$) Ising model, takes the open clusters of independent *bond* percolation, e.g., on \mathbb{Z}^2 , and colors them with random ± 1 signs to define the divide and color spin variables. In Sec. 4, we will consider this model as the density p of open bonds approaches its critical value (we note that in [3] a different phase transition is studied). For independent *site* percolation, e.g., on \mathbb{T} , with say probability p and $1 - p$ for white and black sites, the option for defining the divide and color spin variables that we will use is to “color” *both* the white and black clusters with random signs. It is unclear whether the scaling limit of the critical ($p = 1/2$ on \mathbb{T}) divide and color model corresponds to some known conformal field theory. Note that the limit, in terms of boundaries between clusters of different colors, is a sort of “dilute CLE_6 ” and is conformally invariant, but is not itself described by a CLE since the divide and color model lacks the “domain Markov property.”

Hypothesis 2.1 for FK percolation on \mathbb{Z}^2 . In this case, the behavior of the two-point function is known exactly along the $(1, 1)$ direction from Ising calculations, which yield $\tau_c(y - x) \sim K \|x - y\|^{-1/4}$ (e.g., [46], referred to in [42], and Chap. XI of [25]). Using Lemmas 3.1 and 3.2 one obtains that, up to constants, the two-point function has the same behavior in all directions. Hypothesis 2.1 is then satisfied with $2\theta = 1/4$.

Hypothesis 2.1 for independent percolation. From the analogues of Lemmas 3.1 and 3.2 for independent percolation, we know that $\tau_c^{IN}(z)$ is comparable (up to constants) with $[P(0 \xrightarrow{IN} \partial B(\|z\|/3))]^2$. This immediately gives the desired upper bound for $\tau_c^{IN}(x)$. For the lower bound, it suffices to show that $P(0 \xrightarrow{IN} \partial B(r)) \geq K'''(\varepsilon')^\theta P(0 \xrightarrow{IN} \partial B(\varepsilon'r))$

for some constant $K''' > 0$. Using FKG, we have

$$\begin{aligned}
P(0 \xrightarrow{IN} \partial B(r)) &\geq P(0 \xrightarrow{IN} \partial B(\varepsilon' r) \cap \partial B(\varepsilon' r/2) \xrightarrow{IN} \partial B(r) \cap \text{circ}_{IN}(A(0, \varepsilon' r))) \\
&\geq P(0 \xrightarrow{IN} \partial B(\varepsilon' r)) P(\partial B(\varepsilon' r/2) \xrightarrow{IN} \partial B(r)) P(\text{circ}_{IN}(A(0, \varepsilon' r))) \\
&\geq K''' (\varepsilon')^\theta P(0 \xrightarrow{IN} \partial B(\varepsilon' r)),
\end{aligned}$$

where $P(\text{circ}_{IN}(A(0, \varepsilon' r))) \geq \tilde{K}$ from RSW, and $P(\partial B(\varepsilon' r/2) \xrightarrow{IN} \partial B(r)) \geq \hat{K} (\varepsilon')^\theta$ with $\theta = \alpha - \delta$ for any $\delta > 0$, and α denoting the one-arm exponent.

For site percolation on the triangular lattice, α has been proved to be $5/48$ using SLE computations [23]. For other percolation models (e.g., bond percolation on the square lattice), the five-arm exponent is known to be equal to 2 (see Lemma 2 of [21], Corollary A.8 of [32] and Section 5.2 of [26]), since it can be derived via a general argument that does not use SLE. Using this and the BK inequality [5], we obtain an upper bound of $2/5$ for α . We note, as pointed out to us by P. Nolin, that a more elementary argument from [5] is available showing that $\alpha \leq 1/2$ without use of the five-arm exponent — see Eqn. (2.5) of [14].

4 Discussion

In this paper, we provided a representation (see (5)) for the scaling limit Euclidean random field Φ^0 associated with the $d = 2$ Ising model at its critical point ($T = T_c$, $h = 0$). This field, one of the basic objects of conformal field theory, is the scaling limit of the magnetization field Φ^a (see (1)) on $a\mathbb{Z}^2$ as $a \rightarrow 0$. Φ^0 is represented as a sum $\sum_j \eta_j \mu_j^{FK}$ with random signs η_j and finite measures μ_j^{FK} that are the limits of rescaled area measures associated with the macroscopic Ising-FK clusters. These measures are supported on continuum clusters whose outer boundaries are described by CLE_κ loops with $\kappa = 16/3$. A key to the representation is that natural field strength rescaling (see (2), (3)) insures that for bounded $\Lambda \subset \mathbb{R}^2$, $\sum_j [\mu_j^{FK}(\Lambda)]^2 < \infty$ and hence $\sum_j \eta_j \mu_j^{FK}(\Lambda)$ is convergent (in L_2).

We explained, at the end of Section 2, why the limits μ_j^{SP} of area measures for Ising spin clusters do not appear useful for representing Φ^0 . We also noted, towards the end of Section 3, that a field $\sum_j \eta_j \mu_j^{IN}$ can be constructed using critical clusters from independent in place of FK percolation, but that its physical significance is unclear. We next discuss how the representation (5) could be extended to off-critical models.

Independent percolation, $p \neq p_c$. Here the percolation density p (say of the white sites on \mathbb{T}) converges to the critical density, $p_c (= 1/2)$, appropriately as $a \rightarrow 0$. In this case, the representation of the near-critical field should involve area measures from the near-critical modification of CLE_6 obtained by the approach of [7, 8] and the results of Garban, Pete and Schramm [16, 17]. A feature of that work, which might also be valid in the FK-Ising context, is a natural probabilistic coupling, based on a Poissonian marking of certain pivotal locations, so that the one-parameter family of near-critical models parametrized by the strength of the off-critical perturbation lives on a single probability space. The

appropriate speed at which $p \rightarrow p_c$ is such that the correlation length is bounded away from zero and infinity. Since it is proved in [20] (and [26] — see Theorem 26 there) that crossing probabilities and multi-arm probabilities are comparable (up to constants and up to distances of the order of the correlation length) to those of the critical system, RSW still holds and both Hypothesis 2.1 and Prop. 3.2 can be verified. One of the area measures μ_j^{IN} , corresponding to the unique infinite cluster (white for $p \downarrow p_c$ and black for $p \uparrow p_c$), will now have unbounded support and infinite mass over \mathbb{R}^2 , but its mass in any bounded region Λ will be finite and $\sum_j \eta_j \mu_j^{IN}(\Lambda)$ will be convergent (in L_2 , at least for a bounded Λ not chosen in a way that depends on $\{\eta_j\}$). We note that for *bond* percolation on \mathbb{Z}^2 with p the density of open bonds, it will only be in the $p \downarrow p_c$ near-critical model that some μ_j^{IN} has infinite mass.

FK percolation, $T \neq T_c$. Here one keeps $h = 0$ in the Ising model but lets $T \rightarrow T_c$ appropriately as $a \rightarrow 0$, which is the analogue of $p \rightarrow p_c$ in independent percolation. It is natural to expect that the claims made above for independent percolation still hold in this case. Since one is now considering *bond* (FK) percolation, only for $T \uparrow T_c$ (the analogue of $p \downarrow p_c$ in independent percolation) will there be an infinite mass μ_j^{FK} with unbounded support in \mathbb{R}^2 . Including a random sign η_j for the infinite mass μ_j^{FK} means one is taking the scaling limit of the symmetric mixture of the plus and minus Gibbs measures.

FK percolation, $h \neq 0$. Here one sets $T = T_c$ with $h \neq 0$ and then lets $h \rightarrow 0$ appropriately as $a \rightarrow 0$. Intuitively, in the scaling limit, this should involve formally multiplying the measure describing the critical continuum system by a factor proportional to $\exp(\lambda \int_{\mathbb{R}^2} \Phi^0(z) dz)$. According to (5) the critical Euclidean field is given by the sum of all the $\eta_j \mu_j^{FK}$. Let ν^{FK} denote the marginal distribution of the process $\{\eta_j \mu_j^{FK}\}$ of finite signed measures in the plane, 1_L denote the indicator function of the $L \times L$ square in \mathbb{R}^2 , $Z_L = \int \exp(\lambda \Phi^0(1_L)) d\nu^{FK}$, and finally $d\nu_L^\lambda = (Z_L)^{-1} \exp(\lambda \Phi^0(1_L)) d\nu^{FK}$. We ask: does ν_L^λ converge to some ν^λ as $L \rightarrow \infty$ and is Φ^λ , obtained from ν^λ as the sum of its individual signed measures, the physically correct near-critical Euclidean field? Heuristically, the correct normalization to obtain a nontrivial near-critical scaling limit is such that the correlation length ξ remains bounded away from zero and infinity. Since $\xi \sim h^{-8/15}$ for small h , this gives $h \sim a^{15/8}$, which coincides with the normalization needed to obtain a nontrivial Euclidean field, as can be seen from (2) and the asymptotic behavior of τ_c . Using this observation and the $d = 2$ Ising critical exponent $\delta = 15$ for the magnetization (i.e., $M \sim h^{1/15}$), the rough computation (where \sum_x^L denotes the sum over x in $\Lambda_{L/a}$),

$$\frac{\langle a^{15/8} \sum_x^1 S_x \exp(a^{15/8} \sum_x^L S_x) \rangle_c}{\langle \exp(a^{15/8} \sum_x^L S_x) \rangle_c} \sim a^{-1/8} M(h = a^{15/8}) \sim a^{-1/8} (a^{15/8})^{1/15} = 1,$$

suggests a positive answer to the previous questions.

We conclude this section with brief discussions of the applicability of our approach to higher dimensions, $d > 2$, and to q -state Potts models with $q > 2$. Although the $d = 2$ scaling limit Ising magnetization field Φ^0 should be conformal with close connections to $\text{CLE}_{16/3}$, as we have indicated, very little conformal or SLE machinery was actually used in our analysis. Basically, the two main ingredients were (see Hypothesis 2.1) that

$\tau_c(y-x)$ behaves at long distance like $\|y-x\|^{-\psi}$ with $\psi < d$ and (see Prop. 3.2) that as $a = 1/L' \rightarrow 0$,

$$P(N^a(0, r_1, r_2) \geq 1) = P(B(r_1 L') \xleftrightarrow{FK} B(r_2 L')^c) \leq \lambda < 1. \quad (10)$$

Although such decay of τ_c should be valid for all $d \geq 2$, the crossing probability bound (10) is a different matter and presumably fails above the upper critical dimension (see Appendix A of [1]). When it fails, there can be infinitely many FK clusters with diameter greater than ε in a bounded region and so Prop. 3.1 would not preclude Φ^0 from being a Gaussian (free) field. But it appears that at least for $d = 3$, both (10) and a representation of Φ^0 as a sum of finite measures with random signs ought to be valid.

As pointed out to us by J. Cardy, an analogous representation for the scaling limit magnetization fields of q -state Potts models also ought to be valid, at least for values of q such that for a given d , the phase transition at T_c is second order. The phase transition is believed to be first order for integer $q \geq 3$ when $d \geq 3$ and for $q > 4$ when $d = 2$ — see [45]; this leaves, besides the Ising case, $d = 2$ and $q = 3$ and 4 . We denote the states or colors by $1, 2, \dots, q$ and recall that in the FK representation on the lattice, all sites in an FK cluster have the same color while the different clusters are colored independently with each color equally likely. In the scaling limit, there would be finite measures $\{\mu_j^{FK,q}\}$ and the magnetization field in the color- k direction would be $\sum_j \eta_j^k \mu_j^{FK,q}$ with the η_j^k 's taking the value $+1$ with probability $1/q$ (for the color k) and the value $-1/(q-1)$ with probability $(q-1)/q$ (for any other color). For a fixed k the η_j^k 's would be independent as j varies, but for a fixed j they would be *dependent* as k varies because $\sum_k \eta_j^k = 0$.

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